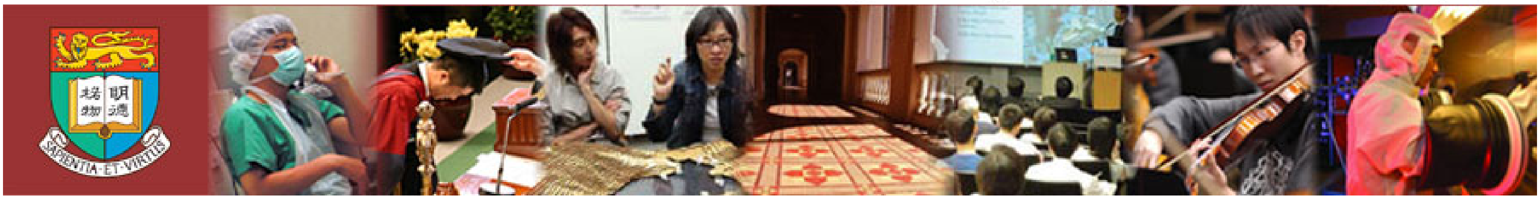
View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by HKU Scholars Hub

The HKU Scholars Hub The University of Hong Kong 香港大學學術庫



Title	**An improved eigendecomposition-based algorithm for frequencies estimation of two sinusoids**
Author(s)	**Liao, B; Chan, SC**
Citation	**IEEE Communications Letters, 2013, v. 17, n. 3, p. 557-560**
Issued Date	**2013**
URL	**http://hdl.handle.net/10722/189096**
Rights	**Creative Commons: Attribution 3.0 Hong Kong License**

An Improved Eigendecomposition-Based Algorithm for Frequencies Estimation of Two Sinusoids

Bin Liao, *Student Member, IEEE,* and Shing-Chow Chan, *Member, IEEE*

***Abstract*—An eigendecomposition-based algorithm for frequencies estimation of two sinusoids has been proposed recently in [1]. It was shown that for a given received vector length M, each frequency ω can be estimated from a set of estimates of $e^{jm\omega}$, the number of which is $\lfloor(M-1)/2\rfloor$. In this letter, an improved eigendecomposition-based algorithm is presented. We show that the frequency can be alternatively estimated from a set of estimates of $e^{j\omega}$. Compared with the previous algorithm, the number of estimates for each frequency is $M-2$, which is larger than $\lfloor(M-1)/2\rfloor$ when $M>3$. Hence, the proposed improved algorithm is expected to achieve a better performance because more estimates of each frequency are available to average out the numerical error. Furthermore, in the proposed algorithm, the frequency is estimated from the estimate of $e^{j\omega}$ directly, instead of $e^{jm\omega}$ indirectly. Therefore, no additional procedures are needed. Numerical examples are provided to illustrate the improvement of the proposed method.**

***Index Terms*—Frequency estimation, eigendecomposition.**

I. Introduction

RECENTLY, an algorithm for frequencies estimation of two sinusoids based on eigendecomposition has been proposed in [1]. Compared with the conventional Pisarenko harmonic decomposition (PHD) method [2], this algorithm can yield more estimates for each frequency, therefore it is possible to average out the numerical error to generate a better estimate. Moreover, this method has much lower complexity than the classical multiple signal classification (MUSIC) algorithm [3], which requires a grid search procedure in the frequency spectrum. The authors also show that this method can achieve better performance than the traditional estimation of signal parameters via rotational invariance techniques (ESPRIT) [4] at reduced complexity.

However, it is worth noting that this eigendecomposition-based algorithm can only produce a total of $\lfloor(M-1)/2\rfloor$ estimates for each frequency, where M is the length of the received vector and $\lfloor\cdot\rfloor$ denotes the floor function. Therefore, its ability to average out the numerical error may be limited. Another issue is that the frequency is indirectly extracted from the estimate $e^{jm\omega}$, which can be either $(\beta_m+\sqrt{\beta_m^2+4\alpha_m})/2$ or $(\beta_m-\sqrt{\beta_m^2+4\alpha_m})/2$ as shown in Eq. (14) of [1]. Hence, an additional procedure should be designed to determine $e^{jm\omega}$. To overcome these limitations, we propose an improved eigendecomposition-based algorithm for frequencies estimation of two sinusoids in this letter. Similar to the previous method [1], in the proposed improved algorithm, an auto-correlation matrix is firstly obtained from the received vectors. Then, eigendecomposition of the auto-correlation matrix is carried out to obtain two principal eigenvectors required for frequency estimation.

