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Title	**Realization of a special class of admittances with one damper and one inerter for mechanical control**
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considered in this paper. Optimal control data schedule is presented in closed-form for a class of systems and some discussions on the optimal schedule for general systems are presented.

Future work along the line of this work include finding the exact optimal schedule for general higher-order systems and LQR control with output feedback, and considering LQG control data scheduling.

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Realization of a Special Class of Admittances with One Damper and One Inerter for Mechanical Control

Michael Z. Q. Chen, Kai Wang, Yun Zou, and James Lam

Abstract—In this note, we investigate the realization problem of a special class of positive-real admittances, which is common in vehicle suspension designs. The number of inerters and dampers is restricted to one in each case and the number of the springs is arbitrary. To solve the problem, we first convert a previous result by [6] to a more direct form. A necessary and sufficient condition for realizability is then derived and explicit circuit arrangements are provided by assuming that the three-port network consisting of only springs after extracting the damper and the inerter has a well-defined impedance. To remove the assumption on the existence of a well-defined impedance, a condition is established on the topological property of the n-port network without a well-defined impedance to obtain an equivalent class of such networks so that the realizability condition is derived with realization. By combining the conditions with and without a well-defined impedance, the final realization result is obtained.

Index Terms—Electric circuits, inerter, mechanical networks, network synthesis, passivity.

I. INTRODUCTION

Passive network synthesis is a classical subject in electrical circuit theory which experienced a "golden era" for the 1930s–1970s [1], [2], [11], [16]. Despite the relative maturity of the field, certain aspects of passive network synthesis are still incomplete. For example, the only general method for transformerless electrical synthesis by Bott and Duffin [2] appears to be highly non-minimal. However, interest in the field has declined despite the relatively recent development in the design of positive real functions [8], [10], [15], [20].

Recently, a new network element, named *inerter* [4], [19], has been introduced with the property that the (equal and opposite) force applied at the terminals is proportional to the relative acceleration between them. Applications of the inerter to vehicle suspension, motorcycle steering control and vibration absorption have been identified with performance advantages demonstrated (see [4] and references therein). One of the main motivations for the inerter was the synthesis of passive mechanical networks. The inerter completes the analogy between electrical networks and mechanical ones (see [19, Fig. 4]), which makes any passive mechanical network realizable with three kinds of passive elements: inerters, dampers, and springs. However, the number of elements for mechanical networks is much more essential than electrical ones. Therefore, given the existing and potential applications of the inerter, interest in passive network synthesis has been revived [5]–[7], [12], [13]. The need for a renewed attempt on the same subject and its fundamental connection to system theory has also been highlighted by Kalman [14].

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The present note is concerned with the realization problem of a special class of positive-real admittances, which is common in vehicle suspension designs. This particular class was first discussed in [19], where it was pointed out that any such positive-real admittance can be realized using one inerter, two dampers, and three springs. This note considers the class of realizations in which the number of dampers and inerters is restricted to one in each case. In our realizations, we impose the condition that no lever (transformer) can be employed since large lever ratios can cause difficulties in practical implementation.

We first present a necessary and sufficient condition for this special class of positive-real admittances to be realizable employing one damper and one inerter by assuming that the impedance of the three-port network after extracting the damper and the inerter is well-defined. In addition, an explicit construction is given comprising two circuit arrangements, one employing four springs and the other two. Furthermore, we give the relationship between the topological property of the n-port network and the fact that its impedance is not well-defined, by which we obtain an equivalent class of networks when the three-port network does not have a well-defined impedance. The realizability condition of this class of network is derived and a corresponding network is given, which removes the constraint on the existence of a well-defined impedance.

II. PROBLEM FORMULATION

The realization of the admittance $Y(s)$ in the form of

$$Y(s) = k \frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)} \quad (1)$$

where $a_0, a_1, d_0, d_1 \geq 0$, and $k > 0$ was first discussed in [19]. Defining the resultant [9] of $p(s) := a_0 s^2 + a_1 s + 1$ and $q(s) := d_0 s^2 + d_1 s + 1$ in s, we have $R_k := (a_0 - d_0)^2 - (a_0 d_1 - a_1 d_0)(a_1 - d_1)$. It was shown that any positive-real $Y(s)$ in the form of (1) can be realized using one inerter, two dampers, and three springs. It is known that many mechanical admittances of suspension struts are in this form (see [4], [6], [19] and references therein).

For mechanical systems, the spring is the easiest element to be realized practically [6]. Thus, the realization problem of admittance (1) when the number of inerters and dampers is restricted to one in each case is meaningful. In [6], it is shown through a counter example that not all positive-real admittances (1) can be realized using one damper and one inerter. The present note addresses the following question: Given a positive-real admittance $Y(s)$ in the form of (1), what additional conditions for $Y(s)$ are needed to be realized with one inerter, one damper, and an arbitrary number of springs but no levers? The final result (Theorem 4.1) of this note provides a necessary and sufficient condition for this realization, which is a neat inequality concerning only with coefficients a_0, a_1, d_0, and d_1. Hence, it will be convenient to test the realizability. In addition, the network configurations that can be used to cover this condition as well as the values of their elements are given in Theorem 3.4, Theorem 3.5, and Lemma 4.5.

