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Positive Definite Solutions of the Nonlinear Matrix Equation $X + A^{\mathrm{H}}\overline{X}^{-1}A = I$

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Abstract

This paper is concerned with the positive definite solutions to the matrix equation $X + A^{\mathrm{H}}\overline{X}^{-1}A = I$ where X is the unknown and A is a given complex matrix. By introducing and studying a matrix operator on complex matrices, it is shown that the existence of positive definite solutions of this class of nonlinear matrix equations is equivalent to the existence of positive definite solutions of the nonlinear matrix equation $W + B^{\mathrm{T}}W^{-1}B = I$ which has been extensively studied in the literature, where B is a real matrix and is uniquely determined by A. It is also shown that if the considered nonlinear matrix equation has a positive definite solution, then it has the maximal and minimal solutions. Bounds of the positive definite solutions are also established in terms of matrix A. Finally some sufficient conditions and necessary conditions for the existence of positive definite solutions of the equations are also proposed.
Keywords: Bound of solutions; Complex matrix; Nonlinear matrix equation; Positive definite solutions.

1 Introduction

Various kinds of matrix equations have received much attention in the literature (see, for example, [4], [7], [8], [11], [12], [13], [15], [14], [16], [19], [27], [28], [29], [34], [38], [39], [40], and the references therein). Especially, the problem of finding fixed points of the nonlinear matrix equation $X + A^*X^{-1}A = Q$ where A and $Q > 0$ are given and X is unknown, has been extensively studied in the last two decades. The interest of studying such problem mainly relies on its applications in many fields such as the analysis of ladder networks [2], dynamic programming [33], control theory [22], stochastic filtering [1] and statistics [32] (see [3], [18] and [19] for detailed introduction). Some generalized forms of the nonlinear matrix equation $X + A^*X^{-1}A = Q$ have received much attention in recent years (see, for example, [6], [17], [24], and [30]).

Among the existing publications in the literature, two kinds of results can be found. The first kind of results concentrate on providing analytical conditions on the existence of positive definite solutions and their corresponding properties. For example, the shorted operator theory was applied in [3] to study the existence of a positive definite solution; necessary and sufficient conditions in terms of symmetric factorizations of some rational matrix-valued function were derived in [18] for the existence of positive definite solutions; and some necessary and sufficient conditions were also derived [35] for the same problem in terms of some factorizations of the coefficient matrices. The other kind of results are mainly concerned with the numerical solutions of this class of nonlinear matrix equations. Basically, this can be accomplished via iterations including inversion-involved iterations [19], [21] and inversion-free iterations [20], [23], [31], [36].

In the present paper, we consider a variation of this well-studied nonlinear equation. We study the nonlinear matrix equation $X + A^*\overline{X}^{-1}A = Q$, which, as we will show in this paper, has totally different solutions from the solutions of $X + A^*X^{-1}A = Q$. In particular, we are interested in the existence of positive definite solutions of such kind of nonlinear matrix equations. Via some specific representations of complex matrices,

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we are able to transform the equation $X+A^*\overline{X}^{-1}A=Q$ into the equation $W+B^*W^{-1}B=P$, where B and P are determined by A and Q, respectively. This allows us to study the original nonlinear matrix equations with the help of the existing results on the equation $X+A^*X^{-1}A=Q$. Other topics of this paper include the estimate of the bounds on the solutions, sufficient conditions, and necessary conditions on guaranteeing a positive definite solution.

The rest of this paper is organized as follows. The problem formulation and some preliminary results to be used are given in Section 2. In Section 3, we present necessary and sufficient conditions for the existence of a positive definite solution of the considered nonlinear matrix equations. Both upper bounds and lower bounds of the solutions will be established in Section 4, while the necessary conditions and sufficient conditions guaranteeing a positive definite solution are given in Section 5. We will draw the conclusions of this paper in Section 6.

Notation: In this paper, for a matrix A, we use $A^{\mathrm{T}}, A^*, \overline{A}$, $\lambda(A)$, $\det(A)$, $\|A\|$ and $\rho(A)$ to denote respectively the transpose, the conjugated transpose, the conjugate, the spectrum, the determinant, the 2-norm, and the spectral radius of A. Moreover, $\omega(A)=\max\{|z|: z=x^*Ax, \|x\|=1\}$ is the numerical radius of A. Finally, the symbol $P>0$ means that P is positive definite, I_n denotes an $n\times n$ identity matrix, $\mathbf{0}$ denotes a zero matrix with appropriate dimensions, and $\mathrm{j}=\sqrt{-1}$.

2 Problem Formulation and Preliminary Results

We consider the following nonlinear matrix equation

$$X+A^*\overline{X}^{-1}A=Q \tag{1}$$

where $Q\in\mathbf{C}^{n\times n}$ is a given positive definite matrix, $A\in\mathbf{C}^{n\times n}$ is a given complex matrix, and $X\in\mathbf{C}^{n\times n}$ is the unknown. In this paper, we are interested in the existence of positive definite solutions of this class of nonlinear matrix equations.

Remark 1 *Similar to [37], we can also consider positive definite solutions of matrix equation*

$$X+A^*X^{-\mathrm{T}}A=Q. \tag{2}$$

However, the positive definite solutions of (2) coincide with the positive definite solutions of equation (1) since $X^{-\mathrm{T}}=\overline{X}^{-*}=\overline{X}^{-1}$.

Via a simple manipulation we can show the following result.

Lemma 1 *Let Q be a positive definite matrix. Then X is a solution of (1) if and only if $Y=Q^{-\frac{1}{2}}XQ^{-\frac{1}{2}}$ is a solution of the following nonlinear matrix equation*

$$I_n=Y+A_Q^*\overline{Y}^{-1}A_Q, \quad A_Q=\overline{Q}^{-\frac{1}{2}}AQ^{-\frac{1}{2}}.$$

Therefore, without loss of generality, we assume hereafter that $Q=I_n$ in (1). We point out that matrix equation (1) has solutions that are totally different from solutions of the following nonlinear matrix equation

$$X+A^*X^{-1}A=I_n. \tag{3}$$

See the following example for illustration.

Example 1 *Consider a nonlinear matrix equation in the form of (1) with $Q=I_2$ and*

$$A=\begin{bmatrix} \frac{1}{4}+\frac{1}{4}\mathrm{j} & \frac{1}{4}\mathrm{j} \\ -\frac{1}{4}\mathrm{j} & \frac{1}{4}-\frac{1}{4}\mathrm{j} \end{bmatrix}.$$

Then according to the results we will give later, we find the maximal positive definite solution of this equation as

$$X_+ = \begin{bmatrix} \frac{1}{2}+\frac{1}{8}\sqrt{6} & -\frac{1}{8}-\frac{1}{8}\mathrm{j} \\ -\frac{1}{8}+\frac{1}{8}\mathrm{j} & \frac{1}{2}+\frac{1}{8}\sqrt{6} \end{bmatrix}.$$

However, according to the results in [18], the maximal positive definite solution of equation (3) can be computed as

$$X_+' = \begin{bmatrix} \frac{1}{8}\sqrt{2}+\frac{1}{2} & -\frac{1}{4}-\frac{1}{8}\sqrt{2}\mathrm{j} \\ -\frac{1}{4}+\frac{1}{8}\sqrt{2}\mathrm{j} & \frac{1}{8}\sqrt{2}+\frac{1}{2} \end{bmatrix}.$$

It is clearly that $X_+ \neq X_+'$.

For a complex matrix $A = A_1 + A_2\mathrm{j} \in \mathbf{C}^{n\times m}$ where $A_1, A_2 \in \mathbf{R}^{n\times m}$, we denote the operators $(\cdot)^{\heartsuit}$ and $(\cdot)^{\diamondsuit}$ as

$$A^{\heartsuit} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}, \quad A^{\diamondsuit} = \begin{bmatrix} A_2 & A_1 \\ A_1 & -A_2 \end{bmatrix}.$$

It follows that both $A^{\heartsuit}$ and $A^{\diamondsuit}$ are real matrices. For further use, we define two unitary matrices E_n and P_n as

$$E_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad P_n = \frac{\sqrt{2}}{2}\begin{bmatrix} \mathrm{j}I_n & I_n \\ I_n & \mathrm{j}I_n \end{bmatrix}. \tag{4}$$

Some basic properties of these matrices are collected as the following lemma whose proof is provided in Appendix B.