In the improved algorithm, each frequency is also estimated from a set of estimates. However, unlike the previous algorithm [1], the estimates are $e^{j\omega}$'s instead of $e^{jm\omega}$'s, so that the frequency can be extracted directly without any additional procedures. More importantly, the improved algorithm can lead to $M-2$ estimates for each frequency. Obviously, we have $M-2>\lfloor(M-1)/2\rfloor$, when $M>3$. It is shown that the performance is improved since more estimates are available to average out the numerical error. For the case of $M=3$, it is found that the improved algorithm is reduced to the previous one [1].

It is also worth mentioning that both the algorithm [1] and the improved one herein are developed for scenarios of two sinusoids, and are applicable to the case of a single sinusoid. However, their generalization to the case, where the number of sinusoids is larger than two, is not straightforward. In such cases, other eigendecomposition-based algorithms such as MUSIC and ESPRIT may be employed. Furthermore, when the number of sinusoids is unknown, an additional procedure for detecting the number of sinusoids, say, using minimum description length (MDL) criterion [5], should be performed.

II. Data Model

The problem of estimating the frequencies of two complex sinusoids in complex additive white Gaussian noise (AWGN) is considered. The observed signal can be modeled as [1]

$$y(k)=A_1e^{j(\omega_1k+\theta_1)}+A_2e^{j(\omega_2k+\theta_2)}+n(k), k=1,2,...,N \tag{1}$$

where $j=\sqrt{-1}$, N is the number of observations, A_1 and A_2 are the amplitudes of the sinusoids, $\omega_1,\omega_2\in[-\pi,\pi)$ and $\theta_1,\theta_2\in[-\pi,\pi)$ are the unknown frequencies and the phases, respectively. $n(k)$ denotes the noise which is assumed to be AWGN with variance σ_n^2. The problem of interest is the estimation of two frequencies ω_1 and ω_2 from the observation sequence $y(1),y(2),...,y(N)$.

To begin with, a received vector $\mathbf{y}(k)$ with a prescribed length M is defined as

$$\begin{aligned}\mathbf{y}(k)&=[y(k),y(k+1),...,y(k+M-1)]^T\\&=\mathbf{h}_1s_1(k)+\mathbf{h}_2s_2(k)+\mathbf{n}(k)\end{aligned} \tag{2}$$

where $k=1,2,...,N-M+1$, $s_n(k)=A_ne^{j(\omega_nk+\theta_n)}$ and $\mathbf{h}_n=[1,e^{j\omega_n},...,e^{j(M-1)\omega_n}]^T$, $n=1,2$, and $\mathbf{n}(k)=$

Manuscript received November 23, 2012. The associate editor coordinating the review of this letter and approving it for publication was P. Salvo Rossi.

The authors are with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong (e-mail: {liaobin, scchan}@eee.hku.hk).

Digital Object Identifier 10.1109/LCOMM.2013.011113.122629

1089-7798/13$31.00 © 2013 IEEE

$[n(k), n(k+1), ..., n(k+M-1)]^T$ is the noise vector. The auto-correlation matrix of the received vector is given by

$$\begin{aligned}\mathbf{R} &= E[\mathbf{y}(k)\mathbf{y}(k)^H] \\ &= A_1^2\mathbf{h}_1\mathbf{h}_1^H + A_2^2\mathbf{h}_2\mathbf{h}_2^H + \sigma_n^2\mathbf{I}\end{aligned} \quad (3)$$

where $\mathbf{I}$ is an $M \times M$ identity matrix. By eigendecomposing $\mathbf{R}$, one can obtain two principal eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, which correspond to two largest eigenvalues of $\mathbf{R}$. Note that, in practice, $\mathbf{R}$ can be estimated as

$$\hat{\mathbf{R}} = \frac{1}{N-M+1}\sum_{k=1}^{N-M+1}\mathbf{y}(k)\mathbf{y}(k)^H. \quad (4)$$

In the following section, we shall show how the two frequencies can be estimated from the principal eigenvectors.