III. REALIZABILITY CONDITIONS WITH A WELL-DEFINED IMPEDANCE

A network with one damper, one inerter and an arbitrary number of springs can be shown in Fig. 1, where X consists of only springs. In this section, we assume that X has a well-defined impedance. Chen and Smith have derived a necessary and sufficient condition for any positive-real admittance to be realized as in Fig. 1 [6].

Lemma 3.1: [6] A positive-real function $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 1, where X
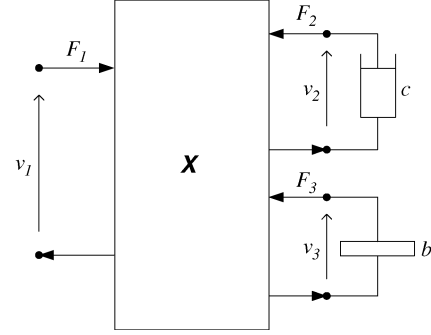

Fig. 1. General one-port containing one damper and one inerter, where b, $c > 0$, and X consists of only springs.

has a well-defined impedance and consists of only springs, if and only if $Y(s)$ can be written in the form of

$$Y(s) = \frac{(R_2 R_3 - R_6^2) s^3 + R_3 s^2 + R_2 s + 1}{s(\det R s^3 + (R_1 R_3 - R_5^2) s^2 + (R_1 R_2 - R_4^2) s + R_1)} \quad (2)$$

where R as defined in

$$R := \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix} \quad (3)$$

is non-negative definite, and the entries of R further satisfy certain conditions such that there exists an invertible matrix $D = \mathrm{diag}\{1, x, y\}$ such that DRD is paramount.

To make it easier to check the realizability condition for admittance $Y(s)$ in the form of (1), it seems natural to convert the admittance $Y(s)$ in the form of (2) to the following form:

$$Y(s) = \frac{\alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + 1}{\beta_4 s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s} \quad (4)$$

so that the conditions are in terms of the coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$. Hence, we have

$$\alpha_3 = R_2 R_3 - R_6^2, \quad \alpha_2 = R_3, \quad \alpha_1 = R_2, \quad \beta_4 = \det(R),$$
$$\beta_3 = R_1 R_3 - R_5^2, \quad \beta_2 = R_1 R_2 - R_4^2, \quad \beta_1 = R_1. \quad (5)$$

We need the following three lemmas to establish Theorem 3.1. Besides, to simplify the expressions, we define the following terms: $W := 2\alpha_1 \alpha_2 \beta_1 + \beta_4 - \alpha_1 \beta_3 - \alpha_2 \beta_2 - \alpha_3 \beta_1$, $W_1 := \alpha_1 \alpha_2 - \alpha_3$, $W_2 := \alpha_2 \beta_1 - \beta_3$, and $W_3 := \alpha_1 \beta_1 - \beta_2$.

Lemma 3.2: Consider any function $Y(s)$ in the form of (4) where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$. $Y(s)$ can also be expressed as (2) with non-negative definite R defined in (3) and the entries of R defined in (5) if and only if $W_1, W_2, W_3 \geq 0$, and $W^2 = 4W_1 W_2 W_3$.

Proof: Sufficiency. Let $R_2 = \alpha_1$, $R_3 = \alpha_2$, and $R_1 = \beta_1$. Since $W_1, W_2, W_3 \geq 0$, then by introducing the variables R_4, R_5, and R_6 we can obtain $\alpha_3 = R_2 R_3 - R_6^2$, $\beta_3 = R_1 R_3 - R_5^2$, and $\beta_2 = R_1 R_2 - R_4^2$, further implying that $R_4 R_5 R_6 = \pm\sqrt{W_1 W_2 W_3}$. By properly assigning the signs of R_4, R_5, and R_6, and because of $W = \pm 2\sqrt{W_1 W_2 W_3}$, we can always guarantee that $W = 2R_4 R_5 R_6$, which can yield $\beta_4 = \det(R)$. Now we obtain all the equations in (5). Hence, we can express (4) as (2). Furthermore, it can be verified that R as defined in (3) is non-negative definite.