Lemma 2 *Let* $A \in \mathbf{C}^{n\times m}$ *and* $B \in \mathbf{C}^{m\times p}$ *be two given complex matrices.*

1. *The following equalities are true.*

$$\begin{array}{lll} (AB)^{\heartsuit} = A^{\heartsuit}B^{\heartsuit}, & (A^{-1})^{\heartsuit} = (A^{\heartsuit})^{-1}, & (A^{\mathrm{T}})^{\heartsuit} = E_m (A^{\heartsuit})^{\mathrm{T}} E_n, \\ (A^*)^{\heartsuit} = (A^{\heartsuit})^{\mathrm{T}}, & (\overline{A})^{\heartsuit} = E_n A^{\heartsuit} E_m, & A^{\diamondsuit} = E_n A^{\heartsuit}. \end{array}$$

2. *Let* P_n *be defined in (4). Then*

$$A^{\heartsuit} = P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_m^*. \tag{5}$$

   Consequently, $A \geq (>)\, 0$ *if and only if* $A^{\heartsuit} \geq (>)\, 0$.

3. $A \in \mathbf{C}^{n\times n}$ *is a normal (unitary) matrix if and only if* $A^{\heartsuit}$ *is a real normal (unitary) matrix.*

4. *For any* $A \in \mathbf{C}^{n\times n}$, *there holds* $\rho(A) = \rho(A^{\heartsuit})$ *and* $\rho(A^{\diamondsuit}) = \rho^{\frac{1}{2}}(A\overline{A})$.

5. *The real matrix* $A^{\diamondsuit}$ *is normal if and only if* A *is con-normal, namely,* $A^*A = \overline{AA^*}$.

6. *For any* $A \in \mathbf{C}^{n\times m}$, *there holds* $\|A^{\diamondsuit}\| = \|A^{\heartsuit}\| = \|A\|$

7. *If* $A \geq 0$, *then* $(A^{\heartsuit})^{\frac{1}{2}} = \left(A^{\frac{1}{2}}\right)^{\heartsuit}$.

Two matrices $A, B \in \mathbf{C}^{n\times n}$ are said to be con-similar if there exists a nonsingular matrix $S \in \mathbf{C}^{n\times n}$ such that $S^{-1}B\overline{S} = A$. The following lemma is borrowed from [4].

Lemma 3 *Let* $J_k(\lambda)$ *denote a* $k\times k$ *Jordan matrix whose diagonal elements are* λ. *Then any matrix* $A \in \mathbf{C}^{n\times n}$ *is con-similar to a direct sum of blocks of the form* $J_k(\lambda)$ *where* $\lambda \geq 0$ *or*

$$\begin{bmatrix} 0 & I_k \\ J_k(\lambda) & 0 \end{bmatrix}$$

where $\lambda < 0$ *or*
mathrmIm $(\lambda) \neq 0$.

The set of complex numbers λ appearing in the $J_k(\lambda)$ in the blocks of the the canonical form defined in Lemma 3 of a matrix A will be called the con-spectrum of A and denoted by $\mathrm{co}\lambda(A)$ [4]. Moreover, the con-spectrum-radius of A will be denoted by

$$\mathrm{co}\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{co}\lambda(A)\}.$$

A property of the con-spectrum-radius is given in the following lemma whose proof will be presented in Appendix C.

Lemma 4 *Let $A \in \mathbf{C}^{n\times n}$ be a given matrix. Then $\mathrm{co}\rho(A) \le (\ge) 1 \Leftrightarrow \rho\left(A\overline{A}\right) \le (\ge) 1$.*

At the end of this section, we recall the well-known Schur complement.

Lemma 5 *Let matrix Φ be defined as*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & \Phi_{22} \end{bmatrix}.$$

Then the following three statements are equivalent:

1. $\Phi > 0$.
2. $\Phi_{11} > 0$ *and* $\Phi_{22} - \Phi_{12}^* \Phi_{11}^{-1} \Phi_{12} > 0$.
3. $\Phi_{22} > 0$ *and* $\Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^* > 0$.

3 Necessary and Sufficient Conditions

In this section, we study necessary and sufficient conditions for the existence of a positive definite solution of the nonlinear matrix equation (1). Firstly we introduce an useful lemma.

Lemma 6 *Assume that A is nonsingular. Then X solves the nonlinear matrix equation (1) if and only if $Y = I_n - \overline{X}$ solves the following nonlinear matrix equation*

$$I_n = Y + A\overline{Y}^{-1}A^*. \tag{6}$$

Proof. Let X be a solution of equation (1), then $A^*\overline{X}^{-1}A = I_n - X$ from which we get $\overline{X}^{-1} = A^{-*}(I_n - X)A^{-1}$. Taking inverses on both sides gives $\overline{X} = A(I_n - X)^{-1}A^*$ which is equivalent to equation (6) by setting $I_n - \overline{X} = Y$. The converse can be shown similarly. ■

Our main result regarding the existence of positive definite solution of equation (1) is presented as follows.

Theorem 1 *The nonlinear matrix equation in (1) has a solution $X > 0$ if and only if the following nonlinear matrix equation*

$$I_{2n} = W + \left(A^{\diamondsuit}\right)^{\mathrm{T}} W^{-1} A^{\diamondsuit}, \tag{7}$$

has a solution $W > 0$. Moreover, the following two statements hold true:

1. *If the nonlinear matrix equation in (1) has a solution $X > 0$, then it must have a maximal positive definite solution X_+. Particularly, if W_+ denotes the maximal solution of the nonlinear matrix equation in (7), then $W_+ = X_+^{\heartsuit}$, or*

$$X_+ = \frac{1}{2}\begin{bmatrix} \mathrm{j}I_n \\ I_n \end{bmatrix}^* W_+ \begin{bmatrix} \mathrm{j}I_n \\ I_n \end{bmatrix}. \tag{8}$$

2. *If A is nonsingular and the nonlinear matrix equation (1) has positive definite solution $X > 0$, then it must have a minimal positive definite solution X_-. Particularly, if W_- denotes the minimal solution of the nonlinear matrix equation in (7), then $W_- = X_-^{\heartsuit}$, or*

$$X_- = \frac{1}{2}\begin{bmatrix} \mathrm{j}I_n \\ I_n \end{bmatrix}^* W_- \begin{bmatrix} \mathrm{j}I_n \\ I_n \end{bmatrix}. \tag{9}$$

Proof. "$\Longrightarrow$" Let $X > 0$ be a solution of equation (1). Taking $(\cdot)^{\heartsuit}$ on both sides of equation (1) and using Lemma 2 gives

$$\begin{aligned} I_n^{\heartsuit} &= X^{\heartsuit} + \left(A^*\overline{X}^{-1}A\right)^{\heartsuit} \\ &= X^{\heartsuit} + \left(A^{\heartsuit}\right)^{\mathrm{T}}\left(\left(\overline{X}\right)^{\heartsuit}\right)^{-1}A^{\heartsuit} \\ &= X^{\heartsuit} + \left(A^{\heartsuit}\right)^{\mathrm{T}}\left(E_nX^{\heartsuit}E_n^{\mathrm{T}}\right)^{-1}A^{\heartsuit} \\ &= X^{\heartsuit} + \left(A^{\diamondsuit}\right)^{\mathrm{T}}\left(X^{\heartsuit}\right)^{-1}A^{\diamondsuit}, \end{aligned} \tag{10}$$

which indicates that $W = X^{\heartsuit} > 0$ is a solution of equation (7).