III. Frequency Estimation Algorithms

A. Eigendecomposition-Based Algorithm [1]

First, we briefly review the eigendecomposition-based algorithm proposed in [1]. With the principal eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, a 2×2 matrix $\mathbf{F}_k$, $k = 1, 2, ..., M-1$, is defined as

$$\mathbf{F}_k = \begin{bmatrix} \mathbf{u}_1(1) & \mathbf{u}_2(1) \\ \mathbf{u}_1(k+1) & \mathbf{u}_2(k+1) \end{bmatrix} \quad (5)$$

where $\mathbf{u}_n(i)$ denotes the ith entry of the vector $\mathbf{u}_n$. It is shown in [1] that $\mathbf{F}_k$ can also be expressed as

$$\mathbf{F}_k = \begin{bmatrix} 1 & 1 \\ e^{jk\omega_1} & e^{jk\omega_2} \end{bmatrix}\mathbf{Z} \quad (6)$$

where $\mathbf{Z}$ is an invertible matrix. According to (5) and (6), one can derive that

$$\mathbf{F}_{2m}\mathbf{F}_m^{-1} = \begin{bmatrix} 1 & 0 \\ -e^{jm\omega_1}e^{jm\omega_2} & e^{jm\omega_1} + e^{jm\omega_2} \end{bmatrix} \quad (7)$$

where $m = 1, 2, ..., L$, and $L = \lfloor (M-1)/2 \rfloor$. Define $\mathbf{D}_m = \mathbf{F}_{2m}\mathbf{F}_m^{-1}$ which can be calculated according to (5), and let $\alpha_m = \mathbf{D}_m(2,1)$ and $\beta_m = \mathbf{D}_m(2,2)$. Then, we have

$$\begin{aligned}-e^{jm\omega_1}e^{jm\omega_2} &= \alpha_m, \\ e^{jm\omega_1} + e^{jm\omega_2} &= \beta_m,\end{aligned} \quad (8)$$

and hence $\{e^{jm\omega_1}, e^{jm\omega_2}\}$ can be calculated as

$$\{e^{jm\omega_1}, e^{jm\omega_2}\} = \frac{\beta_m \pm \sqrt{\beta_m^2 + 4\alpha_m}}{2}. \quad (9)$$

To extract the frequencies from the estimates of $e^{jm\omega_1}, e^{jm\omega_2}$, two vectors are formed as $\mathbf{v}_n = [1, e^{j\omega_n}, ..., e^{jL\omega_n}]$, $n = 1, 2$. Then dividing the $(i+1)$th entry of $\mathbf{v}_n$ by its ith entry, one can get L estimates of $e^{j\omega_n}$ and hence ω_n. By averaging these L estimates, the final estimate of ω_n is obtained.

B. Improved Eigendecomposition-Based Algorithm

In the produced eigendecomposition-based algorithm [1], the number of estimates for each frequency is $\lfloor (M-1)/2 \rfloor$. It is noted that the larger M is, a more reliable solution can be obtained. This is verified by numerical simulation in [1]. However, the complexity will also increase accordingly with M. Furthermore, it can be seen that the frequency ω_n is estimated indirectly from $e^{jm\omega_n}$, which should be correctly chosen from the two solutions as shown in (9). Thus, an additional procedure is required to determine $e^{jm\omega_n}$ as mentioned in [1].

As an alternative, we show that for a given received vector length M, it is possible to produce more estimates for each frequency than the previous algorithm [1]. Therefore, a more accurate estimate of ω_n can be achieved. Also, since the frequency is estimated directly from $e^{j\omega_n}$ instead of $e^{jm\omega_n}$, no additional procedures as in [1] is required.

Recall the subspace principle in [3] and [4], it is known that the signal subspace spans the same space of the principal eigenvectors. In the case considered in this letter, we have

$$span\{[\mathbf{u}_1\ \mathbf{u}_2]\} = span\{[\mathbf{h}_1\ \mathbf{h}_2]\}. \quad (10)$$

This implies that there exists a 2×2 nonsingular matrix $\mathbf{T}$ satisfying [3], [4]

$$[\mathbf{u}_1\ \mathbf{u}_2] = [\mathbf{h}_1\ \mathbf{h}_2]\mathbf{T}. \quad (11)$$

To proceed, we define the matrix $\mathbf{G}_k$ as follows

$$\mathbf{G}_k = \begin{bmatrix} \mathbf{u}_1(k) & \mathbf{u}_2(k) \\ \mathbf{u}_1(k+1) & \mathbf{u}_2(k+1) \end{bmatrix} \quad (12)$$

where $k = 1, 2, ..., M-1$. From (11) and (12), one gets

$$\begin{aligned}\mathbf{G}_k &= \begin{bmatrix} \mathbf{h}_1(k) & \mathbf{h}_2(k) \\ \mathbf{h}_1(k+1) & \mathbf{h}_2(k+1) \end{bmatrix}\mathbf{T} \\ &= \begin{bmatrix} e^{j(k-1)\omega_1} & e^{j(k-1)\omega_2} \\ e^{jk\omega_1} & e^{jk\omega_2} \end{bmatrix}\mathbf{T}.\end{aligned} \quad (13)$$

