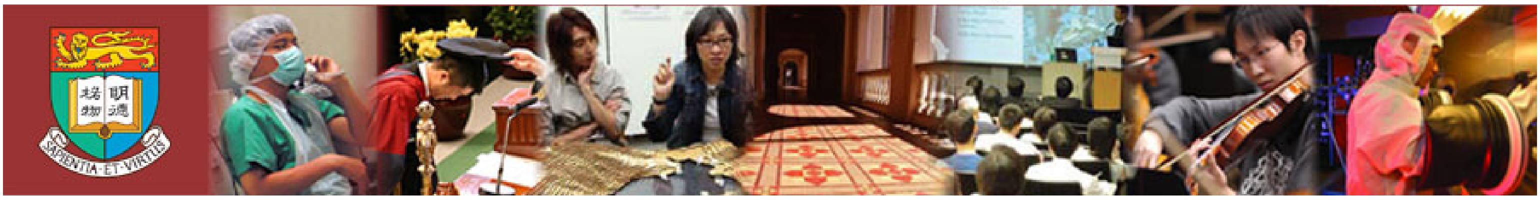

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A UNIFIED "BANG-BANG" PRINCIPLE WITH RESPECT TO $\mathcal{R}$-INVARIANT PERFORMANCE BENCHMARKS*

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Abstract. In recent years, there has been a number of works on finding the optimal selling time of a stock so that the expected ratio of its selling price to a certain benchmark (e.g., its ultimate highest price) over a finite time horizon is maximized. Although formulated in different settings, they all result in a "bang-bang"-type optimal solution, as was originally discovered by Shiryaev, Xu, and Zhou [*Quant. Finance*, 8 (2008), pp. 765–776], which can literally be interpreted as "buy-and-hold" or "sell-at-once" depending on the quality of the stock. In this paper, we first provide three algebraic conditions on a class of benchmarks and call any benchmark satisfying the three conditions an $\mathcal{R}$-invariant performance benchmark. We show that if F is an $\mathcal{R}$-invariant performance benchmark, then the corresponding optimal stopping problem has a "bang-bang"-type optimal solution. Our work here provides a unified proof of all similar problems for Brownian motion considered in the existing literature and also implies new results; in particular, we solve the remaining part (which has not been covered in the literature) of a problem originally formulated by Shiryaev [*Mathematical Finance—Bachelier Congress* (Paris, 2000), Springer, Berlin, 2002, pp. 487–521].

Key words. optimal stopping, buy-and-hold or sell-at-once rule, $\mathcal{R}$-invariant performance benchmarks

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1. Introduction.

In recent years, a considerable amount of work has been devoted to the determination of an optimal stopping time of a drifted Brownian motion beating nonanticipative benchmarks. For instance, motivated by the financial problem of selling a stock at a price "as close as possible" to its highest price over a finite time horizon, Shiryaev, Xu, and Zhou [12] and Du Toit and Peskir [8], using different probabilistic approaches, solved the optimal stopping problem

$$\sup_{\tau\leqq T}\mathbf{E}\left[\exp(-(M_T^{\lambda}-\omega_{\tau}^{\lambda}))\right] \tag{1.1}$$

for some $T>0$, $\omega_\cdot^{\lambda}\triangleq\lambda\cdot+\omega_\cdot$, with $\lambda\in\mathbf{R}$, and $\omega_\cdot$ being a continuous path in $C[0,T]$, $M_T^{\lambda}\triangleq\max_{0\leqq t\leqq T}\omega_t^{\lambda}$, and the expectation is taken with respect to the Wiener measure $\mathbf{P}$ under which $(\omega_t)_{t\geqq 0}$ is a Brownian motion. In (1.1) the supremum is taken over the set of all stopping times τ adapted to the process $(\omega_t)_{t\geqq 0}$ for which $\mathbf{P}(\tau\leqq T)=1$. To the best of our knowledge, the above problem was first formulated in the present form by Shiryaev at the first Bachelier Congress in 2000 (see [11]). In his recent work, Allaart [2] considered the extended problem

$$\sup_{\tau\leqq T}\mathbf{E}\left[f(M_T^{\lambda}-\omega_{\tau}^{\lambda})\right] \tag{1.2}$$

in which f is a nonincreasing and convex function. In both problems (1.1) and (1.2), the ultimate maximum M_T^{λ} can be regarded as a (nonanticipative) benchmark, and one wants to beat this benchmark by stopping a Brownian motion "as close as possible" to it. From a

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PDE point of view, Dai et al. [4] also solved problem (1.1) and considered a similar problem that uses the ultimate minimum of the drifted Brownian motion as the benchmark:

$$\sup_{\tau \leqq T} \mathbf{E}\left[\exp(-(m_T^\lambda - \omega_\tau^\lambda))\right], \tag{1.3}$$

where $m_T^\lambda = \min_{0 \leqq t \leqq T} \omega_t^\lambda$. In [4], problem (1.3) is interpreted as an optimal stock selling problem in which the investor attempts to sell his stock at a price "as far as possible" to its lowest price over a finite time horizon. Using a similar approach, Dai and Zhong [5] also studied the problem

$$\sup_{\tau \leqq T} \mathbf{E}\left[\exp(-(A_T^\lambda - \omega_\tau^\lambda))\right], \tag{1.4}$$

where $A_T^\lambda = \log(T^{-1}\int_0^T \exp(\omega_t^\lambda)\,dt)$ or $A_T^\lambda = T^{-1}\int_0^T \omega_t^\lambda\,dt$; i.e., in (1.4) the benchmark is taken to be the ultimate average (both arithmetic and geometric) of the drifted Brownian motion. While all benchmarks in problems (1.1)–(1.4) are different, a striking result is that they share a common optimal stopping rule which is of "bang-bang" type: if $\lambda \geqq 0$, then $\tau^* = T$ is an optimal stopping time; while if $\lambda \leqq 0$, then $\tau^* = 0$ is an optimal stopping time. Motivated by these results, in our present work, we establish a general form of the benchmark (which we shall call an $\mathcal{R}$-invariant performance benchmark) with respect to which the optimal stopping rule is still of "bang-bang" type; indeed, our present work solves all four problems mentioned above in a unified way and also implies new results which have not yet appeared in the literature.