"$\Longleftarrow$" Let equation (7) have a solution $W > 0$. Then it must have a maximal solution according to Lemma 7 in Appendix A. We denote such maximal solution by W_+. In the following we show that there must exist a matrix $Y > 0$ such that $W_+ = Y^{\heartsuit}$. According to Lemma 8 in Appendix A, we know that

$$W_{k+1} = I_{2n} - \left(A^{\diamondsuit}\right)^{\mathrm{T}}W_k^{-1}A^{\diamondsuit}, \quad W_0 = I_{2n}, \tag{11}$$

converges monotonically to W_+, namely, $0 < W_+ \leq W_{k+1} \leq W_k \leq I_{2n}, k \geq 0$ and

$$\lim_{k\to\infty} W_k = W_+ > 0. \tag{12}$$

We show that, for any integer $k \geq 0$, there exists a matrix $Y_k > 0$ such that

$$W_k = Y_k^{\heartsuit}, \quad \forall k \geq 0. \tag{13}$$

We show this by induction. Clearly, equation (13) holds true for $k = 0$ by setting $Y_0 = I_n$. Assume that (13) is true with $k = s$, say, there exists a $Y_s > 0$ such that $W_s = Y_s^{\heartsuit}$. Then, for $k = s+1$, by applying Lemma 2, we have

$$\begin{aligned} W_{s+1} &= I_{2n} - \left(A^{\diamondsuit}\right)^{\mathrm{T}}\left(Y_s^{\heartsuit}\right)^{-1}A^{\diamondsuit} \\ &= I_{2n} - \left(A^{\heartsuit}\right)^{\mathrm{T}}\left(E_nY_s^{\heartsuit}E_n^{\mathrm{T}}\right)^{-1}A^{\heartsuit} \\ &= I_{2n} - \left(A^*\right)^{\heartsuit}\left(\overline{Y_s}^{-1}\right)^{\heartsuit}A^{\heartsuit} \\ &= I_{2n} - \left(A^*\overline{Y_s}^{-1}A\right)^{\heartsuit} \\ &= Y_{s+1}^{\heartsuit}, \end{aligned}$$

where $Y_{s+1} = I - A^*\overline{Y_s}^{-1}A$. As $W_{s+1} > 0$, we know that $Y_{s+1} > 0$ also. Therefore, (13) is proved by induction.

Hence, it follows from (12) and (13) that there exists a matrix $Y_\infty > 0$ such that

$$W_+ = \lim_{k\to\infty} W_k = \lim_{k\to\infty} Y_k^{\heartsuit} = Y_\infty^{\heartsuit},$$

namely,

$$I_{2n} = Y_\infty^{\heartsuit} + \left(A^{\diamondsuit}\right)^{\mathrm{T}}\left(Y_\infty^{\heartsuit}\right)^{-1}A^{\diamondsuit},$$

which, by using a similar technique used in deriving (10), is equivalent to

$$I_n = Y_\infty + A^*\overline{Y}_\infty^{-1}A.$$

Hence the nonlinear matrix equation (1) has a solution $X = Y_\infty$.

Proof of Item 1: Since for any positive definite solution X of (1), there is a real positive definite solution $X^\heartsuit$ of (7), we must have $X^\heartsuit \leq W_+ = Y_\infty^\heartsuit$ which indicates that $X \leq Y_\infty$. However, $X = Y_\infty$ is also a solution of the nonlinear matrix equation (1). Hence, the nonlinear matrix equation (1) has the maximal solution $X_+ = Y_\infty$. The relation $W_+ = X_+^\heartsuit$ then follows directly.

Proof of Item 2: If the nonlinear matrix equation (1) has a positive definite solution $X > 0$, then, as $X < I_n$, by Lemma 6, the nonlinear matrix equation (6) also has a positive definite solution $I_n - \overline{X}$, which, according to item 1 of this theorem, indicates that equation (6) must have a maximal positive definite solution Y_+. Hence, by applying Lemma 6 again, $X_- \triangleq I_n - \overline{Y}_+$ is also a solution of equation (1). In fact, X_- is the minimal positive definite solution of equation (1). Otherwise assume that $X_* \leq X_-$ is a positive definite solution of (1). Then

$$Y_* = I_n - \overline{X}_* \geq I_n - \overline{X}_- = Y_+,$$

is a positive definite solution of (6). This is impossible since Y_+ is the maximal positive definite solution of equation (6) by assumption.

According to item 1 of this theorem, the maximal positive definite solution Y_+ to equation (6) is related with $Z_+ = Y_+^\heartsuit$ where Z_+ is the maximal positive definite solution of

$$I_{2n} = Z + A^\diamondsuit Z^{-1} \left(A^\diamondsuit\right)^{\mathrm{T}}. \tag{14}$$

As $A^\diamondsuit$ is nonsingular, by applying Lemma 6 again, the maximal positive definite solution Z_+ to equation (14) is related with $Z_+ = I_{2n} - Y_-$ where Y_- is the minimal positive definite solution of equation (7). Hence the minimal positive definite solution X_- to equation (1) satisfies

$$X_-^\heartsuit = (I_n - Y_+)^\heartsuit = I_{2n} - Z_+ = Y_-.$$

The proof is finished. ■

Remark 2 *The nonlinear matrix equation in (7) is in the standard from of (39) with $Q = I_n$ (see Appendix) which has been extensively studied in the literature. Therefore, by adopting the existing results on the nonlinear matrix equation (7), we can get corresponding results on the original nonlinear matrix equation (1).*

According to the proof of Theorem 1, we have the following result regarding iteration based numerical solution of the nonlinear matrix equation (1).

Corollary 1 *Assume that the nonlinear matrix equation (1) has a positive definite solution. Denote the largest solution by X_+. Then the iteration*

$$W_{k+1} = I_{2n} - \left(A^\diamondsuit\right)^{\mathrm{T}} W_k^{-1} A^\diamondsuit, \quad W_0 = I_{2n}, \tag{15}$$

converges and such that $\lim_{k\to\infty} W_k = X_+^\heartsuit$. Moreover, if $\left\|X_+^{-1}\overline{A}\right\| < 1$, then the iteration in (15) converges to $X_+^\heartsuit$ with at least a linear convergence rate, namely, there exists a $k^* > 0$ and a number $0 < \mu < 1$ such that

$$\left\|W_{k+1} - X_+^\heartsuit\right\| \leq \mu \left\|W_k - X_+^\heartsuit\right\|, \quad \forall k \geq k^*. \tag{16}$$

Proof. The convergence of the iteration (15) follows from the proof of Theorem 1. So we need only to show (16). We use the idea found in [36] to prove the result. According to Theorem 1, we have $X_+^\heartsuit = W_+$, the maximal solution to equation (7). Notice that

$$\begin{aligned}\left\|W_{k+1} - X_+^\heartsuit\right\| &= \|W_{k+1} - W_+\| \\ &= \left\|I_{2n} - \left(A^\diamondsuit\right)^{\mathrm{T}} W_k^{-1} A^\diamondsuit - \left(I_{2n} - \left(A^\diamondsuit\right)^{\mathrm{T}} W_+^{-1} A^\diamondsuit\right)\right\| \\ &= \left\|\left(A^\diamondsuit\right)^{\mathrm{T}} \left(W_+^{-1} - W_k^{-1}\right) A^\diamondsuit\right\| \\ &= \left\|\left(W_+^{-1} A^\diamondsuit\right)^{\mathrm{T}} \left(W_k - W_+\right) W_k^{-1} A^\diamondsuit\right\| \\ &\leq \left\|W_+^{-1} A^\diamondsuit\right\| \left\|W_k^{-1} A^\diamondsuit\right\| \|W_k - W_+\|. \end{aligned} \tag{17}$$