Define $\mathbf{J}_m = \mathbf{G}_{m+1}\mathbf{G}_m^{-1}$, where $m = 1, 2, ..., M-2$, we have

$$\begin{aligned}\mathbf{J}_m &= \begin{bmatrix} e^{jm\omega_1} & e^{jm\omega_2} \\ e^{j(m+1)\omega_1} & e^{j(m+1)\omega_2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} e^{j(m-1)\omega_1} & e^{j(m-1)\omega_2} \\ e^{jm\omega_1} & e^{jm\omega_2} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 1 \\ -e^{j\omega_1}e^{j\omega_2} & e^{j\omega_1} + e^{j\omega_2} \end{bmatrix}.\end{aligned} \quad (14)$$

Similar to the derivations in (7)–(9), we first assume that $p_m = \mathbf{J}_m(2,1)$ and $q_m = \mathbf{J}_m(2,2)$, then one gets

$$\begin{aligned}-e^{j\omega_1}e^{j\omega_2} &= p_m, \\ e^{j\omega_1} + e^{j\omega_2} &= q_m,\end{aligned} \quad (15)$$

and hence

$$\{e^{j\omega_1}, e^{j\omega_2}\} = \frac{q_m \pm \sqrt{q_m^2 + 4p_m}}{2}. \quad (16)$$

Consequently, the two frequencies ω_1 and ω_2 can be simply extracted by taking the angle of the estimates $e^{j\omega_1}$ and $e^{j\omega_2}$.

A careful examination of the improved algorithm shows that a total of $M-2$ estimates can be obtained for each frequency. This is larger than that of algorithm [1], which can only offer $\lfloor (M-1)/2 \rfloor$ estimates when $M \geq 3$. It is interesting to notice that if we define the matrix $\mathbf{G}_k$ in (12) equivalently to $\mathbf{F}_k$ in (6), the estimates of $e^{jm\omega_1}$ and $e^{jm\omega_2}$ can be similarly obtained as in (7)–(9). This means that the eigendecomposition-based algorithm can also be derived under the proposed framework. In particular, for the specific case of $M = 3$, we have $M-2 = \lfloor (M-1)/2 \rfloor$, so that the improved algorithm is reduced to the previous one. Also, it can be found that the proposed method is applicable to the case of a single sinusoid. Following the subspace principle, we know that the principal eigenvector of the auto-correlation matrix is proportional to the vector $\mathbf{h}$, i.e.,

$$\mathbf{u} = \gamma \mathbf{h} \tag{17}$$

where γ is a nonzero constant. Readily, we have

$$e^{j\omega} = \frac{\mathbf{h}(k+1)}{\mathbf{h}(k)} = \frac{\mathbf{u}(k+1)}{\mathbf{u}(k)} \tag{18}$$

where $k = 1, 2, ..., M-1$. Therefore, once $\mathbf{u}$ is estimated from $\hat{\mathbf{R}}$, one can get $M-1$ estimates of the frequency ω, and the final estimate can be obtained by averaging them.

IV. Simulation Results

In order to demonstrate the performance of the proposed method, representative examples are provided in this section. First, following the example used in [1], a signal with $\omega_1 =$ 1rad/sample and $\omega_2 =$ 2rad/sample is considered. Moreover, it is assumed that the noise is AWGN, $\theta_1 = 0.2$rad, and $\theta_2 = 0.5$rad. We also assume that $A_1 = A_2$ in the simulations for simplicity. The signal-to-noise ratio (SNR) is defined as SNR $= A_1^2/\sigma_n^2$. The performance of the method is measured in terms of the root-mean-square-error (RMSE) of frequency estimation. More precisely, it is defined as

$$\text{RMSE} = \sqrt{\frac{1}{2K}\sum_{k=1}^{K}\left((\omega_1 - \hat{\omega}_{1,k})^2 + (\omega_2 - \hat{\omega}_{2,k})^2\right)} \tag{19}$$

where K is the number of Monte Carlo experiments, and $\hat{\omega}_{n,k}$ is the estimated frequency of the nth frequency in the kth experiment. In all simulations, we let $K = 500$. In the first example, a total of $N = 200$ observations are collected, and two received vector lengths, i.e., $M = 3$ and 12, are simulated.

