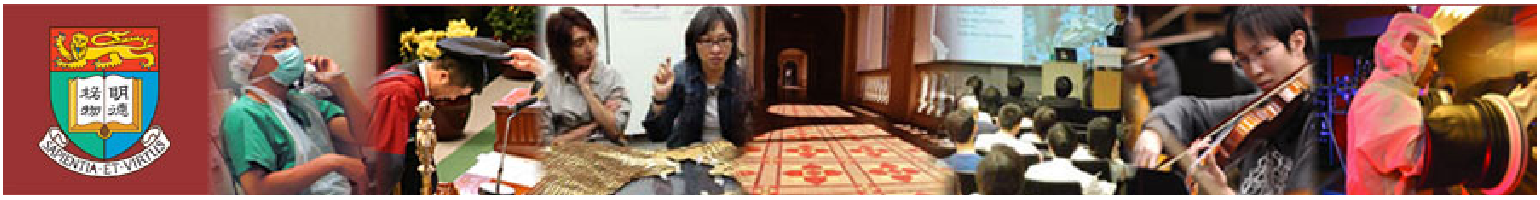

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Control Design of Uncertain Quantum Systems With Fuzzy Estimators

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***Abstract*—An approach of control design using fuzzy estimators (FEs) is proposed for quantum systems with uncertainties. Two types of quantum control problems are considered: 1) control of a pure-state quantum system in the presence of uncertainties and 2) control design of quantum systems with initial mixed states and uncertainties. For the first type of tasks, a partial feedback control scheme with an FE is presented to design controllers. In this scheme, an FE is trained to estimate the quantum state for feedback control of a quantum system, and controlled projective measurement is used to assist in controlling the system. For the second type of quantum control tasks, a probabilistic fuzzy estimator (PFE) is trained to estimate the quantum state for control design of a quantum system with an initial mixed state, and a corresponding control algorithm is proposed to design a control law that drives the system from the mixed state to a target pure state. Two examples of two-spin-$\frac{1}{2}$ systems are also presented and analyzed to demonstrate the process of control design and potential applications of the proposed approach.**

***Index Terms*—Fuzzy estimators (FEs), partial feedback, probabilistic fuzzy logic, quantum control.**

NOMENCLATURE

$\vert\psi\rangle$	State vector (quantum state).
$\widehat{\vert\psi\rangle}$	Estimated quantum state.
$\overline{\vert\psi\rangle}$	Output quantum state.
$\vert\phi\rangle$	Quantum eigenstate.
a^*	Complex conjugate of a.
A^T	Transpose of A.
$A^\dagger$	Adjoint of A.
$\mathrm{tr}(A)$	Trace of A.
$\langle\psi\vert$	Adjoint of $\vert\psi\rangle$.
$\langle\phi\vert\psi\rangle$	Inner product of $\vert\phi\rangle$ and $\vert\psi\rangle$.
ρ	Density operator.
$\widehat{\rho}$	Estimated density operator.
$\overline{\rho}$	Output density operator.
U	Unitary operator.
$\{a_i\}_{i=1}^N$	Set $\{a_1, \ldots, a_N\}$.
$(a_i)_{i=1}^N$	Column vector $(a_1, \ldots, a_N)^T$.
$\Vert A\Vert$	Norm of A.
$\hbar$	Reduced Planck constant.
$\mathbf{R}$	Set of real numbers.
M	Projective measurement.
$\delta\theta$	Uncertainty of θ.
F	Fuzzy set.
$\widetilde{F}$	Probabilistic fuzzy set.
m	Fuzzy membership.
R	Rotation operator for a (probabilistic) fuzzy set.
ι	$\iota = \sqrt{-1}$.
p	Probability.
H	Hamiltonian operator.
E	Set of events.
FE	Fuzzy estimator.
PFE	Probabilistic fuzzy estimator.

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I. INTRODUCTION

RECENT progress in theory and experiments has shown that quantum information technology has many advantages, such as speeding up of large number factorization and enhancing secret communication, over traditional information technology [1]. However, practical applications of quantum information technology are still confronted with some important technical difficulties such as the control of quantum systems in the presence of uncertainties. Developing effective control theory and methods [2] has been recognized as a solution to such difficulties. Some tools (e.g., optimal control and feedback control) from classical control theory have been used to analyze quantum control problems, and some results on quantum control have been presented in [3]–[19] (a comprehensive list can be found in the recent survey [2]). Even with these results, quantum control is still in its infancy, and more effort is necessary to develop a systematic quantum control theory. In practical applications, it is unavoidable that there exist all kinds of uncertainties for fragile quantum control systems. Robustness has been recognized as an important aspect for practical quantum information technology [20], and several methods have been proposed to deal with uncertainties in quantum systems. For example, in [13] an H^∞ controller synthesis problem for a class of quantum linear stochastic systems in the Heisenberg picture has been formulated and solved. In [21] and [22], a sliding mode control approach has been proposed to deal with uncertainties in the system Hamiltonian. In this paper, we propose a control design approach using FEs for a quantum system with uncertainties and present two algorithms for two types of quantum control problems. A common feature of the two control algorithms is that we use an FE to assist in control design of quantum systems. The basic motivation to use FEs for the control of uncertain quantum systems is twofold: 1) There exist similar representations and operators between quantum systems and fuzzy systems [23], [24]. It is possible to use a fuzzy system to simulate the evolution process of a quantum system. 2) Most control methods have difficulties to acquire feedback information without destroying the states of the quantum systems. An FE can use the information from the fuzzy system to provide an estimated feedback signal to guide the quantum control process. In particular, we first present a partial feedback control scheme using an FE for the control of pure-state quantum systems with uncertainties and then propose a control algorithm to drive a quantum system from an initial mixed state to a target pure state where a PFE [25] is used to simulate the mixed state.

For a pure-state quantum control system, its dynamics can be described by a bilinear equation (i.e., Schrödinger equation). Many existing results on quantum control focus on this class of quantum systems [2]. An important issue is the controllability of quantum systems [2]. For a controllable quantum system, we may design a control law to drive the system from a given initial state to a target quantum state. A straightforward approach is open-loop coherent control, where we may design a control Hamiltonian to accomplish a control task. Open-loop coherent control has been widely used to control quantum optical and chemical molecular systems [4]. However, it is usually difficult to design an effective control algorithm for quantum systems with uncertainties. As we know, in classical control theory, feedback is the most important control strategy to deal with uncertainties. In a feedback control system, the control output signal is measured and then fed back for the controller design. Although feedback control theory has been very mature for classical control systems with noises and uncertainties, the feedback control theory at the quantum level is still in its infancy, and only a few classical results can be directly extended to quantum systems [3], [6], [10]. The major challenge of designing feedback control schemes for quantum systems is the proper acquisition of the output signal for the controller design. In quantum feedback control, the feedback strategy may be classical feedback involving measurement (where the controller is a classical controller) or coherent quantum feedback without measurement (i.e., the controller is also a quantum system [13], [26], [27]). Although coherent quantum feedback [12], [13] can accomplish tasks such as entanglement transfer that are not possible using classical feedback, it is difficult to implement in most practical applications. Hence, feedback with measurement is the major approach for quantum feedback control problems. Many approaches have been proposed for quantum feedback control involving measurement, such as Markovian quantum feedback [3] and Bayesian quantum feedback [10], [11], which have also been applied to atomic physics, quantum optics, quantum error correction [28], and other quantum technologies. Quantum measurement in feedback control will unavoidably destroy the coherence in the feedback loop and alter the quantum system's state. One of the most challenging open problems for quantum feedback control is how to get the feedback information properly. In this paper, we propose a feedback control scheme using a controlled discontinuous measurement and a quantum state estimator which is based on fuzzy logic. Usually, the output of the quantum state estimator is fed back for the control design. However, when the estimated state is an almost eigenstate (having a high fidelity with an eigenstate), a quantum measurement will be implemented on the quantum system, where the measurement result is used to construct the controller and regulate the estimator. The feedback information is partially from the controlled quantum system and partially from the state estimator; therefore, this control scheme is called the partial feedback control [24].