More precisely, define $D[0,T]$ to be the space of all piecewise continuous functions γ on $[0,T]$, with at most finitely many jump points $\{t_i\}$ such that for each i

$$\gamma_{t_i} = \lim_{t \downarrow t_i} \gamma_t \quad \text{or} \quad \lim_{t \uparrow t_i} \gamma_t.$$

The reason that we introduce the notion of $D[0,T]$ in addition to $C[0,T]$ is that the permutation specified in Definition 2.1 below may convert an element in $C[0,T]$ to one that can only be defined in $D[0,T]$; in other words, the functional F defined in section 2, in general, maps continuous paths to discontinuous ones. Also let $f\colon \mathbf{R} \to \mathbf{R}$ be a monotone and convex function, $F\colon D[0,T] \to \mathbf{R}$ an $\mathcal{R}$-invariant performance benchmark (see Definition 2.3 in what follows), and then the solution to the optimal stopping problem

$$\sup_{\tau \leqq T} \mathbf{E}\left[f(F(\omega_\cdot^\lambda) - \omega_\tau^\lambda)\right] \tag{1.5}$$

is of "bang-bang" type. That is to say, (i) if f is nonincreasing and convex, then an optimal stopping time is $\tau^* = T$ for $\lambda \geqq 0$ and $\tau^* = 0$ for $\lambda \leqq 0$; while (ii) if f is nondecreasing and convex, then an optimal stopping time is $\tau^* = 0$ for $\lambda \geqq 0$ and $\tau^* = T$ for $\lambda \leqq 0$.

2. Optimal stopping with respect to $\mathcal{R}$-invariant performance benchmarks. In this section, we shall specify conditions on the Wiener functional F so that the solution to problem (1.5) is still of "bang-bang" type. For $t \leqq T$, $y \in D[0,t]$, we shall write y^{-1} to denote the time reversed path of y on $[0,t]$, i.e., $y^{-1}(s) = y(t-s)$;[1] we shall also write y^1 to denote itself. For each $n \in \mathbf{N}$, define

$$\Delta_n \triangleq \left\{(l_1,\ldots,l_n) \in \mathbf{R}^n\colon \sum_{k=1}^n l_k = T\right\}, \tag{2.1}$$

$$\mathcal{H}_n \triangleq \{(y_1,\ldots,y_n)\colon y_k \in D[0,l_k] \text{ for some } (l_1,\ldots,l_n) \in \Delta_n\}. \tag{2.2}$$

[1] Although the time reversed path y^{-1} of y depends on the domain on which the path is reversed, for simplicity of notation, we shall not express this dependence explicitly. Instead, we will always first define the natural domain of y, and then y^{-1} will refer to the time reversed path of y on its natural domain.

And for each $n \in \mathbf{N}$ and $(y_1, \dots, y_n) \in \mathcal{H}_n$, we define

$$(2.3)\quad (y_1 y_2 \cdots y_n)(s) \triangleq \begin{cases} y_1(s) & \text{for } s \in [0, l_1), \\ y_k\Big(s - \sum_{i=1}^{k-1} l_i\Big) & \text{for } s \in \Big[\sum_{i=1}^{k-1} l_i, \sum_{i=1}^{k} l_i\Big),\ k = 2, \dots, n-1, \\ y_n\Big(s - \sum_{i=1}^{n-1} l_i\Big) & \text{for } s \in \Big[\sum_{i=1}^{n-1} l_i, \sum_{i=1}^{n} l_i\Big]. \end{cases}$$

DEFINITION 2.1. *Given a path $y \in D[0,T]$, if there exist $n \in \mathbf{N}$, $(y_1, \dots, y_n) \in \mathcal{H}_n$, a permutation $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$ and $\{p_1, \dots, p_n\} \in \{-1, 1\}^n$ such that for all $s \in [0,T]$, $y(s) = (y_{i_1}^{p_1} y_{i_2}^{p_2} \cdots y_{i_n}^{p_n})(s)$, then we say that $(y_1, \dots, y_n)$ is a representative of y.*

The above definition induces an equivalence relation $\mathcal{R}$ on $D[0,T]$. More precisely, for $y, z \in D[0,T]$, we say y is $\mathcal{R}$-equivalent to z if y and z have a common representative.

DEFINITION 2.2. *A functional $F\colon D[0,T] \to \mathbf{R}$ is called $\mathcal{R}$-invariant if $F(y) = F(z)$ whenever y is $\mathcal{R}$-equivalent to z.*

The following lemma provides a convenient criterion to check whether a Wiener functional is $\mathcal{R}$-invariant or not. For each $t \in [0,T]$, define $F_t\colon D[0,t] \times D[0,T-t] \to \mathbf{R}$ by

$$(2.4)\qquad F_t(y_1, y_2) = F(y_1 y_2),$$

where $y_1 \in D[0,t]$, $y_2 \in D[0,T-t]$, and $y_1 y_2$ is as defined in (2.3).