Necessity: By (5), it can be calculated that $W_1 = R_6^2 \geq 0, W_2 = R_5^2 \geq 0, W_3 = R_4^2 \geq 0$, and $W^2 = 4W_1W_2W_3 = 4R_4^2R_5^2R_6^2$. Thus, the lemma is proven. ∎

Lemma 3.3: Consider a non-negative definite matrix R in the form of (3), and the variables $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, and β_4 as defined in (5). Then, there exists at least one of the first- or second-order minors of R being zero if and only if at least one of the variables $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, W_1, W_2, W_3, (\beta_1 - W/(2W_1)), (\alpha_1 - W/(2W_2))$ and $(\alpha_2 - W/(2W_3))$ is zero.

Proof: Since the variables $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, and β_4 satisfy (5), we have $R_6^2 = W_1, R_5^2 = W_2, R_4^2 = W_3$, and $2R_4R_5R_6 = W$. Consequently, we obtain $R_1 - R_4R_5/R_6 = R_1 - (2R_4R_5R_6)/(2R_6^2) = \beta_1 - W/(2W_1)$. Similarly, $R_2 - R_4R_6/R_5 = \alpha_1 - W/(2W_2)$ and $R_3 - R_5R_6/R_4 = \alpha_2 - W/(2W_3)$. Hence, making use of the above relationship between the entries of R and the coefficients, this lemma can be proven. ∎

It is known from [6, Lemma 3] that for any matrix R satisfying the above lemma, there always exists an invertible matrix $D = \text{diag}\{1, x, y\}$ such that DRD is paramount.

Lemma 3.4: Consider a non-negative definite matrix R as defined in (3), whose first- and second-order minors are all non-zero, and the variables $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, and β_4 as defined in (5). Then, there exists an invertible $D = \text{diag}\{1, x, y\}$ such that DRD is a paramount matrix if and only if one of the following conditions holds:
1) $W < 0$;
2) $W > 0, \beta_1 > (W/(2W_1)), \alpha_1 > (W/(2W_2)), \alpha_2 > (W/(2W_3))$;
3) $W > 0, \alpha_2 < (W/(2W_3)), \beta_4 + \alpha_1\beta_3 + \alpha_3\beta_1 - \alpha_2\beta_2 \geq 0$;
4) $W > 0, \alpha_1 < (W/(2W_2)), \beta_4 + \alpha_2\beta_2 + \alpha_3\beta_1 - \alpha_1\beta_3 \geq 0$;
5) $W > 0, \beta_1 < (W/(2W_1)), \beta_4 + \alpha_1\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_1 \geq 0$.

Proof: The proof is relatively straightforward by showing the equivalence between the condition of [6, Lemma 4] and the condition of this lemma and therefore omitted. ∎

Therefore, the following theorem is obtained, which is equivalent to Lemma 3.1.

Theorem 3.1: A positive-real function $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 1, where X has a well-defined impedance and consists of only springs, if and only if $Y(s)$ can be written in the form of (4), where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0, W_1, W_2, W_3 \geq 0$, and $W^2 = 4W_1W_2W_3$, and the coefficients further satisfy the condition of either Lemma 3.3 or Lemma 3.4.

Proof: Sufficiency. By Lemma 3.2, $Y(s)$ can also be expressed as (2), where R is defined in (3) and is non-negative definite, and the non-negative coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, and β_4 satisfy (5). Furthermore, by [6], if at least one of the first- or second-order minors of R is zero, then there must exist an invertible $D = \text{diag}\{1, x, y\}$ such that DRD is a paramount matrix if the condition of Lemma 3.3 is also satisfied. Finally, by Lemma 3.1, the sufficiency part is proven.

Necessity: By Lemma 3.1, $Y(s)$ can be written in the form of (2) where R as defined in (3) is non-negative definite. Then, it is obvious that $Y(s)$ can also be expressed as (4) with the coefficients defined in (5), which indicates that $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0, W_1, W_2, W_3 \geq 0$, and $W^2 = 4W_1W_2W_3$. Then, we can conclude that the condition of either Lemma 3.3 or Lemma 3.4 is satisfied. ∎

Now we are focusing on the realization of admittance (1). First, it is necessary to show the positive-realness of admittance (1).

Lemma 3.5: [7], [19] A real-rational function $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1 \geq 0$ and $k > 0$ is positive-real if and only if $a_0d_1 - a_1d_0 \geq 0, a_0 - d_0 \geq 0$, and $a_1 - d_1 \geq 0$.

As defined in Section II, $R_k = (a_0 - d_0)^2 - (a_0d_1 - a_1d_0)(a_1 - d_1)$ is the resultant between the numerator and denominator of $Y(s)$. It is

known that $R_k = 0$ if and only if a positive-real $Y(s)$ in the form of (1) can be written in the form

$$Y(s) = k\frac{as + 1}{s(ds + 1)} \tag{6}$$

where $a \geq 0, d \geq 0$, and $a - d \geq 0$ [9]. Therefore, we have the following theorem.