In the first type of quantum control problems, we assume that the initial state is a pure state which can be represented by a complex unit vector (quantum wavefunction). In theory, we know all the information of the quantum state from its wavefunction. However, the initial states of most practical quantum systems are not in pure states. Their states cannot be represented by unit state vectors and are called mixed states. For a mixed state, we have incomplete information, and a density operator is necessary to describe the state [29]. An initial mixed state can be looked as a case that there exist uncertainties in the initial state since we have no complete information about the system's state. For such a quantum system with an initial mixed state, the authors of [30] have proven that open-loop coherent control cannot usually deal with the uncertainties in the initial state. In [8], a feedback stabilization strategy was presented which was based on continuous weak measurement to accomplish the control task to approximately drive the quantum system from an initial mixed state to an eigenstate. In [31], an incoherent control scheme was proposed to deal with the uncertainties in initial states, where a measurement operation is used to drive the system from a mixed state to a pure state, and then, an optimal control is applied to transform the pure state to a target state. For most quantum systems, it is difficult to implement continuous weak measurement, which limits the applications of the feedback stabilization strategy based on continuous measurement. The incoherent control strategy in [31] is easy to implement. However, it cannot deal with other uncertainties in the control process. In this paper, we observe a corresponding relationship between a mixed quantum state and probabilistic fuzzy logic and propose an approach of control design using a PFE for quantum systems with uncertainties and initial mixed states.

This paper is organized as follows. The problem formulation is presented in Section II. In Section III, a partial feedback control scheme using an FE is proposed for the control problem of a pure-state quantum system with uncertainties. In particular, some corresponding relationships between the quantum state and fuzzy logic are established, and a control algorithm is proposed. Section IV presents a control design approach which is based on a PFE to drive a quantum system from an initial mixed state to a target pure state in the presence of uncertainties. Simulated examples are presented to demonstrate the corresponding control algorithm. Concluding remarks are given in Section V.

II. Problem Formulation

In this paper, we consider finite-dimensional (assumed to be N-level) quantum systems. The assumption of finite-dimensional quantum systems is an appropriate approximation in many practical situations, such as quantum bit (qubit) systems. Let us denote the eigenstates of the free Hamiltonian H_0 of an N-level quantum system as $D = \{|\phi_i\rangle\}_{i=1}^N$. An evolving state $|\psi(t)\rangle$ of the controlled system can be expanded in terms of the eigenstates in the set D

$$|\psi(t)\rangle = \sum_{i=1}^{N} c_i(t)|\phi_i\rangle \tag{1}$$

where complex numbers $c_t(t)$ satisfy $\sum_{i=1}^N |c_i(t)|^2 = 1$. Introducing a control $u(t) \in L^2(\mathbf{R})$ acting on the system via a time-independent interaction Hamiltonian H_I and denoting $|\psi(t=0)\rangle$ as $|\psi_0\rangle$, $C(t) = (c_i(t))_{i=1}^N$ evolves according to the Schrödinger equation [32]

$$\iota\hbar\dot{C}(t) = [A + u(t)B]C(t), \qquad C(t=0) = C_0 \tag{2}$$

where $\iota = \sqrt{-1}$, $C_0 = (c_{0i})_{i=1}^N$, $c_{0i} = \langle\phi_i|\psi_0\rangle$, $\sum_{i=1}^N |c_{0i}|^2 = 1$, $\hbar$ is the reduced Planck constant, and the matrices A and B correspond to H_0 and H_I, respectively. We assume that the A matrix is diagonal and the B matrix is Hermitian [32]. In order to avoid trivial control problems, we assume $[A, B] \equiv AB - BA \neq 0$. Equation (2) describes the evolution of a finite-dimensional control system. The propagator $U(t_1 \to t_2)$ is a unitary operator such that for any state $|\psi(t_1)\rangle$ the state $|\psi(t_2)\rangle = U(t_1 \to t_2)|\psi(t_1)\rangle$ is the solution at time $t = t_2$ of (1) and (2) with the initial condition $|\psi(t_1)\rangle$ at time $t = t_1$ [32]. Sometimes, $U(t_1 \to t_2)$ is also simplified as $U(t)$, $t \in [t_1, t_2]$, if the evolution time $t_2 - t_1$ can be neglected when handling these problems.

To control a quantum system from an initial state $|\psi_{\text{initial}}\rangle$ to a desired target state $|\psi_{\text{target}}\rangle$, we need to find a control $u(t) \in L^2(\mathbf{R}), t \in [0, T]$, to constitute a desired propagator $U(t)$. For the discrete case, the control input $u(t)$ is a control sequence of $\{u(1), u(2), \ldots, u(k)\}$, where k is an integer and represents the number of control steps. $u(t)$ can be applied to control the quantum system toward $|\psi_{\text{target}}\rangle$ without measuring the controlled quantum system during the control process. This class of control methods is called the open-loop coherent control since it does not involve measurement and feedback.

Fig. 1 shows an example of open-loop control of a two-spin-$\frac{1}{2}$ system, where $|\psi_{\text{target}}\rangle - |\psi_{\text{initial}}\rangle$ means that we determine the control input by comparing the two quantum states. A spin-$\frac{1}{2}$ system may be used to constitute a quantum bit (qubit) for quantum information industry. It has two eigenstates ($|\uparrow\rangle$ and $|\downarrow\rangle$), and its quantum state can be manipulated using an electromagnetic field. The two-spin-$\frac{1}{2}$ system, as shown in Fig. 1, is represented in terms of Kronecker's product of the single-spin states as

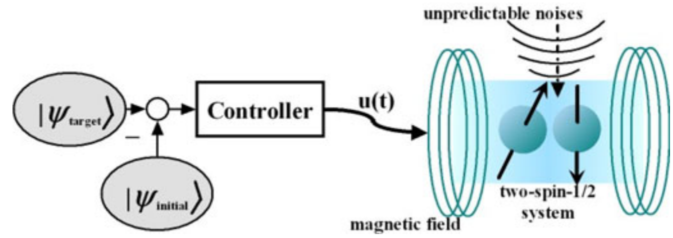

Fig. 1. Open-loop control of a two-spin-$\frac{1}{2}$ quantum system.

$$D = \{|\phi_i\rangle\}_{i=1}^4 = \{|\downarrow\rangle|\downarrow\rangle, |\downarrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\uparrow\rangle|\uparrow\rangle\}$$

$$|\psi\rangle = \sum_{i=1}^{4} c_i(t)|\phi_i\rangle.$$

As shown in Fig. 1, the controller is designed to actuate the magnetic field to drive the system to a desired target state. Generally, open-loop control is effective for some simple quantum control problems. However, when there are unpredictable noises from the environments (see Fig. 1), the control performance will be dramatically deteriorated and the controlled system may even diverge from the desired track. To deal with unavoidable uncertainties, feedback control (closed-loop control) should be adopted instead of open-loop control. In feedback control (except coherent feedback), it is necessary to acquire feedback information by measurement. The measurement will affect or even destroy the measured quantum states. To reduce the possible negative effect of quantum measurement, we use a virtual fuzzy system to simulate the quantum control process to estimate the state of the controlled quantum system. We state the first type of quantum control problems as follows

Control Problem I: For a pure-state quantum system, find a control law using an FE to drive the quantum system from a given initial state to a target state in the presence of uncertainties.

Further, we will consider the quantum system whose initial state is a mixed state. For a mixed state, it is necessary to employ a density operator ρ to describe its state. A density operator ρ is positive and has trace equal to 1. We can usually define a density operator ρ for a mixed state as follows [1]:

$$\rho \equiv \sum_j p_j|\psi_j\rangle\langle\psi_j| \tag{3}$$

where $\langle\psi_j| = (|\psi_j\rangle)^\dagger$, $\sum_j p_j = 1$, and the operation $(\cdot)^\dagger$ refers to the adjoint. That is, we have no exact information for the mixed state. We just know that the mixed state can be looked as a mixture $|\psi_j\rangle$ $(j = 1, 2, \ldots)$ with respective probabilities p_j. For a pure state $|\psi_j\rangle$, its density operator corresponds to $\rho = |\psi_j\rangle\langle\psi_j|$. We observe that there are some similar relationships between the mixed state ρ and the probabilistic fuzzy logic [25]. Hence, it is possible to simulate a quantum control system with an initial mixed state using a probabilistic fuzzy system. Hence, we can state the second type of quantum control problems as follows.