LEMMA 2.1. *A functional $F\colon D[0,T] \to \mathbf{R}$ is $\mathcal{R}$-invariant if and only if $F_t(y_1, y_2) = F_t(y_1, y_2^{-1})$ for each t, $y_1 \in D[0,t]$, and $y_2 \in D[0,T-t]$.*

Proof. First suppose F is $\mathcal{R}$-invariant. Since the path $y_1 y_2^{-1}$ is $\mathcal{R}$-equivalent to $y_1 y_2$, we have $F(y_1 y_2^{-1}) = F(y_1 y_2)$. Now suppose that for each t, $y_1 \in D[0,t]$, and $y_2 \in D[0,T-t]$, we have $F_t(y_1, y_2) = F_t(y_1, y_2^{-1})$, i.e., $F(y_1 y_2) = F(y_1 y_2^{-1})$, then clearly $F(y) = F(y^{-1})$ for any $y \in D[0,T]$. Upon noting that the time reversal process of $y_1 y_2^{-1}$ is $y_2 y_1^{-1}$, we have

$$(2.5)\qquad F(y_1 y_2) = F(y_1 y_2^{-1}) = F(y_2 y_1^{-1}) = F(y_2 y_1),$$

and hence

$$(2.6)\qquad F(y_1 y_2) = F(y_2 y_1^{-1}) = F(y_1^{-1} y_2).$$

To prove F is $\mathcal{R}$-invariant, it suffices to show that for any integer n and $(y_1, \dots, y_n) \in \mathcal{H}_n$, we have

$$(2.7)\qquad F(y_1 \cdots y_k \cdots y_n) = F(y_1 \cdots y_k^{-1} \cdots y_n) \quad \text{for each } k$$

and

$$(2.8)\qquad F(y_1 \cdots y_i y_{i+1} \cdots y_n) = F(y_1 \cdots y_{i+1} y_i \cdots y_n) \quad \text{for each } i.$$

We have already proved (2.7) and (2.8) for the case $n = 2$. For general n, by applying our result for the case $n = 2$, we obtain

$$\begin{aligned} F(y_1 \cdots y_{k-1} y_k \cdots y_n) &= F(y_k \cdots y_n y_1 \cdots y_{k-1}) \\ &= F(y_k^{-1} \cdots y_n y_1 \cdots y_{n-1}) = F(y_1 \cdots y_{n-1} y_k^{-1} \cdots y_n), \end{aligned}$$

so the equality (2.7) is proved. To prove (2.8), first note that if $i = 1$, then by applying (2.5), $F(y_1 y_2 \cdots y_n) = F(y_3 \cdots y_n y_1 y_2)$. Then applying (2.6), $F(y_3 \cdots y_n y_1 y_2) = F((y_3 \cdots y_n)^{-1} y_1 y_2) = F(y_n^{-1} \cdots y_3^{-1} y_1 y_2)$. Further, using (2.7) twice, we see that $F(y_n^{-1} \cdots y_3^{-1} y_1 y_2) = F(y_n^{-1} \cdots y_3^{-1} y_1^{-1} y_2^{-1})$. Finally

$$F(y_n^{-1} \cdots y_3^{-1} y_1^{-1} y_2^{-1}) = F((y_n^{-1} \cdots y_3^{-1} y_1^{-1} y_2^{-1})^{-1}) = F(y_2 y_1 y_3 \cdots y_n).$$

So (2.8) is proved when $i = 1$. For general i

$$\begin{aligned} F(y_1 \cdots y_i y_{i+1} \cdots y_n) &= F(y_i y_{i+1} \cdots y_n y_1 \cdots y_{i-1}) \\ &= F(y_{i+1} y_i \cdots y_n y_1 \cdots y_{i-1}) = F(y_1 \cdots y_{i-1} y_i \cdots y_n), \end{aligned}$$

so the equality (2.8) is also proved.

In his work on problem (1.2), Allaart [2] remarked that the following identity is the key to establishing his result: for each $t \in [0, T]$,

$$(M_t^\lambda - \omega_t^\lambda, \omega_t^\lambda) \overset{d}{=} (M_t^{-\lambda}, -\omega_t^{-\lambda}), \tag{2.9}$$

where $\overset{d}{=}$ means that distributions of two sides coincide under the Wiener measure $\mathbf{P}$. The following lemma extends identity (2.9) from a maximal functional to any $\mathcal{R}$-invariant functional F.

Lemma 2.2. *Let $t \in [0, T]$, $y \in D[0, t]$, $\omega \in C[0, T - t]$. If $F\colon D[0, T] \to \mathbf{R}$ is $\mathcal{R}$-invariant, then for any $\lambda \in \mathbf{R}$*

$$(F_t(y,\ \omega_\cdot^\lambda - \omega_{T-t}^\lambda), \omega_{T-t}^\lambda) \overset{d}{=} (F_t(y,\ \omega_\cdot^{-\lambda}), -\omega_{T-t}^{-\lambda}), \tag{2.10}$$

where $\omega^\lambda = (\lambda s + \omega_s)_{0 \leqq s \leqq T-t}$.