Theorem 3.2: Consider a positive-real function (1), where $a_0, a_1, d_0, d_1 \geq 0$ and $k > 0$. If $R_k := (a_0 - d_0)^2 - (a_0d_1 - a_1d_0)(a_1 - d_1) = 0$, then it can be realized with at most one damper and two springs.

Proof: From the discussion above, it is known that if $R_k = 0$, then $Y(s)$ can be written as (6), where $a \geq 0, d \geq 0$, and $a - d \geq 0$. Furthermore, it is obvious that $Y(s) = k/s + k(a - d)/(ds + 1)$, which is a realization of at most one damper and two springs. ∎

From Theorem 3.2, we see that in order to investigate the realizability conditions of (1), it is only necessary to consider the case when $R_k \neq 0$. Then, the next theorem presents the realizability condition for admittance (1) to be realized in the form of Fig. 1.

Theorem 3.3: Consider a positive-real function

$$Y(s) = k\frac{a_0s^2 + a_1s + 1}{s(d_0s^2 + d_1s + 1)}$$

where $a_0, a_1, d_0, d_1 \geq 0, k > 0$ and $R_k := (a_0 - d_0)^2 - (a_0d_1 - a_1d_0)(a_1 - d_1) \neq 0$. It can be realized as the driving-point admittance of a network in the form of Fig. 1, where X has a well-defined impedance and consists of only springs, if and only if

$$\frac{d_0^2}{(a_0d_1 - a_1d_0)(a_1 - d_1)} \geq 1 \tag{7}$$

or

$$a_0d_1 - a_1d_0 = 0. \tag{8}$$

Proof: Necessity. By Theorem 3.1, it is known that $Y(s)$ in the form of (1) with $R_k \neq 0$ can be expressed as (4) with $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$. Thus, there are two possible cases.

For the first case, the coefficients in (4) can be regarded as follows:

$$\alpha_3 = 0, \; \alpha_2 = a_0, \; \alpha_1 = a_1, \; \beta_4 = 0, \; \beta_3 = \frac{d_0}{k}, \; \beta_2 = \frac{d_1}{k}, \; \beta_1 = \frac{1}{k} \tag{9}$$

which are all non-negative. Furthermore, $W^2 = 4W_1W_2W_3$ yields (8).

For the second case, multiplying the common factor $(Ts + 1)$, the coefficients in (4) can be regarded as

$$\alpha_3 = a_0T, \; \alpha_2 = a_0 + a_1T, \; \alpha_1 = a_1 + T, \; \beta_4 = \frac{d_0T}{k},$$
$$\beta_3 = \frac{(d_0 + d_1T)}{k}, \; \beta_2 = \frac{(d_1 + T)}{k}, \; \beta_1 = \frac{1}{k}, \; T > 0. \tag{10}$$

Moreover, $W^2 = 4W_1W_2W_3$ yields $(a_1 - d_1)T^2 - (a_0d_1 - a_1d_0) = 0$. If $a_1 = d_1$, then we have $a_0d_1 = a_1d_0$, which is the condition derived in the first case. If $a_1 \neq d_1$, then $a_0d_1 \neq a_1d_0$ must hold, which reduces the positive-realness condition to $a_0d_1 - a_1d_0 > 0$, $a_0 - d_0 \geq 0$, and $a_1 - d_1 > 0$, indicating that $a_0, a_1, d_1 > 0$ and

$d_0 \geq 0$. Hence $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 > 0$, and $\beta_4 \geq 0$. Besides, T can be expressed as

$$T = \sqrt{\frac{a_0 d_1 - a_1 d_0}{a_1 - d_1}}. \tag{11}$$

It can be calculated that $W_1 = a_1 T^2 + a_1^2 T + a_0 a_1 > 0$, $W_2 = ((a_1 - d_1)T + (a_0 - d_0))/k > 0, W_3 = (a_1 - d_1)/k > 0, \beta_1 - W/(2W_1) = ((a_1 + d_1)T^2 + 2a_1 d_1 T + (a_0 d_1 + a_1 d_0))/(2ka_1(T^2 + a_1 T + a_0)) > 0, \alpha_1 - W/(2W_2) = ((a_1 - d_1)T^2 + 2(a_0 - d_0)T + (a_0 d_1 - a_1 d_0))/(2((a_1 - d_1)T + (a_0 - d_0))) > 0, \alpha_2 - W/(2W_3) = (a_1 d_0 - a_0 d_1)/(a_1 - d_1) < 0$. Thus, the condition of Lemma 3.3 cannot be satisfied, which indicates that the condition of Lemma 3.4 must hold. We see that $W = 2a_1(\sqrt{(a_0 d_1 - a_1 d_0)(a_1 - d_1)} + (a_0 - d_0))/k > 0$. Consequently, we conclude that only the third condition of Lemma 3.4 holds. Thus, it follows that $\beta_4 + \alpha_1 \beta_3 + \alpha_3 \beta_1 - \alpha_2 \beta_2 \geq 0$. Since the equation $\beta_4 + \alpha_1 \beta_3 + \alpha_3 \beta_1 - \alpha_2 \beta_2 = 2\sqrt{a_0 d_1 - a_1 d_0}(d_0 - \sqrt{(a_0 d_1 - a_1 d_0)(a_1 - d_1)})/(k\sqrt{a_1 - d_1})$ holds, we can obtain (7).