Control Problem II: For a quantum system with an initial mixed state, find a control law using a PFE to drive the quantum system from an initial mixed state ρ to a target pure state in the presence of uncertainties.

III. Control Problem I: Partial Feedback Control with a Fuzzy Estimator

A systematic partial feedback control scheme is proposed in this section to control a quantum system from an initial quantum state to a desired state with the help of an FE. Before the system design and the control algorithm are presented, the fuzzy logic and its relationship to a quantum state are briefly introduced. Then, a method is presented to achieve the translation between quantum operators and fuzzy operators. Finally, a numerical example of a two-spin-$\frac{1}{2}$ system is demonstrated to test the control algorithm.

A. Fuzzy Logic

Fuzzy logic is based on the theory of fuzzy set whose elements have different degrees of membership [33], [34]. An ordinary fuzzy set is a pair (F, m), where $m : F \to [0, 1]$. For each $x \in F$, $m(x)$ is the grade of membership of x:

$$x \in (F, m) \Leftrightarrow x \in F \text{ and } m(x) \neq 0. \tag{4}$$

If $F = \{x_1, x_2, \dots, x_N\}$, the fuzzy set (F, m) can be denoted as

$$(F, m) = \frac{m(x_1)}{x_1} + \frac{m(x_2)}{x_2} + \cdots + \frac{m(x_N)}{x_N} = \sum_{i=1}^{N} \frac{m(x_i)}{x_i}. \tag{5}$$

It is clear that if the distribution of a fuzzy set $m = \{m(x_1), m(x_2), \dots, m(x_N)\}$ is normalized, (5) has the similar representation structure of (1). The unitary operator $U(t)$ on a controlled quantum system can be simulated using the rotation operator on a fuzzy set [23]. Hence, the analog between quantum mechanical systems and fuzzy logic systems makes it suitable to simulate the evolution of quantum systems using fuzzy systems. We define a rotation operation on a fuzzy set as follows.

Definition 1 (Rotation Operation on Fuzzy Set): The rotation of a fuzzy set F is defined using a rotation operator *R* as follows:

$$R(F) = R(F, m) = (F, m') \tag{6}$$

where $F = \{x_i\}_{i=1}^N$, and $m = \{m(x_i)\}_{i=1}^N$. Let

$$\alpha = \left(\sqrt{m(x_i)}e^{\iota\varphi_i}\right)_{i=1}^N$$
$$\beta = R \circ \alpha = (\beta_i)_{i=1}^N$$

where $e^{\iota\varphi_i}$ can be used to simulate the phase in quantum states, and the symbol $\circ$ represents that the vector α is multiplied by the rotation operator *R* and is transformed to β. Then

$$m' = \{m'(x_i)\}_{i=1}^N = \{\beta_i\beta_i^*\}_{i=1}^N.$$

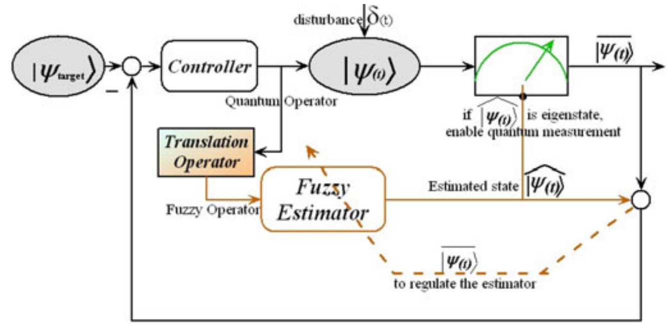

Fig. 2. Partial feedback control structure of pure-state quantum systems using an FE.

B. System Design and Control Algorithm

The partial feedback control scheme for pure-state quantum systems is shown in Fig. 2. In this scheme, an FE is designed to simulate and estimate the controlled quantum system. That is to say, the FE is initialized with the initial state of the controlled quantum system. The FE also runs according to the series of quantum operators on the quantum system. Hence, the FE can track the quantum system and provide an output with corresponding information to estimate the state of the real quantum system to produce a feedback signal. Usually, the estimated state $|\widehat{\psi(t)}\rangle$ is fed back to affect the controller for each control step. However, when the estimated quantum state is near to an eigenstate, a projective measurement M (with the same basis as that of H_0) will be triggered for the controlled quantum system, and the measurement results $\overline{|\psi(t)\rangle}$ will be fed back to construct a controller to drive the quantum system. These measurement results are also used to regulate the FE to eliminate the accumulated simulation errors.

Control Algorithm (Partial feedback control using an FE)

1) Initialize the controlled quantum system with an initial state $|\psi(0)\rangle$, C_0, and a desired target state $|\psi_{\text{target}}\rangle$, C_{target}; initialize the FE with a fuzzy set $F(0)$ according to $|\psi(0)\rangle$.
2) At time t, the controller gives a suitable control corresponding to a quantum operator $U(t)$. Its counterpart of rotation operator for the FE is $R(t)$. Then, drive the quantum system with $U(t)$. At the same time, apply the fuzzy operator $R(t)$ to the FE.
3) Then, the FE gives an estimation $|\widehat{\psi(t)}\rangle$ of the controlled quantum state where
$$\widehat{C(t)} = \beta = (\beta_i)_{i=1}^N.$$
4) Enable the projective measurement if $|\widehat{\psi(t)}\rangle$ is an almost eigenstate, i.e.,
$$\exists i \in \{1, 2, \dots, N\}, \overline{|\psi(t)\rangle} = |\phi_i\rangle, \ \|\overline{C(t)} - \beta\| < \varepsilon$$
where $\overline{C(t)} = (0, 0, \dots, 0, 1, 0, \dots, 0)^T$ with the ith component equal to 1, and $\varepsilon > 0$ is a real number. $\|\cdot\|$ is a norm where for a vector $C = (c_i)_{i=1}^N$, $\|C\| = \sqrt{\sum_{i=1}^N c_i^* c_i}$ and for a matrix A, $\|A\| = \sqrt{\text{tr}(A^*A)}$. Then, the controlled quantum system will collapse to

an eigenstate $\overline{|\psi(t)\rangle} = |\phi_{i'}\rangle,\ i' \in \{1, 2, \ldots, N\}$, $\overline{C(t)} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$ with the i'th component equal to 1. We feed it back to the control system and regulate the FE with $\overline{C(t)}$ to eliminate the accumulated errors; otherwise, feed $|\widehat{\psi(t)}\rangle$ back instead.

5) The control algorithm ends if the quantum state to be fed back is very near to the desired target state, i.e., $\|\overline{C(t)} - C_{\text{target}}\| < \varepsilon$ for the measured feedback or $\|\widehat{C(t)} - C_{\text{target}}\| < \varepsilon$ for the estimated feedback, or go to step 2).

Remark 1: The proposed partial feedback quantum control algorithm has two main features. 1) The estimator is designed with a fuzzy system which has a very similar representation as that of a quantum system. This fuzzy system can simulate the controlled quantum system very well in a natural way. 2) In step "4," when the estimated state $|\widehat{\psi(t)}\rangle$ is an almost eigenstate, the projective measurement will be implemented and the quantum state will collapse to an eigenstate $\overline{|\psi(t)\rangle}$. Although $\overline{|\psi(t)\rangle}$ will be the expected eigenstate with a probability near to 1, it may also have a small probability of failure. Here, the reason that we use a quantum measurement instead of feeding back an estimated eigenstate directly is to regulate the estimator to keep tracking the real system better. Otherwise, the simulation errors of the FE will accumulate along with the control process due to unknown disturbances.

Remark 2: It is clear that the proposed partial feedback control algorithm is a hybrid approach of two control schemes: One is the typical incoherent control [19] using the feedback signal by quantum measurement, and the other is the coherent control using the feedback signal from a classical estimator. We define the quantum control scheme that is presented in this paper as partial feedback control because the feedback signals are partially from the quantum system and partially from the classical estimator. The translation between the quantum control operators and the classical fuzzy operators plays an important role in this control approach.