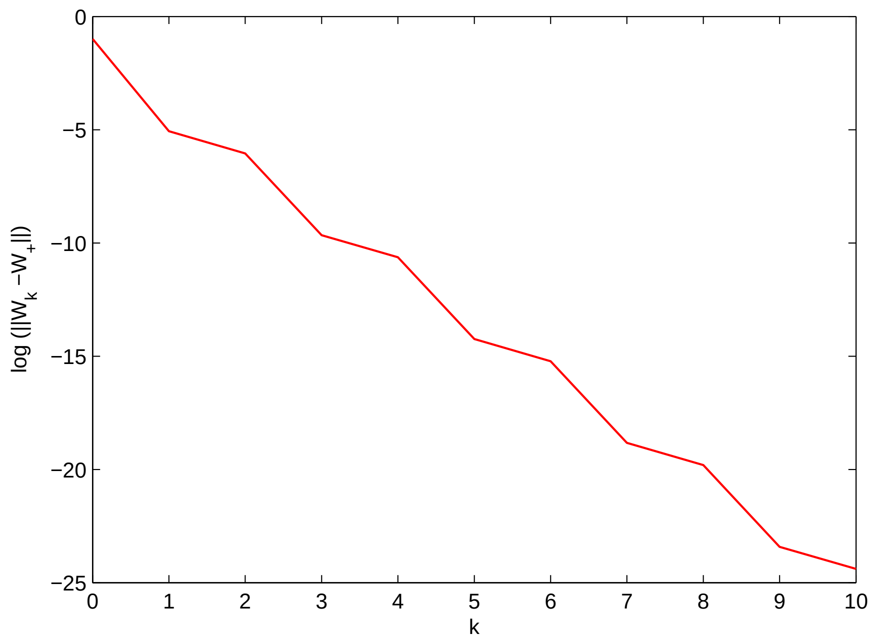

Figure 1: Numerical solution of the nonlinear matrix equation (1) via iteration (15)

On the other hand, by using Lemma 2, we can compute

$$\begin{aligned} \left\|W_+^{-1}A^{\diamondsuit}\right\| &= \left\|\left(X_+^{-1}\right)^{\heartsuit} E_n A^{\heartsuit}\right\| = \left\|E_n \left(X_+^{-1}\right)^{\heartsuit} E_n A^{\heartsuit}\right\| \\ &= \left\|\left(\overline{X}_+^{-1}\right)^{\heartsuit} A^{\heartsuit}\right\| = \left\|\left(\overline{X}_+^{-1} A\right)^{\heartsuit}\right\| \\ &= \left\|\overline{X}_+^{-1} A\right\| = \left\|X_+^{-1}\overline{A}\right\|. \end{aligned}$$

Let $\mu \in (0,1)$ be such that $\left\|W_+^{-1}A^{\diamondsuit}\right\| = \left\|X_+^{-1}\overline{A}\right\| < \sqrt{\mu} < 1$. As $\lim_{k\to\infty} W_k = X_+^{\heartsuit}$, there exists a k^* such that $\left\|W_k^{-1}A^{\diamondsuit}\right\| < \sqrt{\mu} < 1, \forall k \geq k^*$. Hence, the inequality in (17) reduces to (16) immediately. The proof is finished. ■

We emphasize that the condition $\left\|X_+^{-1}\overline{A}\right\| < 1$ is only sufficient for guaranteeing the linear convergence of the iteration in (15) which converges as long as the nonlinear matrix equation (1) has a positive definite solution. By combining Lemma 6 and Corollary 1, we can also present a result regarding obtaining the minimal solution to the nonlinear matrix equation (1). The details are omitted for brevity.

Example 2 *Consider the nonlinear matrix equation in Example 1. By computation we have* $\left\|X_+^{-1}\overline{A}\right\| = 0.614 < 1$. *Then by Corollary 1, we conclude that the corresponding iteration in (15) converges to* $X_+^{\heartsuit}$ *at least linearly. For illustration, the history of the iteration is recorded in Figure 1. From this figure we see that the convergence of the corresponding iteration in (15) is indeed linear. Hence, the estimation in (16) may be nonconservative.*

In the particular case that A is real, we can show the following result.

Corollary 2 *Suppose that* A *is real. If the nonlinear matrix equation (39) has a positive definite solution, then* X_+ *is real. Furthermore, if* A *is nonsingular, then* X_- *is also real. Hence in this case, equation (1) and the following nonlinear matrix equation*

$$I_n = X + A^{\mathrm{T}} X^{-1} A,$$

has the same maximal and minimal positive definite solutions.

Proof. Under the assumption of this corollary, $W_+ = X_+^{\heartsuit}$ and $W_- = X_-^{\heartsuit}$ are respectively the maximal and minimal solution of equation (7). Therefore, we need only to show that W_+ and W_- are in the form of

$$W_+ = \begin{bmatrix} W_{1+} & 0 \\ 0 & W_{1+} \end{bmatrix}, \quad W_- = \begin{bmatrix} W_{1-} & 0 \\ 0 & W_{1-} \end{bmatrix}.$$

Since W_+ is the limit of the iteration in (11), we only need to show that

$$W_k = \begin{bmatrix} W_{1k} & 0 \\ 0 & W_{1k} \end{bmatrix}, \quad \forall k \geq 0. \tag{18}$$

We show this by induction. Clearly, (18) holds true for $k = 0$. Assume that it is true with $k = s$. Then, for $k = s + 1$, we can compute

$$W_{s+1} = \begin{bmatrix} I_n - A_1^{\mathrm{T}} W_{1s} A_1 & 0 \\ 0 & I_n - A_1^{\mathrm{T}} W_{1s} A_1 \end{bmatrix} \triangleq \begin{bmatrix} W_{1(s+1)} & 0 \\ 0 & W_{1(s+1)} \end{bmatrix}.$$

Therefore, (18) is proved by induction. The case W_- can be proved similarly. ■

Combining Theorem 1 and Lemma 10 gives the following corollary.

Corollary 3 *Suppose that A is invertible. Then the nonlinear matrix equation (1) has a positive definite solution if and only if $\omega\left(A^{\Diamond}\right) \leq \frac{1}{2}$.*

Our next theorem presents some properties of the maximal and minimal positive definite solutions of the nonlinear matrix equation (1).

Theorem 2 *Assume that the nonlinear matrix equation in (1) has a solution $X > 0$.*

1. *Let the maximal positive definite solution be X_+. Then X_+ is the unique solution such that $X_+ + s\overline{A}\overline{X}_+^{-1}A$ is invertible for all $s \in \{z : |z| < 1\}$, or equivalently,*

$$\mathrm{co}\rho\left(\overline{X}_+^{-1}A\right) \leq 1. \tag{19}$$

2. *Assume further that A is nonsingular. Let the minimal positive definite solution be X_-. Then X_- is the unique solution such that $X_- + s\overline{A}^*\overline{X}_-^{-1}A^*$ is invertible for all $s \in \{z : |z| > 1\}$, or equivalently,*

$$\mathrm{co}\rho\left(\overline{X}_-^{-1}A^*\right) \geq 1. \tag{20}$$

Proof. *Proof of Item 1*: By Theorem 1, $W_+ = X_+^{\heartsuit}$ is the maximal solution of the nonlinear matrix equation (7). Then according to Lemma 7 in appendix, W_+ is the unique positive definite solution of equation (7) such that $W_+ + \lambda A^{\Diamond}$ is nonsingular for all $\lambda \in \{z : |z| < 1\}$. Since $P_n^* E_n P_n = E_n$, we can compute

$$\begin{aligned} W_+ + \lambda A^{\Diamond} &= X_+^{\heartsuit} + \lambda E_n A^{\heartsuit} \\ &= P_n \begin{bmatrix} X_+ & 0 \\ 0 & \overline{X}_+ \end{bmatrix} P_n^* + \lambda E_n P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_n^* \\ &= P_n \left(\begin{bmatrix} X_+ & 0 \\ 0 & \overline{X}_+ \end{bmatrix} + \lambda P_n^* E_n P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \right) P_n^* \\ &= P_n \left(\begin{bmatrix} X_+ & 0 \\ 0 & \overline{X}_+ \end{bmatrix} + \lambda E_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \right) P_n^* \\ &= P_n \begin{bmatrix} X_+ & \lambda\overline{A} \\ \lambda A & \overline{X}_+ \end{bmatrix} P_n^*, \end{aligned}$$

from which it follows that

$$\det\left(W_+ + \lambda A^{\Diamond}\right) = \det\left(X_+\right)\det\left(X_+ - \lambda^2 \overline{A}\overline{X}_+^{-1}A\right). \tag{21}$$