Sufficiency: When (8) is satisfied, express $Y(s)$ in the form of (4) with the coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$ satisfying (9), indicating that they are all non-negative and the condition of Lemma 3.3 must be satisfied. The positive-realness of $Y(s)$ guarantees that $W_1, W_2, W_3 \geq 0$, and $W^2 = 4W_1 W_2 W_3$ holds because of $a_0 d_1 - a_1 d_0 = 0$.

When (7) is satisfied, combining with the positive-realness of $Y(s)$, we conclude that $a_0, a_1, d_0, d_1 > 0$. Multiply the numerator and the denominator of (1) by a common factor $(Ts+1)$ simultaneously, where T is defined in (11), then $Y(s)$ can be expressed as (4) with the coefficients satisfying (10), which are all positive. From the necessity part of the proof, the condition of Lemma 3.2 and the third condition of Lemma 3.4 hold. Thus, by Theorem 3.1, $Y(s)$ can be realized by the required network of this theorem. ∎

We now provide explicit network constructions that will satisfy the realizability conditions. We treat the two conditions (7) and (8) of Theorem 3.3 in Theorems 3.4 and 3.5, respectively.

Theorem 3.4: Consider a positive-real function $Y(s)$ in the form of (1) where $a_0, a_1, d_0, d_1 \geq 0, k > 0$, and $R_k := (a_0 - d_0)^2 - (a_0 d_1 - a_1 d_0)(a_1 - d_1) \neq 0$. If condition (7) holds, then $Y(s)$ can be realized as in Fig. 2 with

$$k_1 = \frac{ka_0 d_1(a_0 d_1 - a_1 d_0)[(a_1 - d_1)T + (a_0 - d_0)]}{d_0(a_1 - d_1)(d_1 T + d_0)[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]},$$

$$k_2 = \frac{ka_0 a_1 d_1 T[(a_1 - d_1)T + (a_0 - d_0)]^2}{(a_1 - d_1)(d_1 T + d_0)[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]^2},$$

$$k_3 = \frac{ka_0 T[(a_1 - d_1)T + (a_0 - d_0)]}{(d_1 T + d_0)[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]},$$

$$k_4 = \frac{ka_0 T[d_0 T + (a_1 d_0 - a_0 d_1)][(a_1 - d_1)T + (a_0 - d_0)]}{(d_1 T + d_0)[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]^2},$$

$$b = \frac{ka_0^2(a_0 d_1 - a_1 d_0)[(a_1 - d_1)T + (a_0 - d_0)]}{(a_1 - d_1)[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]^2},$$

$$c = \frac{ka_0^2 d_1^2(a_0 d_1 - a_1 d_0)[(a_1 - d_1)T + (a_0 - d_0)]^2}{(a_1 - d_1)^2(d_1 T + d_0)^2[(a_0 - d_0)T + (a_0 d_1 - a_1 d_0)]^2}, \tag{12}$$

where T is defined in (11).

Proof: This theorem can be proven by [6, Lemma 4, Theorem 7] and the sufficiency part of Theorem 3.3. ∎

Theorem 3.5: Consider a positive-real function $Y(s)$ in the form of (1) where $a_0, a_1, d_0, d_1 \geq 0, k > 0$, and $R_k := (a_0 - d_0)^2 - (a_0 d_1 - a_1 d_0)(a_1 - d_1) \neq 0$. If condition (8) holds, then $Y(s)$ can be realized
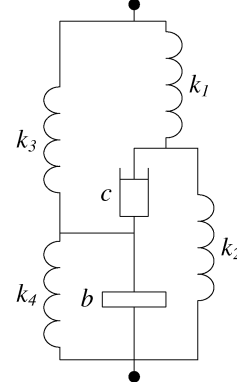

Fig. 2. Network realization of Theorem 3.4.

with at most one inerter, one damper, and two springs (values given in the proof).

Proof: Since $a_0 d_1 = a_1 d_0$, it is implied by $R_k \neq 0$ that $a_0 > d_0$. If $a_0 d_1 = a_1 d_0$ and $a_0 > d_0 = 0$, then $d_1 = 0$. Therefore, $Y(s) = ka_0 s + ka_1 + k/s$, which comprises one spring, one inerter and at most one damper with the values being $k_1 = k > 0, b = ka_0 > 0$, and $c = ka_1 \geq 0$, respectively.