Proof. Write $\upsilon_s = \omega_s^\lambda - \omega_{T-t}^\lambda$ for $s \in [0, T - t]$; then

$$(\upsilon_s^{-1})_{0 \leqq s \leqq T-t} = (\omega_{T-t-s}^\lambda - \omega_{T-t}^\lambda)_{0 \leqq s \leqq T-t} \overset{d}{=} (\omega_s^{-\lambda})_{0 \leqq s \leqq T-t},$$

which in the case $s = T - t$ gives $\omega_{T-t}^\lambda \overset{d}{=} -\omega_{T-t}^{-\lambda}$. Therefore,

$$(\omega_{T-t-s}^\lambda - \omega_{T-t}^\lambda, \omega_{T-t}^\lambda)_{0 \leqq s \leqq T-t} \overset{d}{=} (\omega_s^{-\lambda}, -\omega_{T-t}^{-\lambda})_{0 \leqq s \leqq T-t}. \tag{2.11}$$

Observe that

$$\begin{aligned} (F_t(y, \omega_\cdot^\lambda - \omega_{T-t}^\lambda), \omega_{T-t}^\lambda) &= (F_t(y, \upsilon_\cdot), \omega_{T-t}^\lambda) = (F_t(y, \upsilon_\cdot^{-1}), \omega_{T-t}^\lambda) \\ &= (F_t(y, \omega_{T-t-\cdot}^\lambda - \omega_{T-t}^\lambda), \omega_{T-t}^\lambda), \end{aligned} \tag{2.12}$$

where in the second equality the $\mathcal{R}$-invariance of F and Lemma 2.1 have been applied. Combining (2.11) and (2.12), we obtain (2.10). Lemma 2.2 is proved.

Definition 2.3. *A functional $F\colon D[0, T] \to \mathbf{R}$ is called a performance benchmark if it satisfies the following two conditions*:

(C1) *F is translation invariant in the sense that for any $c \in \mathbf{R}$ and $y \in D[0, T]$, we have $F(y + c) = F(y) + c$*;

(C2) *F is nondecreasing in the sense that for any two paths $y_1, y_2 \in D[0, T]$ with $y_1(t) \geqq y_2(t)$ for all $t \in [0, T]$, we have $F(y_1) \geqq F(y_2)$.*

It is called an $\mathcal{R}$-invariant performance benchmark if in addition to (C1) *and* (C2), *it also satisfies that*

(C3) *F is $\mathcal{R}$-invariant.*

Throughout the rest of this paper, we shall assume that the functional F in problem (1.5) is an $\mathcal{R}$-invariant performance benchmark.

For each $t \in [0, T]$, define $D_t\colon D[0, t] \to \mathbf{R}$ and $G_t\colon D[0, t] \to \mathbf{R}$ by

$$D_t(y) = \mathbf{E}\left[f(F_t(y, \omega_\cdot^{-\lambda}) - \omega_{T-t}^{-\lambda})\right], \tag{2.13}$$

$$G_t(y) = \mathbf{E}\left[f(F_t(y, \omega_\cdot^\lambda))\right], \tag{2.14}$$

where $\omega^\lambda = (\lambda s + \omega_s)_{0 \leqq s \leqq T-t}$. The following lemma is the key result for establishing the main theorem.

Lemma 2.3. *Under either one of the following conditions, $D_t(y) \geqq G_t(y)$ for any $t \in [0, T]$ and $y \in D[0, t]$:*

1. $\lambda \leqq 0$ *and* f *is nonincreasing and convex;*
2. $\lambda \geqq 0$ *and* f *is nondecreasing and convex.*

Proof. See the appendix.

COROLLARY 2.1. *Under either one of the conditions of Lemma* 2.3,

$$\mathbf{E}\left[f(F_t(y,\omega_\cdot^{-\lambda}) - \omega_{T-t}^{-\lambda})\right] \geqq \mathbf{E}[f(F_t(y,\omega_\cdot^{-\lambda}))]$$

for any $t \in [0,T]$ *and* $y \in D[0,t]$.

Proof. Under both conditions, using condition (C2),

$$G_t(y) = \mathbf{E}\left[f(F_t(y,\omega_\cdot^{\lambda}))\right] \geqq \mathbf{E}\left[f(F_t(y,\omega_\cdot^{-\lambda}))\right],$$

and hence by Lemma 2.3, $\mathbf{E}\left[f(F_t(y,\omega_\cdot^{-\lambda}) - \omega_{T-t}^{-\lambda})\right] = D_t(y) \geqq G_t(y) \geqq \mathbf{E}\left[f(F_t(y,\omega_\cdot^{-\lambda}))\right]$.

We now prove the main result of this paper.

THEOREM 2.1. *Let* $f\colon \mathbf{R} \to \mathbf{R}$ *be a monotone and convex function, and let* $F\colon D[0,T] \to \mathbf{R}$ *be an* $\mathcal{R}$*-invariant performance benchmark. Consider the optimal stopping problem* (1.5).

1. *If* f *is nonincreasing, then an optimal stopping time is* $\tau^* = T$ *for the case* $\lambda \geqq 0$, *and is* $\tau^* = 0$ *for the case* $\lambda \leqq 0$.
2. *If* f *is nondecreasing, then the optimal stopping time is* $\tau^* = 0$ *for the case* $\lambda \geqq 0$, *and is* $\tau^* = T$ *for the case* $\lambda \leqq 0$.

Proof. The idea of the proof below is essentially due to Du Toit and Peskir [8].