C. Operator Translation

In order to provide real-time state estimation using an FE for the control of a quantum system, it is important to get an accurate mapping from the allowed quantum control operators to the counterpart fuzzy operators. For system (2), the propagator $U(t \to t')$ is a unitary operator that drives a state $|\psi(t)\rangle$ to the state $|\psi(t')\rangle$ through the controller, where

$$|\psi(t')\rangle = U(t \to t')|\psi(t)\rangle.$$

To simulate this process, the fuzzy rotation operation is utilized, and it can be formulated as

$$F(t') = R(t \to t')F(t).$$

Since $F_{(t)}$ (or $F(t)$) has the same representation structure as $|\psi(t)\rangle$, we let

$$R(t \to t') = U(t \to t').$$

Then as long as $U(t \to t')$ is known, we can get an accurate mapping from $U(t \to t')$ to $R(t \to t')$. Otherwise, a training process should be carried out before controlling the quantum system. The training of quantum/fuzzy operator mapping may be implemented by an initial quantum control model with adaptation and learning from multiple real experiments. In this paper, our interest on partial feedback control is from the control strategy rather than quantum system modeling. Therefore, it is assumed that the mapping of operator translation is known in this paper. The training methods for operator translation will be investigated in future work.

Remark 3: Even with the assumption that the operator translation is accurate, the fuzzy system may not be able to simulate the controlled quantum system completely, because there are unavoidable stochastic noises $\delta(t)$ at time t. There may exist errors for the FE at each control step. Because of the unknown noises (uncertainties), the quantum control operator and the quantum state are not known exactly. Let us denote by $\delta|\psi(t)\rangle$ the uncertainty of $|\psi(t)\rangle$ and by $\delta U(t \to t')$ the uncertainty of $U(t \to t')$. Therefore, the quantum state evolution will be

$$\begin{aligned} &|\psi(t')\rangle + \delta|\psi(t')\rangle \\ &= (U(t \to t') + \delta U(t \to t'))(|\psi(t)\rangle + \delta|\psi(t)\rangle). \end{aligned} \tag{7}$$

We assume that this control operator is still a unitary matrix. For example, the Hamiltonian uncertainties in [21] and unitary errors in [20] satisfy this assumption. By matrix and perturbation theory [35], the norm of the uncertainty $\delta|\psi(t')\rangle$ can be bounded using those of $\delta|\psi(t)\rangle$ and $\delta U(t \to t')$ by the following relationship:

$$\begin{aligned} \|\delta|\psi(t')\rangle\| \le \|\delta|\psi(t)\rangle\| + \|\delta U(t \to t')\| \\ + \|\delta|\psi(t)\rangle\|\|U(t \to t')\|. \end{aligned} \tag{8}$$

Remark 4: In the proposed control scheme, we use an FE to produce the feedback signal. The process of feedback information acquisition does not destroy the state of the controlled quantum system. When the estimated state is an almost eigenstate, the projective measurement is triggered. From (8), we know that the measurement process can eliminate the error in the state at time t [i.e., $\|\delta|\psi(t)\rangle\| = 0$ in (8)]. Compared with open-loop control, the mechanism of dynamic partial feedback improves the robustness of the control system. The following numerical examples will illustrate the robust performance. However, a strict proof of the stability and robustness for the partial feedback control scheme remains open, which is related to specific uncertainties, given control operations and permissible errors.

D. Numerical Example

To demonstrate the proposed approach, several numerical experiments are carried out on a multispin-$\frac{1}{2}$ system. For a two-spin-$\frac{1}{2}$ system, its state can be formulated as

$$\begin{aligned} |\psi(t)\rangle = c_1(t)|\downarrow\rangle|\downarrow\rangle + c_2(t)|\downarrow\rangle|\uparrow\rangle \\ + c_3(t)|\uparrow\rangle|\downarrow\rangle + c_4(t)|\uparrow\rangle|\uparrow\rangle \end{aligned}$$

where $c_i(t)$ $(i = 1, 2, 3, 4)$ are the complex numbers that satisfy

$$|c_1(t)|^2 + |c_2(t)|^2 + |c_3(t)|^2 + |c_4(t)|^2 = 1.$$

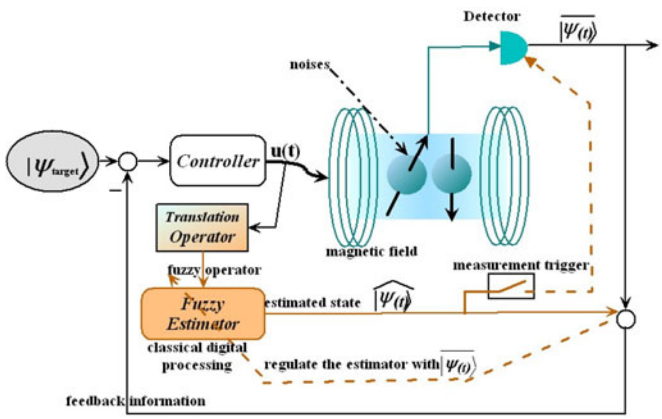

Fig. 3. Experimental configuration of the partial feedback control using an FE.

Denoting $C(t) = (c_i(t))_{i=1}^4$, it evolves according to the Schrödinger equation [32]:

$$\iota\hbar\dot{C}(t) = [A + u(t)B]C(t).$$

The translation of $C(t)$ to $C(t')$ is determined by

$$C(t') = e^{-\frac{\iota(A+u(t)B)}{\hbar}}C(t).$$

The unitary matrix

$$U(t) = e^{-\frac{\iota(A+u(t)B)}{\hbar}}$$

drives the quantum system from $C(t)$ to $C(t')$ by the control $u(t)$. For the specific simulation, we assume that the allowed electromagnetic field controls can construct the following unitary transformations:

$$U(\theta(t)) = \{U_k(\theta(t))\},\ k = 1, 2, 3$$

$$U_1(\theta(t)) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 & 0 \\ 0 & 0 & \iota/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & \iota/\sqrt{2} \end{pmatrix}$$

$$U_2(\theta(t)) = \begin{pmatrix} \iota/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 1/\sqrt{2} & 0 & 0 & \iota/\sqrt{2} \end{pmatrix}$$

$$U_3(\theta(t)) = \begin{pmatrix} \iota/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & \iota/\sqrt{2} & 0 & 0 \\ 0 & 0 & \cos\theta(t) & -\sin\theta(t) \\ 0 & 0 & \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

where $\theta(t) = 0, \pm\frac{\pi}{8}, \pm\frac{\pi}{4}$. During the quantum control process, there are unpredictable disturbances, i.e.,

$$U(\theta'(t)) = U(\theta(t)) + \delta U(t) = U(\theta(t) + \delta\theta(t)).$$

The demonstration of the experimental configuration is shown in Fig. 3. The FE is implemented using a traditional computer.

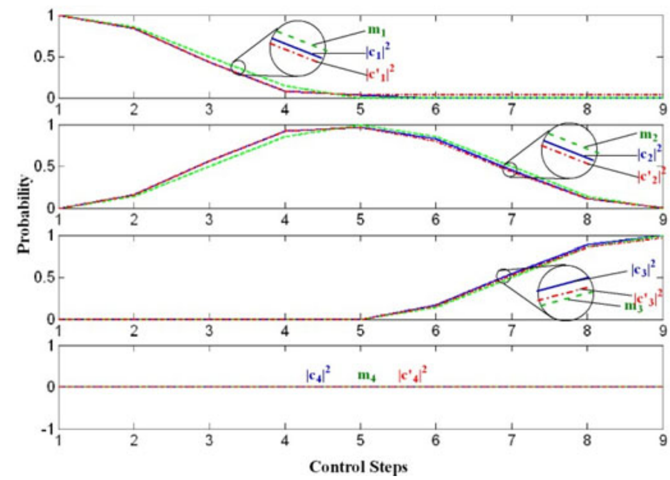

Fig. 4. Control performance of the two-spin-$\frac{1}{2}$ system from the initial state $C_0 = (1, 0, 0, 0)^T$ to the target state $C_{\text{target}} = (0, 0, 1, 0)^T$.