If $a_0 d_1 = a_1 d_0$ and $a_0 > d_0 > 0$, then we have $Y(s) = k/s + 1/(d_0 s/(k(a_0 - d_0)) + 1/(k(a_0 - d_0)s + k(a_1 - d_1)))$, which comprises two springs, one inerter, and at most one damper with the values being $k_1 = k > 0, k_2 = k(a_0 - d_0)/d_0 > 0, b = k(a_0 - d_0) > 0$, and $c = k(a_1 - d_1) \geq 0$, respectively. ∎

IV. FINAL REALIZABILITY CONDITIONS

The results presented in previous sections are under the assumption that X in Fig. 1 has a well-defined impedance since the result in [6] is under this assumption as well as other results [5]. This section mainly investigates the realization problem of $Y(s)$ when this constraint is removed in order to solve the realization problem of $Y(s)$ completely.

For any n-port network N_n without levers (transformers), we can formulate a graph $\mathcal{G}$ of N_n named *augmented graph* by letting each element or each port correspond to an edge of a graph, and each velocity node correspond to a vertex of the graph. Define the graph consisting of all the edges corresponding to ports as *port graph* $\mathcal{G}_p$. $\mathcal{G}$ can always be regarded as being connected, for otherwise we can obtain one by letting one velocity node of a component be common with that of another. Basic notions of graph theory are referred to [17].

Lemma 4.1: Consider an n-port network N_n consisting of at most three kinds of elements, which are springs, dampers, and inerters. Then, the impedance of network N_n is not well-defined if and only if there exists a cut-set of $\mathcal{G}$ as a subgraph of $\mathcal{G}_p$.

Proof: Necessity. Assume that such cut-set does not exist. By [17, p. 33], $\mathcal{G}_p$ is made part of the complement of a tree of $\mathcal{G}$. Then, from [3], we see that $A_1 D A_1^T$ is nonsingular and the impedance is of the form $Z = A_2^T(A_1 D A_1^T)^{-1} A_2$, where $A = [A_1, A_2]$ is the reduced incidence matrix of $\mathcal{G}$ with the columns of A_1 corresponding to the elements and the columns of A_2 corresponding to the ports, and D is the positive definite diagonal matrix whose diagonal entries are the admittances of all the elements. This contradicts with the assumption.

Sufficiency: Assume that the impedance of N_n exists, then any $\hat{\mathbf{F}} = [\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n]^T$ vector is permitted. Since there is a cut-set of $\mathcal{G}$ as a subgraph of $\mathcal{G}_p$, a constraint on the entries of $\hat{\mathbf{F}} = [\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n]^T$ is immediately obtained [17]. This contradicts with the assumption. ∎
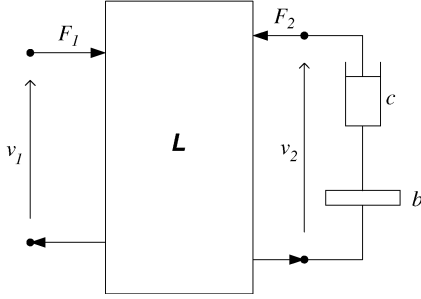
Fig. 3. General network discussed in Lemma 4.2, where $b, c > 0$, and L has a well-defined impedance and consists of only springs.

Lemma 4.2: Consider a positive-real function $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1 \geq 0$, and $k > 0$. $Y(s)$ can be realized as the admittance of a network N as shown in Fig. 1 with the impedance of X being not well-defined and its augmented graph $\mathcal{G}$ nonseparable, if and only if it is the admittance of the network as shown in Fig. 3, where L has a well-defined impedance, whose augmented graph is nonseparable, and consists of only springs.

Proof: Sufficiency. It is obvious that the network in Fig. 3 with the augmented graph of L nonseparable is a special case of Fig. 1. Besides, since it is obvious that the two edges of $\mathcal{G}_p$ corresponding to ports terminated with the damper and the inerter constitute a cut-set, by Lemma 4.1 the impedance of X is not well-defined.

Necessity: It is obvious that $Z(s) = Y^{-1}(s)$ has a zero at $s = 0$. By [18, Theorem 2], there must be a path (a, a') consisting of only springs, where a and a' are two terminals of N.

We will prove that the two edges of the port graph $\mathcal{G}_p$ of X corresponding to the ports terminated with the inerter and the damper constitute a cut-set of the augmented graph $\mathcal{G}$. By Lemma 4.1, we know that there must exist a cut-set among the three edges of $\mathcal{G}_p$. It is shown in [17, Theorem 3-3] that any nonseparable graph with at least two edges must be cyclically connected, which means that any two vertices in $\mathcal{G}$ can be placed in a circuit. Then, it implies that the number of edges of any cut-set of $\mathcal{G}$ is at least two. Therefore, it suffices to show that any such cut-set can never contain the edge e_p that corresponds to the only port of N whose two terminals are a and a'. Assume that there exists such cut-set $\mathcal{C}$ among those three edges containing the edge e_p, then by the property of the cut-set, $\mathcal{C}$ separates $\mathcal{G}$ into two parts, which results in terminals a and a' located in each part. Then, all the paths (a, a') of the network in Fig. 1 must contain a damper, or an inerter, or both, which contradicts with the conclusion stated at the beginning of the necessity part.