Consider the case when f is nonincreasing and convex. First let $\lambda \geqq 0$. It suffices to establish the following inequality:

$$\mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_T^{\lambda}\big)\right] \geqq \mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_\tau^{\lambda}\big)\right] \tag{2.15}$$

for any stopping time τ adapted to $(\mathcal{F}_t)_{t\geqq 0}$ with $\mathbf{P}(\tau \leqq T) = 1$. For this, it suffices to prove

$$\mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_T^{\lambda}\big) \,\big|\, \mathcal{F}_\tau\right] \geqq \mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_\tau^{\lambda}\big) \,\big|\, \mathcal{F}_\tau\right],$$

which, upon using condition (C1), is equivalent to

$$\begin{aligned}&\mathbf{E}\left[f\big(F_\tau((\omega_s^{\lambda})_{0\leqq s\leqq \tau} - \omega_\tau^{\lambda}, (\omega_s^{\lambda} - \omega_\tau^{\lambda})_{\tau\leqq s\leqq T}) - (\omega_T^{\lambda} - \omega_\tau^{\lambda})\big) \,\big|\, \mathcal{F}_\tau\right]\\ &\quad\geqq \mathbf{E}\left[f\big(F_\tau((\omega_s^{\lambda})_{0\leqq s\leqq \tau} - \omega_\tau^{\lambda}, (\omega_s^{\lambda} - \omega_\tau^{\lambda})_{\tau\leqq s\leqq T})\big) \,\big|\, \mathcal{F}_\tau\right].\end{aligned} \tag{2.16}$$

Using the strong Markov property of Brownian motion, (2.16) becomes

$$\begin{aligned}&\mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t}) - \omega_{T-t}^{\lambda}\big)\right]\Big|_{t=\tau,\, y=(\omega_s^{\lambda})_{0\leqq s\leqq \tau} - \omega_\tau^{\lambda}}\\ &\quad\geqq \mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t})\big)\right]\Big|_{t=\tau,\, y=(\omega_s^{\lambda})_{0\leqq s\leqq \tau} - \omega_\tau^{\lambda}},\end{aligned}$$

which is equivalent to

$$\mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t}) - \omega_{T-t}^{\lambda}\big)\right] \geqq \mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t})\big)\right] \tag{2.17}$$

with $t = \tau(\omega)$ and $y = (\omega_s^{\lambda})_{0\leqq s\leqq \tau} - \omega_\tau^{\lambda}$. Inequality (2.17) follows from case 1 of Corollary 2.1 with $-\lambda$ being replaced by λ. Therefore, inequality (2.15) is established. Further, let $\lambda \leqq 0$, and we need to show that

$$\mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T})\big)\right] \geqq \mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_\tau^{\lambda}\big)\right] \tag{2.18}$$

for any stopping time τ adapted to $(\mathcal{F}_t)_{t\geqq 0}$ with $\mathbf{P}(\tau \leqq T) = 1$. Using Lemma 2.2, we see that the above inequality is equivalent to

$$\mathbf{E}\left[f\big(F((\omega_s^{-\lambda})_{0\leqq s\leqq T} - \omega_T^{-\lambda})\big)\right] \geqq \mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_\tau^{\lambda}\big)\right]. \tag{2.19}$$

To prove (2.19), it suffices to prove

$$\mathbf{E}\left[f\big(F((\omega_s^{-\lambda})_{0\leqq s\leqq T}) - \omega_T^{-\lambda}\big) \,\big|\, \mathcal{F}_\tau\right] \geqq \mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T}) - \omega_\tau^{\lambda}\big) \,\big|\, \mathcal{F}_\tau\right],$$

which, upon using condition (C1) and the strong Markov property of Brownian motion, is equivalent to

$$\begin{aligned}&\mathbf{E}\left[f\big(F_t(y,(\omega_s^{-\lambda})_{0\leqq s\leqq T-t})-\omega_{T-t}^{-\lambda}\big)\right]\big|_{t=\tau,y=(\omega_s^{-\lambda})_{0\leqq s\leqq\tau}-\omega_\tau^{-\lambda}}\\&\quad\geqq \mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t})\big)\right]\big|_{t=\tau,y=(\omega_s^{\lambda})_{0\leqq s\leqq\tau}-\omega_\tau^{\lambda}},\end{aligned}$$

that is,

$$D_t((\omega_s^{-\lambda})_{0\leqq s\leqq t}-\omega_t^{-\lambda})\geqq G_t((\omega_s^{\lambda})_{0\leqq s\leqq t}-\omega_t^{\lambda}) \tag{2.20}$$

with $t=\tau(\omega)$. Using condition (C2) and the fact that f is nonincreasing, we see that if $y_i\in D[0,t]$ with $y_1(s)\geqq y_2(s)$ for $s\in[0,t]$, then $G_t(y_1)\leqq G_t(y_2)$. Define $y_1=(\omega_s^{\lambda})_{0\leqq s\leqq t}-\omega_t^{\lambda}$ and $y_2=(\omega_s^{-\lambda})_{0\leqq s\leqq t}-\omega_t^{-\lambda}$. Since $\lambda\leqq 0$, we have $y_1(s)=\omega_s^{\lambda}-\omega_t^{\lambda}\geqq\omega_s^{-\lambda}-\omega_t^{-\lambda}=y_2(s)$ for $s\in[0,t]$, and therefore,

$$G_t((\omega_u^{-\lambda})_{0\leqq u\leqq t}-\omega_t^{-\lambda})\geqq G_t((\omega_u^{\lambda})_{0\leqq u\leqq t}-\omega_t^{\lambda}). \tag{2.21}$$

Combining case 1 of Lemma 2.3 and (2.21), we prove inequality (2.20) and hence inequality (2.18).