During the control process, the output of the controller $u(t)$ activates the magnetic field to control the two-spin-$\frac{1}{2}$ system. At the same time, $u(t)$ is transmitted to a corresponding fuzzy operator and acts on the FE. When the estimated state $\widehat{|\psi(t)\rangle}$ from the FE indicates that the quantum state is very near to an eigenstate, the quantum measurement is triggered. For all the simulated experiments in this section, the experimental settings are as follows: The stochastic disturbances are a noise with a uniform distribution on the interval $[-10\%\ \theta(t), 10\%\ \theta(t)]$; the fuzzy set is initialized as $\alpha = C_0$, $R(t) = U(t)$; the threshold of triggering quantum measurement is set as $\varepsilon = 0.05$.

In the first group of simulation experiments, the initial quantum state is $C_0 = (1, 0, 0, 0)^T$, and the target quantum state $C_{\text{target}} = (0, 0, 1, 0)^T$. The control process using the proposed approach is shown in Figs. 4 and 5. In Fig. 4, m_j are the memberships of the corresponding fuzzy set, $|c_j|^2$ are the probabilities of quantum eigenstates of the controlled system using the proposed partial feedback control, and $|c'_j|^2$ are the probabilities of quantum eigenstates of the controlled system using the open-loop control, where $j = 1, 2, 3, 4$. Fig. 4 demonstrates that all the probability amplitudes converge to that of the target state. Compared with the open-loop control method, the partial feedback control that is presented in this paper performs much better and is more robust to unpredictable noises with the feedback information. To compare the control performance more clearly, Fig. 5 shows the state distances of the controlled quantum state to the target state. The state distance between two quantum states with the same eigenstates $|\psi^a\rangle = \sum_{i=1}^N c_i^a|\phi_i\rangle$ and $|\psi^b\rangle = \sum_{i=1}^N c_i^b|\phi_i\rangle$ is defined as

$$\text{Dist}_{ab} = \sum_{i=1}^N |c_i^a - c_i^b|^2. \tag{9}$$

From Fig. 5, it is clear that it is very difficult to control the quantum system to the target state, while the proposed partial feedback control method performs better and can control the quantum system to the target state with a permissible error.

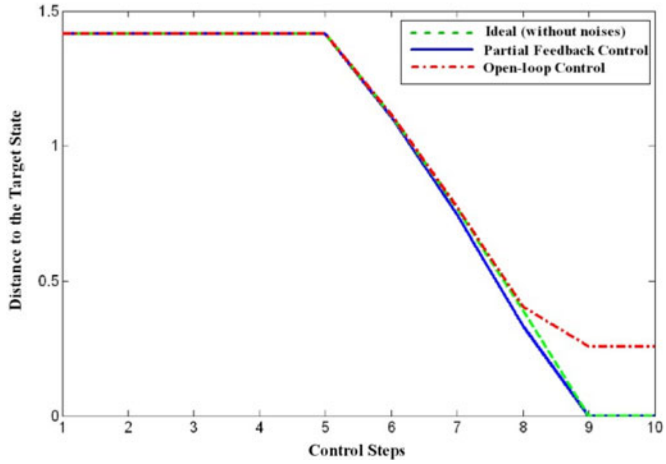

Fig. 5. State distances to the target state $C_{\mathrm{target}} = (0, 0, 1, 0)^T$ in the control process.

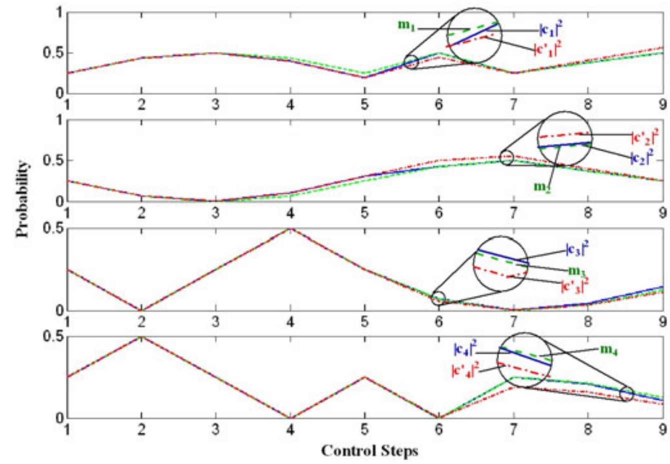

Fig. 6. Control performance of the two-spin-$\frac{1}{2}$ system from the initial state $C_0 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\iota)^T$ to the target state $C_{\mathrm{target}} = (\frac{\sqrt{2}}{2}\iota, -\frac{1}{2}\iota, -\frac{\sqrt{2}}{4}\iota, \frac{\sqrt{2}}{4}\iota)^T$.

In the second group of experiments, the initial quantum state is $C_0 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\iota)^T$, and the target quantum state is $C_{\mathrm{target}} = (\frac{\sqrt{2}}{2}\iota, -\frac{1}{2}\iota, -\frac{\sqrt{2}}{4}\iota, \frac{\sqrt{2}}{4}\iota)^T$. This group of experiments is more difficult to control because there are fewer eigenstates appearing during the control process, and less quantum measurement can be taken. The control results using the proposed approach are shown in Figs. 6 and 7. In Fig. 6, m_j are the memberships of the corresponding fuzzy set, $|c_j|^2$ are the probabilities of quantum eigenstates of the controlled system using the proposed partial feedback control, and $|c_j'|^2$ are the probabilities of quantum eigenstates of the controlled system using the open-loop control, where $j = 1, 2, 3, 4$. These experimental results also show that the partial feedback control approach performs well and evidently improves the performance of open-loop control.

From all these simulated results, it is clear that no matter the initial state and target state are eigenstates or not, the partial feedback control algorithm using an FE successfully drives the pure-state quantum system to its desired target state with a permissible error. The results also show that the proposed control method is robust to unknown disturbances.

IV. Control Problem II: Control Design with a Probabilistic Fuzzy Estimator

Now we consider the control problem II. To implement a partial feedback control for a quantum system with an initial mixed state, a probabilistic fuzzy logic system is used to model the quantum state and the stochastic uncertainties as well [24].

A. Probabilistic Fuzzy Logic

A probabilistic fuzzy logic system is different from an ordinary fuzzy logic system, and it uses probabilistic fuzzy sets instead of ordinary fuzzy sets to capture the information with stochastic uncertainties [36], [37]. Similar to the definition of probabilistic fuzzy set in [25] and [38], the concept of probabilistic fuzzy set is defined as follows.

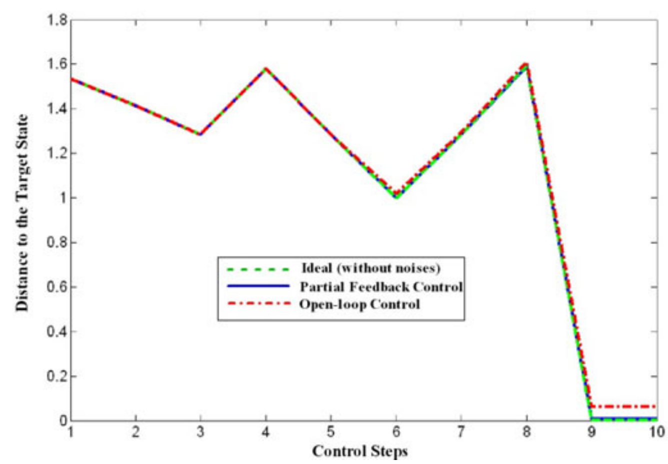

Fig. 7. State distances to the target state $C_{\mathrm{target}} = (\frac{\sqrt{2}}{2}\iota, -\frac{1}{2}\iota, -\frac{\sqrt{2}}{4}\iota, \frac{\sqrt{2}}{4}\iota)$ in the control process.