Then, by the property of cut-set and the analysis of all the possible cases, the network in Fig. 1 can always be equivalent to that in Fig. 3, where L contains only springs. Since it has been shown that any cut-set of $\mathcal{G}$ never contains e_p, we can prove that L must have a well-defined impedance because no cut-set is contained in its port graph. ∎

We now present a necessary and sufficient condition for any positive-real function to be the admittance of the network in Fig. 3 in Lemma 4.4 based on Lemma 4.3. The realization is then given in Lemma 4.5.

lemma 4.3: Consider any 2×2 symmetric matrix Q. If Q is non-negative definite, then there must exist an invertible matrix $D = \mathrm{diag}\{1, x\}$ with $x > 0$ such that DQD is paramount.

Proof: This lemma can be easily proven by the definition of paramountcy and the property of non-negative definiteness. ∎

Lemma 4.4: A positive-real function $Y(s)$ can be realized as the driving-point admittance of a network in the form of Fig. 3, where L

has a well-defined impedance and consists of only springs, if and only if $Y(s)$ can be written in the form of

$$Y(s) = \frac{\alpha_2 s^2 + \alpha_1 s + 1}{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s} \qquad (13)$$

where $\alpha_2, \beta_2, \beta_3 \geq 0$ and $\alpha_1, \beta_1 > 0$, and the coefficients satisfy $\alpha_1 \beta_1 - \beta_2 = 0$.

Proof: Necessity. The admittance of the network in Fig. 3 can be calculated as

$$\frac{\hat{F}_1}{\hat{v}_1} = \frac{\left(\frac{b}{c}\right) c P_2 s^2 + \left(\frac{b}{c}\right) s + 1}{\left(\frac{b}{c}\right)\left(c P_1 P_2 - c P_3^2\right) s^3 + \left(\frac{b}{c}\right) P_1 s^2 + P_1 s} \qquad (14)$$

where $b, c > 0$, and

$$P := \begin{bmatrix} P_1 & P_3 \\ P_3 & P_2 \end{bmatrix} \qquad (15)$$

is the impedance of L, which is paramount by [6]. Furthermore, (14) can be expressed as

$$Y(s) = \frac{m Q_2 s^2 + m s + 1}{m\left(Q_1 Q_2 - Q_3^2\right) s^3 + m Q_1 s^2 + Q_1 s} \qquad (16)$$

where $m = b/c$, and Q is obtained through

$$Q := \begin{bmatrix} Q_1 & Q_3 \\ Q_3 & Q_2 \end{bmatrix} = T \begin{bmatrix} P_1 & P_3 \\ P_3 & P_2 \end{bmatrix} T \qquad (17)$$

where $T = \mathrm{diag}\{1, \sqrt{c}\}$. Hence, Q must be non-negative definite. Letting $\alpha_1 = m$, $\alpha_2 = m Q_2$, $\beta_1 = Q_1$, $\beta_2 = m Q_1$, and $\beta_3 = m(Q_1 Q_2 - Q_3^2)$, we have $\alpha_2, \beta_1, \beta_2, \beta_3 \geq 0$, $\alpha_1 > 0$, and $\alpha_1 \beta_1 = \beta_2 = m Q_1$. By the positive-realness of Q, $\beta_1 = 0$ leads to $\beta_2 = \beta_3 = 0$, which implies the nonexistence of $Y(s)$. Then, we have $\beta_1 > 0$.

Sufficiency: It suffices to show that $Y(s)$ can be expressed as (14), where $b, c > 0$ and P as defined in (15) is paramount. Formulate a function in the form of (16), where $m = \alpha_1$, $Q_1 = \beta_1$, $Q_2 = \alpha_2/\alpha_1$, and $Q_3^2 = (\alpha_2 \beta_1 - \beta_3)/\alpha_1$, which must exist because the positive-realness of $Y(s)$ guarantees that $\alpha_2 \beta_1 - \beta_3 \geq 0$. Consequently, it is verified that Q as defined in (17) is non-negative definite. Furthermore, since $\alpha_1 \beta_1 - \beta_2 = 0$ holds, it can be calculated that $\alpha_1 = m$, $\alpha_2 = m Q_2$, $\beta_1 = Q_1$, $\beta_2 = m Q_1$, and $\beta_3 = m(Q_1 Q_2 - Q_3^2)$ hold, indicating that $Y(s)$ can also be expressed as (16) with $m > 0$, and Q as defined in (17) being non-negative definite. By Lemma 4.3, there must exist an invertible matrix $D = \mathrm{diag}\{1, x\}$, where $x > 0$ such that

$$P := \begin{bmatrix} P_1 & P_3 \\ P_3 & P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$

is paramount. Therefore, we have $Q_1 = P_1$, $Q_2 = P_2/x^2$, and $Q_3 = P_3/x$, making (16) become

$$Y(s) = \frac{m\left(\frac{1}{x^2}\right) P_2 s^2 + m s + 1}{m\left(\left(\frac{1}{x^2}\right) P_1 P_2 - \left(\frac{1}{x^2}\right) P_3^2\right) s^3 + m P_1 s^2 + P_1 s}.$$