Hence, $\det\left(W_+ + \lambda A^\diamondsuit\right)$ is nonzero for all $\lambda \in \{z : |z| < 1\}$ if and only if $\det(X_+ - \lambda^2 \overline{A X_+^{-1}} A)$ is nonzero for all $\lambda \in \{z : |z| < 1\}$. Notice that

$$\forall \lambda \in \{z : |z| < 1\} \Leftrightarrow \forall s \triangleq -\lambda^2 \in \{z : |z| < 1\}.$$

The first conclusion then follows directly. Moreover, it follows from (21) that

$$\begin{aligned}\det\left(W_+ + \lambda A^\diamondsuit\right) &= \det\left(X_+\right) \det\left(I_n + s\overline{A X_+^{-1}} A X_+^{-1}\right) \det\left(X_+\right) \\ &= \left(\det\left(X_+\right)\right)^2 \det\left(I_n + s\overline{A X_+^{-1}} A X_+^{-1}\right),\end{aligned}$$

where $s \in \{z : |z| < 1\}$. Hence $\det\left(W_+ + \lambda A^\diamondsuit\right)$ is nonzero for all $\lambda \in \{z : |z| < 1\}$ if and only if the matrix $\overline{A X_+^{-1}} A X_+^{-1}$ has no poles $\alpha \in \{z : |z| > 1\}$ which is equivalent to

$$1 \geq \rho\left(\overline{A X_+^{-1}} A X_+^{-1}\right) = \rho\left(\left(\overline{X}_+^{-1} A\right)\left(\overline{\overline{X}_+^{-1} A}\right)\right) = \rho\left(\left(\overline{\overline{X}_+^{-1} A}\right)\left(\overline{X}_+^{-1} A\right)\right).$$

By applying Lemma 4, the above inequality is equivalent to (19). Assume that there exists another positive definite solution X_* such that $\mathrm{co}\rho\left(\overline{X}_+^{-1} A\right) \leq 1$. As the above process is invertible, we can show that $W_* = X_*^\heartsuit$ is such that $W_+ + \lambda A^\diamondsuit$ is nonsingular for all $\lambda \in \{z : |z| < 1\}$ which contradicts with Lemma 7 in appendix.

Proof of Item 2: By Theorem 1, $W_- = X_-^\heartsuit$ is the minimal solution of the nonlinear matrix equation (7). Then according to Lemma 7 in appendix, W_- is the unique positive definite solution of equation (7) such that $W_- + \lambda\left(A^\diamondsuit\right)^{\mathrm{T}}$ is nonsingular for all $\lambda \in \{z : |z| > 1\}$. Similar to the proof of item 1 of this theorem, via some computation, we have

$$W_- + \lambda\left(A^\diamondsuit\right)^{\mathrm{T}} = P_n \begin{bmatrix} X_- & \lambda \overline{A}^* \\ \lambda A^* & \overline{X}_- \end{bmatrix} P_n^*.$$

The remaining is similar to the proof of item 1 and is omitted for brevity. The proof is complete. ■

4 Bound of the Positive Definite Solutions

In this section, we study the bound of the positive definite solutions of the nonlinear matrix equation (1). Let $\{H_k\}_{k=1}^\infty$ be generated recursively as follows:

$$H_1 = I_n, \quad H_{k+1} = \begin{cases} \left[\begin{array}{c|c} H_k & \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & I_n \end{array}\right], & k \text{ is odd}, \\ \left[\begin{array}{c|c} H_k & \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & \overline{A} \end{bmatrix} & I_n \end{array}\right], & k \text{ is even}. \end{cases} \tag{22}$$

Here, if $k = 2$, the zero matrices in H_2 are obviously of zero dimension. Then we can prove the following result.

Theorem 3 *If the nonlinear matrix equation (1) has a positive definite solution, then $H_k > 0, \forall k \geq 1$, and for any integer $k \geq 1$, there holds*

$$X > \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^* H_k^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \triangleq S_k, \quad k \text{ is even}, \tag{23}$$

and

$$X > \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^* \overline{H}_k^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \triangleq S_k, \quad k \text{ is odd}, \tag{24}$$

in which, if $k=1$, the zero matrices involved are of zero dimensions. Moreover, S_k is nondecreasing, namely,

$$S_{k+1} \geq S_k, \quad \forall k \geq 1. \tag{25}$$

Proof. Since $X = I_n - A^* \overline{X}^{-1} A > 0$, via a Schur implement, we have

$$0 < \begin{bmatrix} I_n & A^* \\ A & \overline{X} \end{bmatrix} = \begin{bmatrix} I_n & A^* \\ A & I_n - \overline{A}^* X^{-1} \overline{A} \end{bmatrix} = H_2 - \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix} X^{-1} \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix}^*,$$

which indicates that $H_2 > 0$. Applying another Schur complement on the above inequality gives

$$\begin{aligned} 0 < \left[\begin{array}{c|c} H_2 & \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix} \\ \hline \begin{bmatrix} 0 & \overline{A} \end{bmatrix} & X \end{array}\right] &= \left[\begin{array}{c|c} H_2 & \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix} \\ \hline \begin{bmatrix} 0 & \overline{A} \end{bmatrix} & I_n - A^* \overline{X}^{-1} A \end{array}\right] \\ &= H_3 - \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \overline{X}^{-1} \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix}^*, \end{aligned}$$

which indicates that $H_3 > 0$. By using Schur complement again, the above inequality can be equivalently rewritten as

$$\begin{aligned} 0 < \left[\begin{array}{c|c} H_3 & \begin{bmatrix} 0 \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & \overline{X} \end{array}\right] &= \left[\begin{array}{c|c} H_3 & \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & I_n - \overline{A}^* X^{-1} \overline{A} \end{array}\right] \\ &= H_4 - \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} X^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^*, \end{aligned}$$

which indicates that $H_4 > 0$. Repeating the above process gives $H_k > 0, \forall k \geq 1$.

On the other hand, from the above development, we see that, if k is even, then

$$\left[\begin{array}{c|c} H_k & \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix} \\ \hline \begin{bmatrix} 0 & \overline{A} \end{bmatrix} & X \end{array}\right] > 0,$$

and if k if odd, then

$$\left[\begin{array}{c|c} H_k & \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & \overline{X} \end{array}\right] > 0,$$

where, if $k=1$, the zero matrices are of zero dimension. We notice that the above two inequalities are respectively equivalent to (23) and (24) via Schur complements.

We finally show (25). We only prove the case that k is odd since the case that k is even can be proven similarly. As $k+1$ is even, we have

$$\begin{aligned} S_{k+1} &= \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^* H_{k+1}^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^* \left[\begin{array}{c|c} H_k & \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & I_n \end{array}\right]^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}. \end{aligned} \tag{26}$$

We notice that, if k is odd, then

$$\begin{aligned} \left[\begin{array}{c|c} H_k & \begin{bmatrix} \mathbf{0} \\ A^* \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A \end{bmatrix} & I_n \end{array}\right] &= \left[\begin{array}{c|c} I_n & \begin{bmatrix} A^* & \mathbf{0} \end{bmatrix} \\ \hline \begin{bmatrix} A \\ \mathbf{0} \end{bmatrix} & \overline{H}_k \end{array}\right] \\ &= \begin{bmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{12}^* & \left(\overline{H}_k - \begin{bmatrix} A \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} A^* & \mathbf{0} \end{bmatrix} \right)^{-1} \end{bmatrix}^{-1}, \end{aligned}$$

where ∇_{11} and ∇_{12} are two matrices that are of no concern. With this, the relation in (26) can be continued as

$$\begin{aligned} S_{k+1} &= \left[\begin{array}{c} \mathbf{0} \\ \hline \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right] \end{array}\right]^* \left[\begin{array}{c|c} I_n & \left[\begin{array}{cc} A^* & \mathbf{0} \end{array}\right] \\ \hline \left[\begin{array}{c} A \\ \mathbf{0} \end{array}\right] & \overline{H}_k \end{array}\right]^{-1} \left[\begin{array}{c} \mathbf{0} \\ \hline \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right] \end{array}\right] \\ &= \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right]^* \left(\overline{H}_k - \left[\begin{array}{c} A \\ \mathbf{0} \end{array}\right] \left[\begin{array}{cc} A^* & \mathbf{0} \end{array}\right]\right)^{-1} \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right] \\ &\geq \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right]^* \overline{H}_k^{-1} \left[\begin{array}{c} \mathbf{0} \\ \overline{A}^* \end{array}\right] \\ &= S_k. \end{aligned}$$