Consider the case when f is nondecreasing and convex. First let $\lambda\leqq 0$; we shall show that

$$\mathbf{E}\left[f\big(F((\omega^{\lambda})_{0\leqq s\leqq T})-\omega_T^{\lambda}\big)\right]\geqq\mathbf{E}\left[f\big(F((\omega^{\lambda})_{0\leqq s\leqq T})-\omega_\tau^{\lambda}\big)\right] \tag{2.22}$$

for any stopping time τ adapted to $(\mathcal{F}_t)_{t\geqq 0}$ with $\mathbf{P}(\tau\leqq T)=1$. Using arguments similar to those as above, it is enough to prove

$$\mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t})-\omega_{T-t}^{\lambda}\big)\right]\geqq\mathbf{E}\left[f\big(F_t(y,(\omega_s^{\lambda})_{0\leqq s\leqq T-t})\big)\right] \tag{2.23}$$

with $t=\tau(\omega)$ and $y=(\omega_s^{\lambda})_{0\leqq s\leqq\tau}-\omega_\tau^{\lambda}$. Since inequality (2.23) follows from case 2 in Corollary 2.1 with $-\lambda$ replaced by λ, inequality (2.22) is proved. Further, let $\lambda\geqq 0$; we need to prove that

$$\mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T})\big)\right]\geqq\mathbf{E}\left[f\big(F((\omega_s^{\lambda})_{0\leqq s\leqq T})-\omega_\tau^{\lambda}\big)\right] \tag{2.24}$$

for any stopping time τ adapted to $(\mathcal{F}_t)_{t\geqq 0}$ with $\mathbf{P}(\tau\leqq T)=1$. Using arguments similar to those in the first case, it suffices to prove

$$D_t((\omega_s^{-\lambda})_{0\leqq s\leqq t}-\omega_t^{-\lambda})\geqq G_t((\omega_s^{\lambda})_{0\leqq s\leqq t}-\omega_t^{\lambda}). \tag{2.25}$$

Using condition (C2) and the fact that f is nondecreasing, we see that if $y_i\in D[0,t]$ with $y_1(s)\geqq y_2(s)$ for $s\in[0,t]$, then $G_t(y_1)\geqq G_t(y_2)$. Define $y_1=(\omega_s^{-\lambda})_{0\leqq s\leqq t}-\omega_t^{-\lambda}$ and $y_2=(\omega_s^{\lambda})_{0\leqq s\leqq t}-\omega_t^{\lambda}$. Since $\lambda\geqq 0$, we have $y_1(s)=(\omega_s^{-\lambda})_{0\leqq s\leqq t}-\omega_t^{-\lambda}\geqq(\omega_s^{\lambda})_{0\leqq s\leqq t}-\omega_t^{\lambda}=y_2(s)$ for $s\in[0,t]$, and therefore,

$$G_t((\omega_u^{-\lambda})_{0\leqq u\leqq t}-\omega_t^{-\lambda})\geqq G_t((\omega_u^{\lambda})_{0\leqq u\leqq t}-\omega_t^{\lambda}). \tag{2.26}$$

Combining case 2 of Lemma 2.3 and (2.26), we can prove inequality (2.25) and hence inequality (2.24).

3. Applications of "bang-bang" principle with respect to $\mathcal{R}$-invariant performance benchmarks.

3.1. Existing results in the literature. We now demonstrate that our theorem includes the four concrete problems stated in section 1.

1. If $F(y) = \max_{0\leqq t\leqq T} y_t$ for $y \in D[0,T]$, we can see that F satisfies both conditions (C1) and (C2). Since for $y \in D[0,t]$ and $z \in D[0,T-t]$, $F_t(y,z) = (\max_{0\leqq s\leqq t} y_s) \vee (\max_{0\leqq s\leqq T-t} z_s)$ and $\max_{0\leqq s\leqq T-t} z_s = \max_{0\leqq s\leqq T-t} z_s^{-1}$, by Lemma 2.1 F also satisfies condition (C3). By taking $f(x) = e^{-x}$, it yields problem (1.1), while by taking f to be any nonincreasing convex function, it yields problem (1.2).

2. If $F(y) = \min_{0\leqq t\leqq T} y_t$ for $y \in D[0,T]$, then for $y \in D[0,t]$ and $z \in D[0,T-t]$,

$$F_t(y,z) = \Big(\min_{0\leqq s\leqq t} y_s\Big) \wedge \Big(\min_{0\leqq s\leqq T-t} z_s\Big).$$

Upon the fact that $\min_{0\leqq s\leqq T-t} z_s = \min_{0\leqq s\leqq T-t} z_s^{-1}$, we can also see that F satisfies conditions (C1)–(C3). By taking $f(x) = e^{-x}$, we also arrive at problem (1.3).

3. If $F(y) = T^{-1}\int_0^T y_t\,dt$ for $y \in D[0,T]$, then for $y \in D[0,t]$ and $z \in D[0,T-t]$,

$$F_t(y,z) = \frac{1}{T}\int_0^t y_s\,ds + \frac{1}{T}\int_0^{T-t} z_s\,ds.$$

Since $\int_0^{T-t} z_s\,ds = \int_0^{T-t} z_s^{-1}\,ds$, we can again see that this F also satisfies conditions (C1)–(C3). On the other hand, if $F(y) = \log(T^{-1}\int_0^T \exp(y_t)\,dt)$ for $y \in D[0,T]$, then for $y \in D[0,t]$ and $z \in D[0,T-t]$,

$$F_t(y,z) = \log\left(\frac{1}{T}\bigg(\int_0^t \exp(y_s)\,ds + \int_0^{T-t}\exp(z_s)\,ds\bigg)\right).$$

Again $\int_0^{T-t} e^{z_s}\,ds = \int_0^{T-t} e^{z_s^{-1}}\,ds$, and hence F satisfies conditions (C1)–(C3). Taking $f(x) = e^{-x}$, we recover problem (1.4).