Definition 2 (Probabilistic Fuzzy Set [38]): The probabilistic fuzzy set $\widetilde{F}$ is expressed by a probability space (E, P), where $E = \{E_j = (F, m_j)\}$ is the set of all possible events $x \in (F, m_j)$, and $P = \{p_j = P(E_j)\}$ is the corresponding set of probabilities. For all element event $E_j \in E$

$$p_j = P(E_j) \geq 0, \quad P\left(\sum E_j\right) = \sum P(E_j), \quad P(E) = 1. \tag{10}$$

The probabilistic fuzzy set $\widetilde{F}$ can be expressed as the union of finite ordinary fuzzy sets with a probability distribution as follows:

$$\widetilde{F} = \bigcup_{x \in (F,m)} ((F, m), P). \tag{11}$$

Fig. 8 shows a 3-D illustration of a probabilistic fuzzy set $\widetilde{F_1}$ (discrete case):

$$\widetilde{F}_1 = \bigcup_{j=1,2,3} \{((F, m_j), p_j)\} \tag{12}$$

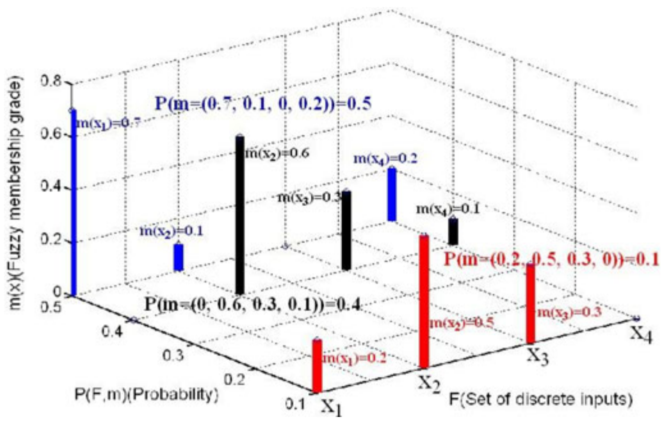

Fig. 8. Three-dimensional illustration of probabilistic fuzzy sets.

where $F = \{x_1, x_2, x_3, x_4\}$, $p_1 = P(m_1) = 0.5, m_1 = \{0.7, 0.1, 0, 0.2\}$, $p_2 = P(m_2) = 0.4, m_2 = \{0, 0.6, 0.3, 0.1\}$, $p_3 = P(m_3) = 0.1$, and $m_3 = \{0.2, 0.5, 0.3, 0\}$.

Similar to the ordinary fuzzy logic system, operations such as fuzzification, inference engine, and defuzzification can be defined based on the probabilistic fuzzy set introduced earlier. Some set-theoretic and logic operations, such as intersection and union, can also be specified for a probabilistic fuzzy set. In particular, to simulate the quantum operations on quantum mixed states, we can also define a unitary operator R on the probabilistic fuzzy set to change the membership function of the probabilistic fuzzy set, which is an extension of the rotation operation on the ordinary fuzzy set.

Definition 3 (Rotation Operation): The rotation of a probabilistic fuzzy set $\widetilde{F}$ is defined using a rotation operator *R* as follows:

$$R(\widetilde{F}) = R\left(\bigcup_{x\in(F,m)} ((F,m),P)\right) = \bigcup_{x\in(F,m')} ((F,m'),P) \tag{13}$$

where $F = \{x_i\}_{i=1}^N$, and $m = \{m(x_i)\}_{i=1}^N$. Let

$$\alpha = (\sqrt{m(x_i)}e^{\iota\varphi_i})_{i=1}^N$$
$$\beta = R \circ \alpha = (\beta_i)_{i=1}^N$$

where $e^{\iota\varphi_i}$ can be used to simulate the phase in quantum states. Then

$$m' = \{m'(x_i)\}_{i=1}^N = \{\beta_i\beta_i^*\}_{i=1}^N.$$

Example 1: One rotation operator *R* is represented with the following rotation matrix:

$$R = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & 1/2 \\ 0 & 0 & -1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Now, we apply *R* to $\widetilde{F_1}$ with $\alpha_i = \sqrt{m(x_i)}e^{\iota\varphi_i}$, $i = 1, 2, 3$. For simplicity, α and β are set as real vectors. We can calculate to get $\alpha_1 = \{0.8367, 0.3162, 0, 0.4472\}$, $\alpha_2 = \{0, 0.7746, 0.5477, 0.3162\}$, and $\alpha_3 = \{0.4472, 0.7071, 0.5477, 0\}$. Then

$$\beta_1 = R \circ \alpha_1 = \{0.8152, -0.3680, 0.2236, 0.3873\}$$
$$m'_1 = \{0.6646, 0.1354, 0.05, 0.15\}$$
$$\beta_2 = R \circ \alpha_2 = \{0.5477, 0.5477, 0.6325, 0\}$$
$$m'_2 = \{0.3, 0.3, 0.4, 0\}$$
$$\beta_3 = R \circ \alpha_3 = \{0.8162, 0.1838, 0.4743, -0.2739\}$$
$$m'_3 = \{0.6662, 0.0338, 0.225, 0.075\}$$
$$\widetilde{F_1'} = R\left(\bigcup_{j=1,2,3} \{((F,m_j),p_j)\}\right) = \bigcup_{j=1,2,3} \{((F,m'_j),p_j)\}. \tag{14}$$

As the statement of quantum mixed states in the problem formulation, a mixed state must be described by a density operator ρ. By a comparison of relative counterparts between a quantum mixed state and a probabilistic fuzzy system (as shown in Table I), we will show that a PFE is suitable to estimate the evolution of a quantum mixed state, and a PFE can provide an effective estimation of the state of the controlled quantum system.

In Table I, the symbol $M = \{M_i = |\phi_i\rangle\langle\phi_i| : i = 1, \dots, N\}$ represents a measurement operator, and it is a projective measurement operator in this paper. From the comparison shown in Table I, it is clear that when the quantum mixed state ρ evolves with the unitary transformation U, this process can be simulated by a counterpart of probabilistic fuzzy system $\widetilde{F}$ with the rotation transformation R. The FE provides an estimate $\widehat{\rho}$ of the quantum system's state. The partial feedback control scheme of a quantum system with an initial mixed state using a PFE is shown in Fig. 9.

B. System Design and Control Algorithm

Control Algorithm (Partial feedback control using a PFE)

1) Initialize the controlled quantum system with an initial state $\rho(0) : \{p_j, |\psi_j\rangle\}$ and a desired target state ρ_{target}; initialize the PFE with a probabilistic fuzzy set $\widetilde{F}(0)$ according to $\rho(0)$.
2) At time t, the controller gives a suitable control which is denoted as a quantum operator $U(t)$. Its counterpart of a rotation operator for the FE is $R(t)$. Then, drive the quantum system with $U(t)$. At the same time, apply the fuzzy operator $R(t)$ to the FE:

$$\rho(t') = U(t)\rho(t)U^\dagger(t)$$
$$\widetilde{F}(t') = R(t)(\widetilde{F}(t)) = \bigcup_j \{((F,m'_j),p_j)\}.$$

3) Then, the PFE gives an estimate $\widehat{\rho(t)}$ of the controlled quantum system

$$\widehat{\rho(t)} = \sum_j p_j |\widehat{\psi_j}\rangle\langle\widehat{\psi_j}|$$

TABLE I
COMPARISON BETWEEN A QUANTUM MIXED STATE AND A PROBABILISTIC FUZZY SYSTEM

Compared items	Quantum mixed state	Probabilistic fuzzy system
Definition	distribution of pure states $\{(p_j, \lvert\psi_j\rangle)\}$	distribution on ordinary fuzzy sets $\{(p_j, (F, m_j))\}$
Mathematical Representation	$\rho \equiv \sum_j p_j \lvert\psi_j\rangle\langle\psi_j\rvert$ where $\lvert\psi_j\rangle = \sum_{i=1}^{N} c_i^j(t)\lvert\phi_i\rangle$	$\widetilde{F} = \bigcup_j \{((F, m_j), p_j)\}$ where $(F, m_j) = \sum_{i=1}^{N} \frac{m_j(x_i)}{x_i}$
Operator	unitary operator U $U(\rho) \to \rho' = \sum_j p_j U\lvert\psi_j\rangle\langle\psi_j\rvert U^\dagger = U\rho U^\dagger$	rotation operator R $R(\widetilde{F}) \to \widetilde{F}' = \bigcup_j \{((F, m_j'), p_j)\}$
Probability of getting $\lvert\phi_i\rangle$ or x_i	$p(\lvert\phi_i\rangle) = \sum_j p_j \mathrm{tr}(M_i \lvert\psi_j\rangle\langle\psi_j\rvert) = \mathrm{tr}(M\rho)$	$p(x_i) = \sum_j p_j m_j(x_i)$