In the above development, the zero matrices involved are of zero dimension if $k = 1$. The proof is finished. ■

Since S_k is bounded above, there must exist a positive definite matrix $S_\infty < I_n$ such that

$$\lim_{k\to\infty} S_k = S_\infty.$$

Obviously, we have $X > S_\infty$. It is not clear whether $S_\infty = X_-$. Moreover, it is clear that the larger the k, the better the S_k gives a lower bound of X. However, large k may lead to numerical problems. By choosing some special values in k, the following corollary can be obtained.

Corollary 4 *If the nonlinear matrix equation (1) has a positive definite solution, then A satisfies*

$$I_n > AA^* + \overline{A^* A}. \tag{27}$$

Moreover, the solution X satisfies the following inequalities

$$X > S_1 = \overline{AA^*}, \tag{28}$$

$$X > S_2 = \overline{A}\left(I_n - AA^*\right)^{-1} A^{\mathrm{T}}, \tag{29}$$

$$X > S_3 = \overline{A}\left(I_n - A\left(I_n - \overline{AA^*}\right)^{-1} A^*\right)^{-1} A^{\mathrm{T}}. \tag{30}$$

Proof. Let $k = 2$. We get from Theorem 3 that $H_2 > 0$, which, via a Schur complement, is equivalent to $I_n > AA^*$. Similarly, by letting $k = 3$, we get from $H_3 > 0$ that

$$0 < \left[\begin{array}{ccc} I_n & A^* & \mathbf{0} \\ A & I_n & \overline{A}^* \\ \mathbf{0} & \overline{A} & I_n \end{array}\right],$$

which, via a Schur complement, implies

$$I_n - \left[\begin{array}{cc} 0 & \overline{A} \end{array}\right] \left[\begin{array}{cc} I_n & A^* \\ A & I_n \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \overline{A}^* \end{array}\right] = I_n - \overline{A}\left(I_n - AA^*\right)^{-1} A^{\mathrm{T}} > 0.$$

By applying Schur complement again, the above inequality is equivalent to

$$\left[\begin{array}{cc} I_n & \overline{A} \\ A^{\mathrm{T}} & I_n - AA^* \end{array}\right] > 0.$$

However, another Schur complement indicates that the above inequality implies $A^* A + \overline{AA^*} < I_n$.

Now if we set $k = 1$, we get from (24) that $\overline{X} > AA^* = \overline{S}_1$ (or $X > \overline{A}A^{\mathrm{T}} = S_1$) which is (28); if we set $k = 2$, we get from (23) that

$$\begin{aligned} X > S_2 &= \left[\begin{array}{cc} 0 & \overline{A} \end{array}\right] \left[\begin{array}{cc} I_n & A^* \\ A & I_n \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \overline{A}^* \end{array}\right] \\ &= \left[\begin{array}{cc} 0 & \overline{A} \end{array}\right] \left[\begin{array}{cc} \nabla_{11} & \nabla_{12} \\ \nabla_{12}^* & \left(I_n - AA^*\right)^{-1} \end{array}\right] \left[\begin{array}{c} 0 \\ \overline{A}^* \end{array}\right] \\ &= \overline{A}\left(I_n - AA^*\right)^{-1} A^{\mathrm{T}}, \end{aligned}$$

which is (29). Here ∇_{ij} denotes the elements that are of no concern. Moreover, if we set $k = 3$, we know from (24) that

$$
\begin{aligned}
X > S_3 &= \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix}^* \overline{H}_3^{-1} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \overline{A} \end{bmatrix} \begin{bmatrix} I_n & \overline{A}^* & \mathbf{0} \\ \overline{A} & I_n & A^* \\ \mathbf{0} & A & I_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \overline{A}^* \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \overline{A} \end{bmatrix} \begin{bmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{12}^* & \left(I_n - A\left(I_n - \overline{AA^*}\right)^{-1} A^*\right)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \overline{A}^* \end{bmatrix} \\
&= \overline{A}\left(I_n - A\left(I_n - \overline{AA^*}\right)^{-1} A^*\right)^{-1} A^{\mathrm{T}},
\end{aligned}
$$

which is just (30). The proof is finished. ■

With the help of Lemma 6 and Theorem 3, we can also obtain upper bounds of X. To this end, we let $\{G_k\}_{k=1}^{\infty}$ be generated as (22) where A^* is replaced with A, namely,

$$
G_1 = I_n, \quad G_{k+1} = \begin{cases} \left[\begin{array}{c|c} G_k & \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & A^* \end{bmatrix} & I_n \end{array}\right], & k \text{ is odd} \\ \left[\begin{array}{c|c} G_k & \begin{bmatrix} \mathbf{0} \\ \overline{A} \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & \overline{A}^* \end{bmatrix} & I_n \end{array}\right], & k \text{ is even.} \end{cases}
$$

Again, if $k = 2$, the zero matrices in G_2 are obviously of zero dimension.

Theorem 4 *Assume that A is nonsingular and the nonlinear matrix equation in (1) has a positive definite solution. Then $G_k > 0, \forall k \geq 1$, and for any integer $k \geq 1$, there holds*

$$
X < I_n - \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix}^* \overline{G}_k^{-1} \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix} \triangleq R_k, \quad k \textit{ is even,} \tag{31}
$$

and

$$
X < I_n - \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix}^* G_k^{-1} \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix} \triangleq R_k, \quad k \textit{ is odd.} \tag{32}
$$

in which, if $k = 1$, the zero matrices involved are of zero dimension. Moreover, R_k is non-increasing, namely,

$$
R_{k+1} \leq R_k, \quad \forall k \geq 1. \tag{33}
$$

Proof. Since A is nonsingular, by Lemma 6, equation (1) is equivalent to equation (6) with $Y = I - \overline{X}$. Applying Theorem 3 on equation (6) gives $G_k > 0, k \geq 1$ and

$$
\begin{aligned}
Y &> \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix}^* G_k^{-1} \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix} \triangleq S'_k, \quad k \text{ is even,} \\
Y &> \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix}^* \overline{G}_k^{-1} \begin{bmatrix} \mathbf{0} \\ A \end{bmatrix} \triangleq S'_k, \quad k \text{ is odd,}
\end{aligned}
$$

which are respectively equivalent to (31) and (32) by using $Y = I - \overline{X}$. Finally, $R_k = I_n - S'_k$ is non-increasing because S'_k is nondecreasing according to Theorem 3. The proof is finished. ■

It is also clear that the larger the k, the better the R_k gives an upper bound of X. However, large k may lead to numerical problems. Choosing some special values in k gives the following corollary.

Corollary 5 *If the nonlinear matrix equation in (1) has a positive definite solution and A is nonsingular, then A satisfies*

$$I_n > A^*A + \overline{AA^*}.$$

Moreover, the solution X satisfies the following inequalities

$$\begin{aligned} X &< R_1 = I_n - A^*A, \\ X &< R_2 = I_n - A^*\left(I_n - \overline{A^*A}\right)^{-1}A, \\ X &< R_3 = I_n - A^*\left(I_n - A^{\mathrm{T}}\left(I_n - A^*A\right)^{-1}\overline{A}\right)^{-1}A. \end{aligned}$$

5 Sufficient Conditions and Necessary Conditions

In this section, we present some necessary conditions and sufficient conditions for the existence of a positive definite solutions of the nonlinear matrix equation (1).