Letting $c = 1/x^2$ and $b = mc$ gives (14), where $b, c > 0$ and P is paramount. ∎
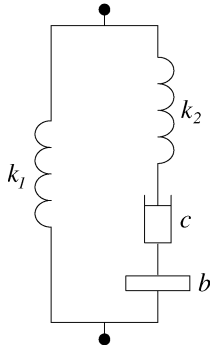
Fig. 4.   Network with X not well-defined when expressed in the form of Fig. 1.

Lemma 4.5: Consider a positive-real function $Y(s)$ in the form of (1) with $a_0, a_1, d_0, d_1 \geq 0$, $k > 0$, and $R_k \neq 0$. $Y(s)$ can be realized as the driving-point admittance of a network as shown in Fig. 3, where L has a well-defined impedance and consists of only springs, if and only if $a_1 = d_1 > 0$. Moreover, if the condition holds, then it can be realized by the network shown in Fig. 4 with the values of the elements given below

$$k_1 = k, \; k_2 = \frac{(a_0 - d_0)k}{d_0}, \; b = (a_0 - d_0)k, \; c = \frac{(a_0 - d_0)k}{a_1}. \quad (18)$$

Proof: The condition can be easily derived from Lemma 4.4. Furthermore, the admittance of the network shown in Fig. 4 is calculated to be $Y(s) = (bc(k_1 + k_2)s^2 + bk_1k_2s + ck_1k_2)/(s(bcs^2 + bk_2s + ck_2))$. Substituting (18), whose values are non-negative, into the above equation results in (1) because of $a_1 = d_1 > 0$. Therefore, this lemma is proven.   ∎

Now, the final condition is presented in Theorem 4.1.

Theorem 4.1: Consider a positive-real function $Y(s)$ in the form of (1) where $a_0, a_1, d_0, d_1 \geq 0$, $k > 0$, and $R_k := (a_0 - d_0)^2 - (a_1 d_0 - a_1 d_0)(a_1 - d_1) \neq 0$. It can be realized as the driving-point admittance of a network consisting of one inerter, one damper, and an arbitrary number of springs, that is, the network shown in Fig. 1, if and only if

$$(a_0 d_1 - a_1 d_0)(a_1 - d_1) - d_0^2 \leq 0. \quad (19)$$

Proof: Necessity. When X has a well-defined impedance, then by Theorem 3.3 it implies (7) or (8), that is, $d_0^2/[(a_0 d_1 - a_1 d_0)(a_1 - d_1)] \geq 1$ or $a_0 d_1 - a_1 d_0 = 0$ holds. When X does not has a well-defined impedance and its augmented graph $\mathcal{G}$ is nonseparable, then by Lemma 4.5 we have $a_1 = d_1 > 0$. When X does not has a well-defined impedance and its augmented graph $\mathcal{G}$ is separable, we can always obtain an equivalent network N' in the form of Fig. 1, which satisfies either of the above two cases. Combining the above conditions, we obtain (19).

Sufficiency: Since condition (19) holds, it implies that at least one of the conditions of Theorem 3.3 and Lemma 4.5 must hold. Then, admittance (1) can be realized by the required network by Theorem 3.4, Theorem 3.5, and Lemma 4.5.   ∎

V. Conclusion

This note has studied a realization problem for one special class of admittances, which is widely used in passive suspension design. The number of inerters and dampers is restricted to one in each case and the number of the springs is arbitrary. To solve the problem, we first converted a previous result by Chen and Smith [6] into a more direct form. Then, a necessary and sufficient condition for realizability with the assumption that the three-port network X consisting of only springs has a well-defined impedance was derived. Furthermore, explicit circuit arrangements were provided to cover the realizability conditions. Furthermore, a relationship between the topological property of the n-port network and the fact that its impedance is not well-defined was provided. Consequently, considering the property of this class of admittance and the above relationship, we obtained an equivalent general network when the impedance of X is not well-defined through the use of graph theory. Then, a necessary and sufficient condition for the realizability of this kind of networks was derived and a network construction covering this condition was presented. Finally, combining this condition with the previous one yielded the final realization without any assumption on the existence of a well-defined impedance.

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