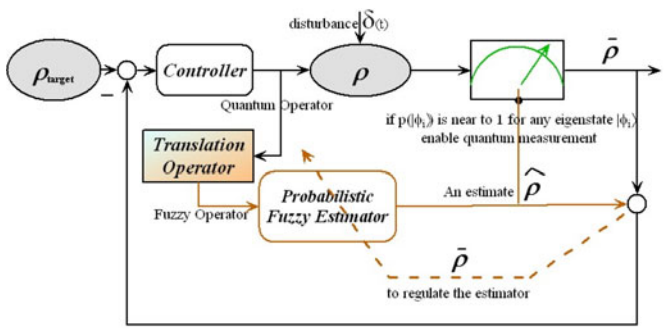

Fig. 9. Partial feedback control structure of a quantum system with an initial mixed state using a PFE.

where

$$\widehat{|\psi_j\rangle} = \sum_i \beta_i |\phi_i\rangle.$$

4) Trigger the projective measurement if $|p(|\phi_i\rangle) - 1| < \varepsilon$ with $p(|\phi_i\rangle) = \mathrm{tr}(M_i \widehat{\rho(t)})$, where $\varepsilon > 0$ is a real number. Then, the controlled quantum state will collapse to an eigenstate $|\phi_{i'}\rangle$, $i' \in \{1, 2, \ldots, N\}$, whose density matrix corresponds to $\overline{\rho}$, feed it back to the quantum system, and regulate the FE to eliminate the accumulated errors of the PFE; otherwise, feed $\widehat{\rho(t)}$ back instead.
5) The control algorithm ends if the feedback quantum state is very near to the desired target state, i.e., $\|\overline{\rho(t)} - \rho_{\text{target}}\| < \varepsilon'$ for the measured feedback or $\|\widehat{\rho(t)} - \rho_{\text{target}}\| < \varepsilon'$ for the estimated feedback, where $\varepsilon' > 0$ is a positive real number or go to step 2).

Remark 5: The operator translation and error analysis are similar to those in Section III. Besides describing a single quantum system, the mixed state ρ is often used to describe a quantum ensemble [1]. A quantum ensemble $\rho = \sum_j p_j |\psi_j\rangle\langle\psi|$ can be looked as a mixture of a number of pure states $|\psi_j\rangle$ with respective probabilities p_j. Hence, it is also possible to develop quantum control methods using a PFE for the control of quantum ensemble [39].

C. Numerical Example

To test the proposed partial feedback control approach using a PFE, several numerical experiments of a two-spin-$\frac{1}{2}$ system are also presented. For these experiments, the following unitary transformations are taken as the allowed electromagnetic field controls:

$$U(\theta(t)) = \{U_k(\theta(t))\},\ k = 1, 2, 3$$

$$U_1(\theta(t)) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 & 0 \\ 0 & 0 & -\iota/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -\iota/\sqrt{2} \end{pmatrix}$$

$$U_2(\theta(t)) = \begin{pmatrix} -\iota/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 1/\sqrt{2} & 0 & 0 & -\iota/\sqrt{2} \end{pmatrix}$$

$$U_3(\theta(t)) = \begin{pmatrix} -\iota/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -\iota/\sqrt{2} & 0 & 0 \\ 0 & 0 & \cos\theta(t) & -\sin\theta(t) \\ 0 & 0 & \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

where $\theta(t) = 0, \pm\frac{\pi}{8}, \pm\frac{\pi}{4}$.

The unpredictable disturbances during the quantum control process are denoted as

$$U(\theta'(t)) = U(\theta(t)) + \delta U(t) = U(\theta(t) + \delta\theta(t)).$$

To demonstrate the control performance, the state distance between two quantum mixed states is defined using the density matrix. For two quantum states that are described with ρ^a and ρ^b, the state distance between them is defined as

$$Dist_{ab} = \|\rho^a - \rho^b\|.$$

The experimental settings are as follows: The stochastic disturbances are a noise with a uniform distribution on the interval $[-10\%\ \theta(t), 10\%\ \theta(t)]$; $R(t) = U(t)$; the threshold of triggering quantum measurement is set as $\varepsilon = 0.05$; and the algorithm ending criteria $\varepsilon' = 0.02$. The initial quantum mixed state is $\{(p_j, |\psi_j\rangle)\}, j = 1, 2, 3$, where

$$p_1 = 0.6, \quad C_1 = (-0.4619 - 0.1913\iota, 0.6036 + 0.2500\iota$$
$$0.2500 + 0.1036\iota, -0.1913 + 0.4619\iota)^T$$

$$p_2 = 0.3, \quad C_2 = (0.0698 - 0.4951\iota, -0.2162 + 0.6469\iota$$
$$-0.1628 + 0.0912\iota, -0.3994 - 0.3008\iota)^T$$

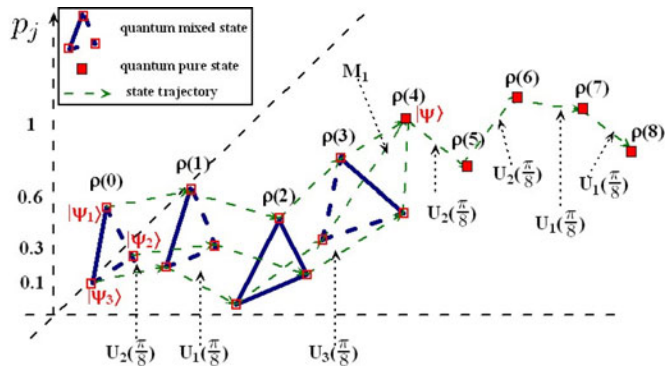

Fig. 10. Trajectory of the two-spin-$\frac{1}{2}$ system from the initial mixed state $\rho(0)$ to the target state $\rho(8)$.

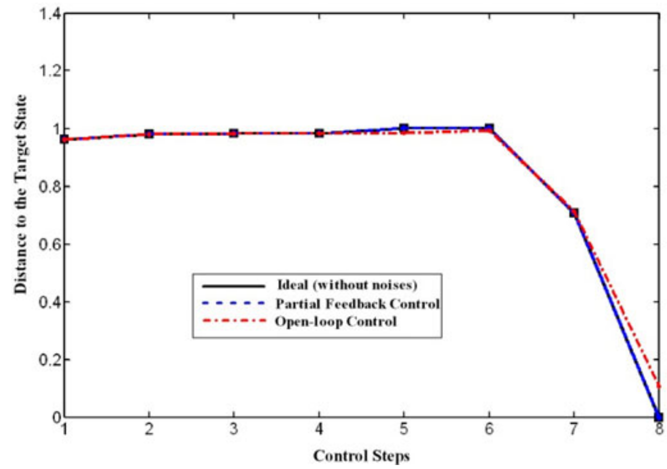

Fig. 11. Control performance of the two-spin-$\frac{1}{2}$ system with the initial mixed state using a PFE.

$$p_3 = 0.1, \quad C_3 = (0.3967 - 0.3044\iota, -0.1647 + 0.6477\iota - 0.2147 - 0.0853\iota, -0.4957 + 0.0653\iota)^T \quad (15)$$

and the density matrix is

$$\rho(0) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \quad (16)$$

where $\rho_{11} = 0.2500 + 0.0000\iota$, $\rho_{12} = -0.3228 - 0.0021\iota$, $\rho_{13} = -0.1040 + 0.0322\iota$, $\rho_{14} = 0.0147 + 0.2281\iota$, $\rho_{21} = -0.3228 + 0.0021\iota$, $\rho_{22} = 0.4403 - 0.0000\iota$, $\rho_{23} = 0.1323 - 0.0410\iota$, $\rho_{24} = -0.0201 - 0.3240\iota$, $\rho_{31} = -0.1040 - 0.0322\iota$, $\rho_{32} = 0.1323 + 0.0410\iota$, $\rho_{33} = 0.0597 + 0.0000\iota$, $\rho_{34} = 0.0214 - 0.1012\iota$, $\rho_{41} = 0.0147 - 0.2281\iota$, $\rho_{42} = -0.0201 + 0.3240\iota$, $\rho_{43} = 0.0214 + 0.1012\iota$, and $\rho_{44} = 0.2500 + 0.0000\iota$. The target state is a quantum pure state $C_{\text{target}} = (0, 0, -1, 0)^T$, and its density matrix is

$$\rho_{\text{target}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The simulation experimental results is shown in Figs. 10 and 11. In Fig. 10, this control task is achieved by the proposed partial feedback control approach. The fuzzy rules used are defined as follows.