Theorem 5 *If the nonlinear matrix equation (1) has a positive definite solution, then $\rho\left(A\overline{A}\right) \le \frac{1}{4}$, $\|A\| < 1$ and*

$$\rho\left(\left(A \pm \overline{A}^*\right)\left(\overline{A \pm \overline{A}^*}\right)\right) \le 1, \tag{34}$$

which can be equivalently rewritten as $\mathrm{co}\rho\left(A \pm \overline{A}^*\right) \le 1$.

Proof. It follows from Theorem 1 and Lemma 11 in appendix that if the nonlinear matrix equation (1) has a positive definite solution, then $\rho\left(A^{\diamondsuit}\right) \le \frac{1}{2}$, $\left\|A^{\diamondsuit}\right\| < 1$ and

$$\rho\left(A^{\diamondsuit} \pm \left(A^{\diamondsuit}\right)^{\mathrm{T}}\right) \le 1. \tag{35}$$

Clearly, by applying Lemma 2, $\rho\left(A^{\diamondsuit}\right) \le \frac{1}{2}$ is equivalent to $\rho\left(A\overline{A}\right) \le \frac{1}{4}$ and $\left\|A^{\diamondsuit}\right\| < 1$ is equivalent to $\|A\| < 1$ (we point out that $\|A\| < 1$ also follows directly from (27)). We next show that (35) is equivalent to (34). By virtue of Lemma 2 and in view of $P_n^*E_nP_n = E_n$, we get

$$\begin{aligned} \rho\left(A^{\diamondsuit} \pm \left(A^{\diamondsuit}\right)^{\mathrm{T}}\right) &= \rho\left(E_nA^{\heartsuit} \pm \left(A^{\heartsuit}\right)^{\mathrm{T}}E_n\right) \\ &= \rho\left(E_nP_n\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}P_n^* \pm P_n\begin{bmatrix} A^* & 0 \\ 0 & \overline{A}^* \end{bmatrix}P_n^*E_n\right) \\ &= \rho\left(P_n^*E_nP_n\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \pm \begin{bmatrix} A^* & 0 \\ 0 & \overline{A}^* \end{bmatrix}P_n^*E_nP_n\right) \\ &= \rho\left(E_n\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \pm \begin{bmatrix} A^* & 0 \\ 0 & \overline{A}^* \end{bmatrix}E_n\right) \\ &= \rho\left(\begin{bmatrix} 0 & \overline{A} \pm A^* \\ A \pm \overline{A}^* & 0 \end{bmatrix}\right) \\ &= \rho^{\frac{1}{2}}\left(\begin{bmatrix} 0 & \overline{A} \pm A^* \\ A \pm \overline{A}^* & 0 \end{bmatrix}^2\right) \\ &= \rho^{\frac{1}{2}}\left(\begin{bmatrix} \left(\overline{A} \pm A^*\right)\left(A \pm \overline{A}^*\right) & 0 \\ 0 & \left(A \pm \overline{A}^*\right)\left(\overline{A} \pm A^*\right) \end{bmatrix}\right) \\ &= \rho^{\frac{1}{2}}\left(\left(A \pm \overline{A}^*\right)\left(\overline{A} \pm A^*\right)\right), \end{aligned}$$

which is the desired relation. Finally, the equivalence between (34) and $\mathrm{co}\rho\left(A \pm \overline{A}^*\right) \le 1$ follows from Lemma 4. The proof is finished. ■

Theorem 6 *The nonlinear matrix equation (1) has a positive definite solution provided*

$$\|A\| \leq \frac{1}{2}. \tag{36}$$

Moreover, if A is con-normal, then the nonlinear matrix equation (1) has a positive definite solution if and only if A satisfies (36). In this case, the maximal solution is given by

$$X_{+} = \frac{1}{2}\left(I_n + (I_n - 4A^*A)^{\frac{1}{2}}\right). \tag{37}$$

If A is further assumed to be nonsingular, then the minimal solution can be expressed as

$$X_{-} = \frac{1}{2}\left(I_n - (I_n - 4A^*A)^{\frac{1}{2}}\right). \tag{38}$$

Proof. If $\|A\| \leq \frac{1}{2}$, then by Lemma 2, we know that $\left\|A^{\diamondsuit}\right\| \leq \frac{1}{2}$, which, by using Lemma 12, indicates that equation (7) has positive definite solution. This is further equivalent to the existence of positive definite solution of equation (1) in view of Theorem 1. The case that A is con-normal can be shown similarly. We next show (37). Notice that, according to Lemma 12, the maximal solution of equation (7) is given as

$$\begin{aligned}
W_{+} &= \frac{1}{2}\left(I_{2n} + \left(I_{2n} - 4\left(A^{\diamondsuit}\right)^{\mathrm{T}} A^{\diamondsuit}\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(I_{2n} + \left(I_{2n} - 4\left(A^{\heartsuit}\right)^{\mathrm{T}} E_n^{\mathrm{T}} E_n A^{\heartsuit}\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(I_{2n} + \left(I_{2n} - 4\left(A^{\heartsuit}\right)^{\mathrm{T}} A^{\heartsuit}\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(I_{2n} + \left(I_{2n} - 4\left(A^*A\right)^{\heartsuit}\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(I_n + (I_n - 4A^*A)^{\frac{1}{2}}\right)^{\heartsuit},
\end{aligned}$$

which, according to Theorem 1, implies (37). The equation (38) can be shown similarly. The proof is done. ∎

6 Conclusion

This paper has studied the existence of a positive definite solutions to the nonlinear matrix equation $X + A^*\overline{X}^{-1}A = Q$. With the help of some operators associated with complex matrices, we have shown that the existence of a positive definite solution of this type of nonlinear matrix equation is equivalent to the existence of a positive definite solution of a nonlinear matrix equation in the form of $W + B^{\mathrm{T}}W^{-1}B = I$ where B is real and is determined by A. Since the later nonlinear matrix equation has been well studied in the literature, properties of the original nonlinear matrix equations can be established based on the existing results on the transformed nonlinear matrix equations. Moreover, with the help of Schur complement, we have shown in this paper some upper bounds and lower bounds on the solutions to the nonlinear matrix equations. Simultaneously, some easily tested sufficient conditions and necessary conditions for the existence of positive definite solution of the nonlinear equations have also been established. We point out that, by combining the results obtained in this paper and the existing results on numerical computation of solutions to the standard nonlinear matrix equation $W + B^{\mathrm{T}}W^{-1}B = I$, numerical reliable algorithms can be built for computing the positive definite solutions to the original nonlinear matrix equation, which is currently under study.

Appendix

A: Solutions of Matrix Equation $X + A^*X^{-1}A = Q$

In this subsection, we recall some basic results regarding positive definite solutions of the following matrix equation

$$X + A^*X^{-1}A = Q. \tag{39}$$

Lemma 7 *(Theorem 3.4 in [18]) Suppose that $Q > 0$ and assume that he nonlinear matrix equation (39) has a positive definite solution. Then it has a maximal and minimal solution X_+ and X_-, respectively. Moreover, X_+ is the unique solution for which $X + \lambda A$ is invertible for all $|\lambda| < 1$, while X_- is the unique solution for which $X + \lambda A^*$ is invertible for all $|\lambda| > 1$.*

Lemma 8 *(Algorithm 4.1 in [18]) Suppose that $Q = I$. If the nonlinear matrix equation (39) has a positive definite solution, then the iteration*

$$X_{k+1} = I_n - A^*X_k^{-1}A, \quad X_0 = I_n,$$

converges to the maximal solution X_+, namely, $\lim_{k\to\infty} X_k = X_+$.