Thus we see that problems (1.1)–(1.4) are included in case 1 of Theorem 2.1, i.e., when f is nonincreasing and convex. Furthermore, Theorem 2.1 also shows that in the case when f is nondecreasing and convex, the solutions to problems (1.1)–(1.4) are still of "bang-bang" type yet in a reverse manner.

3.2. New results in the literature. In addition to these four problems as quoted in section 1, our theorem also ensures some new results which are still absent in the literature.

1. Let $F(y) = \max_{0\leqq t\leqq T} y_t$ for $y \in D[0,T]$, and $f(x) = -x^p$ for $0 < p < 1$; then problem (1.5) becomes

$$\inf_{\tau\leqq T} \mathbf{E}[(M_T^\lambda - \omega_\tau^\lambda)^p]. \tag{3.1}$$

Since we have seen that the maximum functional satisfies conditions (C1)–(C3), and that $x \mapsto -x^p$ is nonincreasing and convex, Theorem 2.1 suggests that the solution to problem (3.1) is given by $\tau^* = T$ for the case $\lambda \geqq 0$ and $\tau^* = 0$ for the case $\lambda \leqq 0$. Problem (3.1) for all $p > 0$ was first formulated by Shiryaev (see [11]). For the two cases (i) $p \geqq 1$ and $\lambda = 0$ and (ii) $p = 2$ and $\lambda \in \mathbf{R}$, problem (3.1) has already been solved in the literature (see [9], [10], [7]), with optimal stopping times given by the first hitting times of the process $M_\cdot^\lambda - \omega_\cdot^\lambda$ to some nontrivial time-dependent boundaries. Also note that (see [10, Remark 2.3]) for the case when $0 < p < 1$ and $\lambda = 0$, $\tau^* = 0$ is the optimal stopping rule for problem (3.1). Thus our Theorem 2.1 supplements existing results by solving for the case when $0 < p < 1$ and $\lambda \in \mathbf{R}$.

2. For $0 < \alpha < 1$, $y \in D[0,T]$, define

$$F^\alpha(y) \triangleq \inf\left\{x > 0\colon \int_0^T \mathbf{1}_{\{y_s\leqq x\}}\,ds \geqq \alpha T\right\}.$$

$F^\alpha(y)$ is called the α-quantile of the path y. For discussions on the distribution of F^α under $\mathbf{P}$, its representation in terms of maximum and minimum, and its application on α-quantile option, one can see [1] and [6] for details. Since for any $c \in \mathbf{R}$,

$$\int_0^T \mathbf{1}_{\{y_s \leqq x\}}\, ds = \int_0^T \mathbf{1}_{\{y_s + c \leqq x + c\}}\, ds,$$

F^α clearly satisfies condition (C1). If $y_1(s) \geqq y_2(s)$ for $s \in [0,T]$, then for each $x \geqq F^\alpha(y_1)$, by definition, we have $\int_0^T \mathbf{1}_{\{y_1 \leqq x\}}\, ds \geqq \alpha T$, and hence

$$\int_0^T \mathbf{1}_{\{y_2 \leqq x\}}\, ds \geqq \int_0^T \mathbf{1}_{\{y_1 \leqq x\}}\, ds \geqq \alpha T,$$

which implies $x \geqq F^\alpha(y_2)$. This shows that F^α satisfies condition (C2). Finally for $y_1 \in D[0,t]$ and $y_2 \in D[0,T-t]$,

$$\begin{aligned}\int_0^T \mathbf{1}_{\{(y_1 y_2)(s) \leqq x\}}\, ds &= \int_0^t \mathbf{1}_{\{y_1(s) \leqq x\}}\, ds + \int_0^{T-t} \mathbf{1}_{\{y_2(s) \leqq x\}}\, ds \\ &= \int_0^t \mathbf{1}_{\{y_1(s) \leqq x\}}\, ds + \int_0^{T-t} \mathbf{1}_{\{y_2^{-1}(s) \leqq x\}}\, ds = \int_0^T \mathbf{1}_{\{y_1 y_2^{-1}(s) \leqq x\}}\, ds,\end{aligned}$$

hence $F_t^\alpha(y_1 y_2) = F_t^\alpha(y_1 y_2^{-1})$, and therefore F^α also satisfies condition (C3). If we take $f(x) = e^{-x}$, Theorem 2.1 suggests that the solution to the problem

$$\sup_{\tau \leqq T} \mathbf{E}\left[\exp\left(-\left(F^\alpha(\omega^\lambda) - \omega_\tau^\lambda\right)\right)\right] \tag{3.2}$$

is given by $\tau^* = T$ for the case $\lambda \geqq 0$ and $\tau^* = 0$ for the case $\lambda \leqq 0$; i.e., if an investor attempts to sell his stock with reference to the α-quantile, then the optimal selling strategy is still buy-and-hold for superior stock and sell-at-once for inferior stock.

We note that if we replace the Brownian motion in problem (1.5) by Bernoulli random walks, then results similar to those in Theorem 2.1 can also be obtained by similar arguments. See [13] and [2] on the "bang-bang" principle for the Bernoulli random walk. Allaart [3] also extended the "bang-bang" principle to a general class of random walks and Lévy processes.