Rule ℓ $(\ell = 1, 2, 3, 4)$: IF $|\text{tr}(M_\ell \widehat{\rho(t)}) - 1| < \varepsilon = 0.05$, THEN trigger the projective measurement M_ℓ;

Rule (i, j) $(j = 1, \ldots, 15)$: IF $|\text{tr}(M_\ell \widehat{\rho(t)}) - 1| \geq \varepsilon = 0.05$ and the estimated quantum state $\widehat{\rho(t)}$ is $\widetilde{F}_i$, THEN the control output is U_j.

Here, $\widetilde{F}_i$ is the predefined probabilistic fuzzy set (in this numerical example, we choose 16 probabilistic fuzzy sets, i.e., $i = 1, \ldots, 16$), and the control rules are attained by experience according to the real quantum control system. For example, nine specific probabilistic fuzzy sets $\widetilde{F}_i$ that are used in this numerical example can be found in Table II. However, for general quantum control problems, 16 probabilistic fuzzy sets are far from enough, and many more probabilistic fuzzy sets are required to simulate the quantum system. In addition, in more complex control problems, it is another key task to obtain the fuzzy rules to determine the control operations. We may employ a learning process to accomplish such a task (e.g., using quantum reinforcement learning in [40]). The detailed learning process and more simulation experiments for more complex control problems will be presented in our future work. Here, we only focus on a simple example aiming to demonstrate the basic idea of the proposed approach. The simulation result gives a decision of a control sequence $\{U_2(\frac{\pi}{8}), U_1(\frac{\pi}{8}), U_3(\frac{\pi}{8}), U_2(\frac{\pi}{8}), U_2(\frac{\pi}{8}), U_1(\frac{\pi}{8}), U_1(\frac{\pi}{8})\}$ and a projective measurement M_1. Fig. 10 demonstrates the trajectory of the two-spin-$\frac{1}{2}$ system that evolves from the initial mixed state $\rho(0)$ to the target state $\rho(8)$ under the partial feedback control scheme. This control task is achieved by a control decision of a control sequence $\{U_2(\frac{\pi}{8}), U_1(\frac{\pi}{8}), U_3(\frac{\pi}{8}), U_2(\frac{\pi}{8}), U_2(\frac{\pi}{8}), U_1(\frac{\pi}{8}), U_1(\frac{\pi}{8})\}$ with a projective measurement M_1. As shown in Fig. 10, the projective measurement M_1 is triggered on the quantum state $\rho(3)$ because $|\text{tr}(M_1 \widehat{\rho(3)}) - 1| = 0.04 < \varepsilon$.

The change of memberships during the control process is shown in Table II, where m_j denotes the membership regarding $|\psi_j\rangle$ in a quantum mixed state, $\{R_1(\frac{\pi}{8}), R_2(\frac{\pi}{8}), R_3(\frac{\pi}{8})\}$ are the fuzzy rotation operations that correspond to the electromagnetic field controls $\{U_1(\frac{\pi}{8}), U_2(\frac{\pi}{8}), U_3(\frac{\pi}{8})\}$, respectively, M_1 denotes the measurement operation $M_1 = |\phi_1\rangle\langle\phi_1|$, and $I(\rho_0)$ provides the initial membership regarding the initial quantum mixed state. For $m_1 = m_2 = m_3$ (corresponding to a pure state), we omit p_1, p_2, p_3 since their values do not affect the following results. Fig. 11 gives more results of the performance comparison between the proposed method and an open-loop control method. These results demonstrate the success of the proposed partial feedback control scheme for the quantum system with an initial mixed state with the help of a PFE. It evidently improves the performance of open-loop control and helps with finding more suitable controls as well. These experimental results further

TABLE II
MEMBERSHIP FUNCTIONS OF A PFE DURING THE CONTROL PROCESS

R	Probabilistic Fuzzy System $\widetilde{F}_i=\bigcup_j\{((F,m_j),p_j)\}$, $(F,m_j)=\sum_{k=1}^{4}\frac{m_j(x_k)}{x_k}$
$I(\rho_0)$	$p_1=0.6, m_1=\{0.2500, 0.4268, 0.0732, 0.2500\}$
	$p_2=0.3, m_2=\{0.2500, 0.4652, 0.0348, 0.2500\}$
	$p_3=0.1, m_3=\{0.2500, 0.4466, 0.0534, 0.2500\}$
$R_2(\frac{\pi}{8})$	$p_1=0.6, m_1=\{0.5000, 0.4973, 0.0027, 0.0000\}$
	$p_2=0.3, m_2=\{0.4688, 0.4539, 0.0461, 0.0312\}$
	$p_3=0.1, m_3=\{0.3750, 0.3512, 0.1488, 0.1250\}$
$R_1(\frac{\pi}{8})$	$p_1=0.6, m_1=\{0.4995, 0.4979, 0.0013, 0.0013\}$
	$p_2=0.3, m_2=\{0.4708, 0.4522, 0.0692, 0.0078\}$
	$p_3=0.1, m_3=\{0.3792, 0.3476, 0.0328, 0.2404\}$
$R_3(\frac{\pi}{8})$	$p_1=0.6, m_1=\{0.9995, 0.0000, 0.0003, 0.0002\}$
	$p_2=0.3, m_2=\{0.9315, 0.0000, 0.0680, 0.0005\}$
	$p_3=0.1, m_3=\{0.7398, 0.0001, 0.0008, 0.2593\}$
M_1	$m_1=m_2=m_3=\{1.0000, 0, 0, 0\}$
$R_2(\frac{\pi}{8})$	$m_1=m_2=m_3=\{0.5000, 0, 0, 0.5000\}$
$R_2(\frac{\pi}{8})$	$m_1=m_2=m_3=\{0, 0, 0, 1.0000\}$
$R_1(\frac{\pi}{8})$	$m_1=m_2=m_3=\{0, 0, 0.5000, 0.5000\}$
$R_1(\frac{\pi}{8})$	$m_1=m_2=m_3=\{0, 0, 1.0000, 0\}$

prove that the proposed partial feedback control method using FEs provides a useful tool for the control design of quantum systems with uncertainties.

V. CONCLUSION

Robust control of quantum systems is an important issue for the development of practical quantum technology. This paper has proposed an approach of control design for quantum systems with uncertainties, where an FE is used to estimate the quantum state of the controlled quantum system. Two types of quantum control problems have been investigated. For a pure-sate quantum system, we have presented a partial feedback control scheme where a fuzzy system is trained to estimate quantum states, and a controlled projective measurement is also used to assist in controlling quantum systems with uncertainties. The main advantage is that this scheme can avoid the direct measurement on quantum systems and can obtain feedback information from an FE as well. For a quantum system with an initial mixed state and uncertainties, a PFE is trained to estimate quantum mixed states for control design. We have demonstrated the proposed method by several simulated examples. Our future work will focus on the control of complex quantum systems and other control design methods using intelligent computation techniques.

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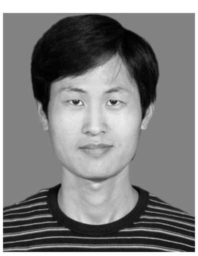

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