Lemma 9 *(Theorem 8.1 in [18]) Suppose that $Q = I$ and A is real. If the nonlinear matrix equation (39) has a positive definite solution, then X_+ is real. Furthermore, if A is nonsingular, then X_- is also real.*

Lemma 10 *(Theorem 5.1 in [18]) Suppose that A is invertible. Then the nonlinear matrix equation (39) has a positive definite solution if and only if $\omega(A) \leq \frac{1}{2}$.*

Lemma 11 *(Theorem 7 in [19] and Theorem 3.1 in [35]) If the nonlinear matrix equation (39) has a positive definite solution, then $\rho(A) \leq \frac{1}{2}, \|A\| < 1$ and $\rho(A \pm A^*) \leq 1$.*

Lemma 12 *(Theorem 11 and Theorem 13 in [19]) The nonlinear matrix equation (39) has a positive definite solution provided $\|A\| \leq \frac{1}{2}$. Moreover, if A is normal, then the nonlinear matrix equation (39) has a positive definite solution if and only if $\|A\| \leq \frac{1}{2}$. In this case, the maximal solution is given by*

$$X_+ = \frac{1}{2}\left(I_n + \left(I_n - 4A^{\mathrm{T}}A\right)^{\frac{1}{2}}\right).$$

Furthermore, if A is nonsingular, then

$$X_- = \frac{1}{2}\left(I_n - \left(I_n - 4A^{\mathrm{T}}A\right)^{\frac{1}{2}}\right).$$

B: Proof of Lemma 2

Proof of Item 1: These equalities can be verified directly by definition.

Proof of Item 2. This result follows from Lemma 9 in [37].

Proof of Item 3: We need only to show that A is a normal matrix if and only if $A^\heartsuit$ is a real normal matrix since unitary matrix is a special case of normal matrix. If $A^\heartsuit$ is a real normal matrix, then $\left(A^\heartsuit\right)^{\mathrm{T}} A^\heartsuit = A^\heartsuit \left(A^\heartsuit\right)^{\mathrm{T}}$. However, $A^\heartsuit \left(A^\heartsuit\right)^{\mathrm{T}} = (AA^*)^\heartsuit$ and $\left(A^\heartsuit\right)^{\mathrm{T}} A^\heartsuit = (A^*A)^\heartsuit$. Hence we have $(AA^*)^\heartsuit = (A^*A)^\heartsuit$ and consequently $AA^* = A^*A$, that is, A is a normal matrix. The converse can be shown similarly.

Proof of Item 4: From item 2 of this lemma, we obtain

$$\begin{aligned}\rho\left(A^{\heartsuit}\right) &= \rho\left(P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_n^*\right) \\ &= \rho\left(\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}\right) \\ &= \max\left\{\rho\left(A\right), \rho\left(\overline{A}\right)\right\} \\ &= \rho\left(A\right).\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}\rho\left(A^{\diamondsuit}\right) = \rho\left(E_n A^{\heartsuit}\right) &= \rho\left(E_n P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_n^*\right) \\ &= \rho\left(P_n^* E_n P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_n^* P_n\right) \\ &= \rho\left(E_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}\right) = \rho\left(\begin{bmatrix} 0 & \overline{A} \\ A & 0 \end{bmatrix}\right) \\ &= \rho^{\frac{1}{2}}\left(\begin{bmatrix} 0 & \overline{A} \\ A & 0 \end{bmatrix}\begin{bmatrix} 0 & \overline{A} \\ A & 0 \end{bmatrix}\right) \\ &= \rho^{\frac{1}{2}}\left(\begin{bmatrix} \overline{A}A & 0 \\ 0 & A\overline{A} \end{bmatrix}\right) = \rho^{\frac{1}{2}}\left(\overline{A}A\right).\end{aligned}$$

Proof of Item 5: By definition, we can compute

$$\begin{aligned}\left(A^{\diamondsuit}\right)^{\mathrm{T}} A^{\diamondsuit} = \left(E_n A^{\heartsuit}\right)^{\mathrm{T}} E_n A^{\heartsuit} &= \left(A^{\heartsuit}\right)^{\mathrm{T}} E_n^{\mathrm{T}} E_n A^{\heartsuit} \\ &= \left(A^{\heartsuit}\right)^{\mathrm{T}} A^{\heartsuit} = \left(A^*\right)^{\heartsuit} A^{\heartsuit} = \left(A^* A\right)^{\heartsuit},\end{aligned}$$

and similarly,

$$\begin{aligned}A^{\diamondsuit}\left(A^{\diamondsuit}\right)^{\mathrm{T}} = E_n A^{\heartsuit}\left(E_n A^{\heartsuit}\right)^{\mathrm{T}} &= E_n A^{\heartsuit}\left(A^{\heartsuit}\right)^{\mathrm{T}} E_n^{\mathrm{T}} \\ &= E_n A^{\heartsuit}\left(A^*\right)^{\heartsuit} E_n^{\mathrm{T}} = E_n\left(AA^*\right)^{\heartsuit} E_n^{\mathrm{T}} = \left(\overline{AA^*}\right)^{\heartsuit}.\end{aligned}$$

Clearly, $A^{\diamondsuit}$ is a normal matrix if and only if $\left(A^{\diamondsuit}\right)^{\mathrm{T}} A^{\diamondsuit} = A^{\diamondsuit}\left(A^{\diamondsuit}\right)^{\mathrm{T}}$, which is equivalent to $\left(\overline{AA^*}\right)^{\heartsuit} = \left(A^*A\right)^{\heartsuit}$, namely, $A^*A = \overline{AA^*}$.

Proof of Item 6: By using item 1 of this lemma, we obtain

$$\left\|A^{\diamondsuit}\right\| = \left\|E_n A^{\heartsuit}\right\| = \left\|A^{\heartsuit}\right\| = \left\|P_n \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} P_m^*\right\| = \left\|\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}\right\| = \left\|A\right\|.$$

Proof of Item 7: Let U be a unitary matrix such that $A = UDU^*$ where D is a real diagonal positive semi-definite matrix. Then

$$\begin{aligned}\left(A^{\heartsuit}\right)^{\frac{1}{2}} = \left(\left(UDU^*\right)^{\heartsuit}\right)^{\frac{1}{2}} &= \left(U^{\heartsuit}\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\left(U^{\heartsuit}\right)^{\mathrm{T}}\right)^{\frac{1}{2}} \\ &= U^{\heartsuit}\begin{bmatrix} D^{\frac{1}{2}} & 0 \\ 0 & D^{\frac{1}{2}} \end{bmatrix}\left(U^{\heartsuit}\right)^{\mathrm{T}} = \left(UD^{\frac{1}{2}}U^*\right)^{\heartsuit} = \left(A^{\frac{1}{2}}\right)^{\heartsuit}.\end{aligned}$$

C: Proof of Lemma 4

We only show the case "$\leq$". According to the results in [4], we know that

(1). for $\lambda \geq 0, \lambda \in \mathrm{co}\lambda\left(A\right) \Leftrightarrow \lambda^2 \in \lambda\left(A\overline{A}\right)$.

(2). for $\lambda < 0, \lambda \in \mathrm{co}\lambda(A) \Leftrightarrow \lambda \in \lambda\left(A\overline{A}\right)$.

(3). for $\mathrm{Im}(\lambda) \neq 0, \lambda \in \mathrm{co}\lambda(A) \Leftrightarrow \lambda, \overline{\lambda} \in \lambda\left(A\overline{A}\right)$.

Let s be an arbitrary eigenvalue of $A\overline{A}$. Then $|s| \leq 1$. Consider three cases. Case 1: $1 \geq s \geq 0$. Then it follows that $\lambda = \sqrt{s} \in \mathrm{co}\lambda(A)$ and hence $|\lambda| \leq 1$. Case 2: $-1 \leq s < 0$. Then we see that $\lambda = s \in \mathrm{co}\lambda(A)$ and hence $|\lambda| \leq 1$. Case 3: $\mathrm{Im}(\lambda) \neq 0$. In this case, we see that either $\lambda = s \in \mathrm{co}\lambda(A)$ or $\lambda = \overline{s} \in \mathrm{co}\lambda(A)$. In both cases, there holds $|\lambda| = |s| \leq 1$. The proof is completed.

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