Appendix. Proof of Lemma 2.3. To prove the lemma, we extend Allaart's argument (see [2]) to the case of an $\mathcal{R}$-invariant performance benchmark. Using condition (C1) and Lemma 2.2, respectively, we obtain that

$$D_t(y) = \mathbf{E}\left[f(F_t(y - \omega_{T-t}^{-\lambda}, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda}))\right], \quad G_t(y) = \mathbf{E}\left[f(F_t(y, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda}))\right],$$

where $\omega^{-\lambda} = (\omega_s^{-\lambda})_{0 \leqq s \leqq T-t}$. Writing $\Omega = C[0, T-t]$, we have

$$\begin{aligned}D_t(y) - G_t(y) &= \int_\Omega \left(f(F_t(y - \omega_{T-t}^{-\lambda}, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda})) - f(F_t(y, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda}))\right) d\mathbf{P}(\omega) \\ &= \int_\Omega \left(f(F_t(y - \omega_{T-t}^{-\lambda}, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda})) - f(F_t(y, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda}))\right) \cdot \mathbf{1}_{\{\omega_{T-t}^{-\lambda} \geqq 0\}}\, d\mathbf{P}(\omega) \\ &\quad + \int_\Omega \left(f(F_t(y - \omega_{T-t}^{-\lambda}, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda})) - f(F_t(y, \omega_\cdot^{-\lambda} - \omega_{T-t}^{-\lambda}))\right) \cdot \mathbf{1}_{\{\omega_{T-t}^{-\lambda} \leqq 0\}}\, d\mathbf{P}(\omega) \\ &\triangleq I^+ + I^-.\end{aligned}$$

Note that

$$\begin{aligned}I^- &= \int_\Omega \left(f(F_t(y + \omega_{T-t}^{\lambda}, \omega_\cdot^{\lambda})) - f(F_t(y, \omega_\cdot^{\lambda}))\right) \cdot \mathbf{1}_{\{\omega_{T-t}^{\lambda} \geqq 0\}}\, d\mathbf{P}(\omega) \\ &= \int_\Omega \left(f(F_t(y + \omega_{T-t}, \omega_\cdot)) - f(F_t(y, \omega_\cdot))\right) \cdot \mathbf{1}_{\{\omega_{T-t} \geqq 0\}} \\ &\qquad \times \exp\left\{\lambda \omega_{T-t} - \frac{\lambda^2}{2}(T-t)\right\} d\mathbf{P}(\omega),\end{aligned} \tag{A.1}$$

where the first equality is a consequence of Lemma 2.2 and the second equality is a consequence of the Girsanov transform. On the other hand

$$I^+ = \int_\Omega \big(f(F_t(y-\omega_{T-t}, \omega_\cdot - \omega_{T-t})) - f(F_t(y, \omega_\cdot - \omega_{T-t}))\big) \cdot \mathbf{1}_{\{\omega_{T-t} \geqq 0\}} \times \exp\bigg\{ -\lambda\omega_{T-t} - \frac{\lambda^2}{2}(T-t)\bigg\} d\mathbf{P}(\omega).$$

Combining these results, we obtain that

$$I^+ + I^- = \int_\Omega \psi(\omega) \cdot \mathbf{1}_{\{\omega_{T-t} \geqq 0\}} e^{-\lambda^2(T-t)/2}\, d\mathbf{P}(\omega),$$

where

$$\begin{aligned}\psi(\omega) = {} & \big(f(F_t(y-\omega_{T-t}, \omega - \omega_{T-t})) - f(F_t(y, \omega - \omega_{T-t}))\big)e^{-\lambda\omega_{T-t}} \\ & + \big(f(F_t(y+\omega_{T-t}, \omega)) - f(F_t(y,\omega))\big)e^{\lambda\omega_{T-t}}.\end{aligned} \tag{A.2}$$

For the case when $\lambda \leqq 0$ and f is nonincreasing and convex, on $\{\omega_{T-t} \geqq 0\}$,

$$f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t})) - f(F_t(y, \omega-\omega_{T-t})) \geqq 0,$$
$$f(F_t(y+\omega_{T-t}, \omega)) - f(F_t(y,\omega)) \leqq 0,$$

then upon using condition (C1) and the fact that $\lambda \leqq 0$, we have

$$\begin{aligned}\psi(\omega) \geqq {} & f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t})) - f(F_t(y, \omega-\omega_{T-t})) \\ & + f(F_t(y+\omega_{T-t}, \omega)) - f(F_t(y,\omega)) \\ = {} & (f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t})) - f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t}) + \omega_{T-t})) \\ & - (f(F_t(y, \omega-\omega_{T-t})) - f(F_t(y, \omega-\omega_{T-t}) + \omega_{T-t})) \geqq 0,\end{aligned}$$

where the last inequality holds, since f is convex. Therefore, in this case we proved $I^+ + I^- \geqq 0$ and hence $D_t(y) \geqq G_t(y)$. For the case when $\lambda \geqq 0$ and f is nondecreasing and convex, on $\{\omega_{T-t} \geqq 0\}$,

$$f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t})) - f(F_t(y, \omega-\omega_{T-t})) \leqq 0,$$
$$f(F_t(y+\omega_{T-t}, \omega)) - f(F_t(y,\omega)) \geqq 0,$$

then upon using condition (C1) and the fact that $\lambda \geqq 0$, we have

$$\begin{aligned}\psi(\omega) \geqq {} & f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t})) - f(F_t(y, \omega-\omega_{T-t})) \\ & + f(F_t(y+\omega_{T-t}, \omega)) - f(F_t(y,\omega)) \\ = {} & (f(F_t(y, \omega-\omega_{T-t}) + \omega_{T-t}) - f(F_t(y, \omega-\omega_{T-t}))) \\ & - (f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t}) + \omega_{T-t}) - f(F_t(y-\omega_{T-t}, \omega-\omega_{T-t}))) \geqq 0,\end{aligned}$$

where the last inequality holds, since f is convex. Therefore in this case we also have $I^+ + I^- \geqq 0$ and hence $D_t(y) \geqq G_t(y)$. Lemma 2.3 is proved.

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