Fig. 1 shows the resultant RMSEs versus SNR. For comparison, the eigendecomposition-based algorithm [1], MUSIC, ESPRIT, and PHD are tested, the corresponding Cramér-Rao bound (CRB) [7] is also computed. It can be seen from Fig. 1(a) that, for the specific case of $M = 3$, all of the tested methods perform similarly. Moreover, the improved algorithm offers the same performance as the eigendecomposition-based algorithm. This is because these two algorithms are actually equivalent in this case as discussed earlier. However, for a larger M, e.g., $M = 12$, as shown in Fig. 1(b), the proposed algorithm is able to offer some improvement compared with the earlier one, due to the fact that more estimates can be obtained in our proposed algorithm to average out the numerical error.

It is worth noting that the RMSEs of all tested algorithms including the improved one do not attain the CRB even at high SNRs. This implies that for the tested vector lengths these estimators are not efficient in the case of large number of observations. To this end, the algorithms are evaluated in the case of $N = 32$. Fig. 2 shows the resultant RMSEs versus SNR for $M = 8$ and $M = 12$. Obviously, it can be seen that when $M = 8$, the frequencies can be efficiently estimated by all methods, except the PHD. When $M = 12$, all methods, except the algorithm [1] and PHD, can also perform very well. An interesting observation is that the algorithm [1] performs better in the case of $M = 8$ than the case of $M = 12$. This contradicts to the results in the case of a large N. A possible explanation is that, in the case of a small N, the estimate of auto-correlation matrix is relatively poorer for a larger M, and the algorithm [1] is more sensitive to the error in this matrix.

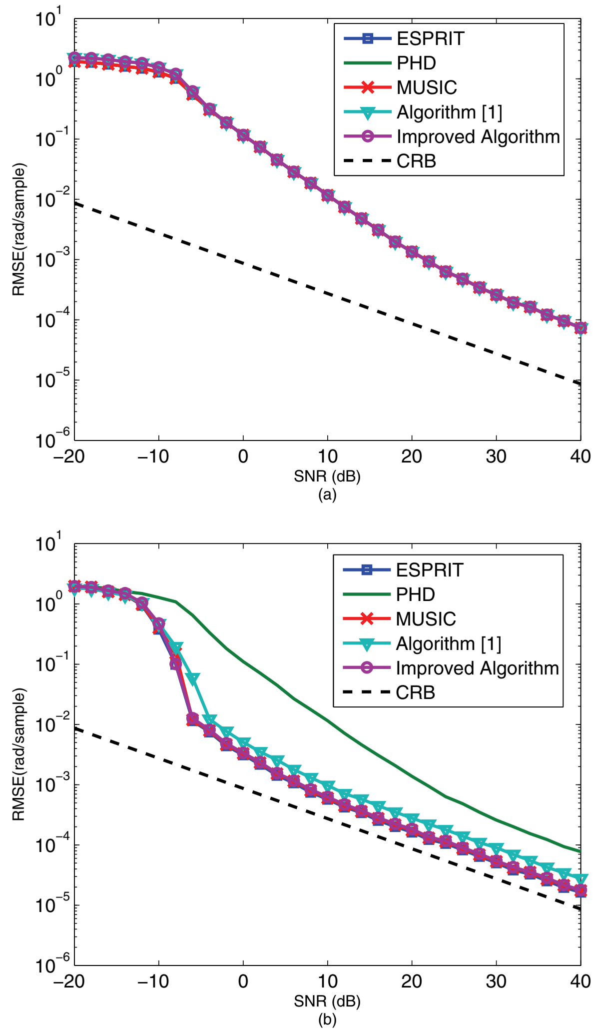


Fig. 1. Resultant RMSEs versus SNR, $N = 200$, $\omega_1 =$ 1rad/sample and $\omega_2 =$ 2rad/sample. (a) $M = 3$, (b) $M = 12$.

Next, we test the performance of the algorithm in the case of two close frequencies. Following the above example, we assume that $N = 32$, $M = 8$, and the two frequencies are changed to be $\omega_1 =$ 1rad/sample and $\omega_2 =$ 1.2rad/sample. Fig. 3 shows the RMSEs versus SNR. It can be noticed that the MUSIC and ESPRIT algorithms have higher ability to resolve close frequencies especially at relatively lower SNRs. We can also notice that the proposed algorithm is able to offer improved performance compared with the algorithm [1].

Finally, the performance of the proposed algorithm is evaluated in the case of a single sinusoid. Assume that $\omega = 2$rad/sample, $\theta = 0.5$rad, $N = 32$, and $M = 8$. As a representative, the DFT-based algorithm proposed for single frequency estimation in [6] is tested for comparison. The algorithm [1] is not tested in this example since it produces a pair of symmetric frequencies, one is positive and the other is negative, and hence, an additional procedure is

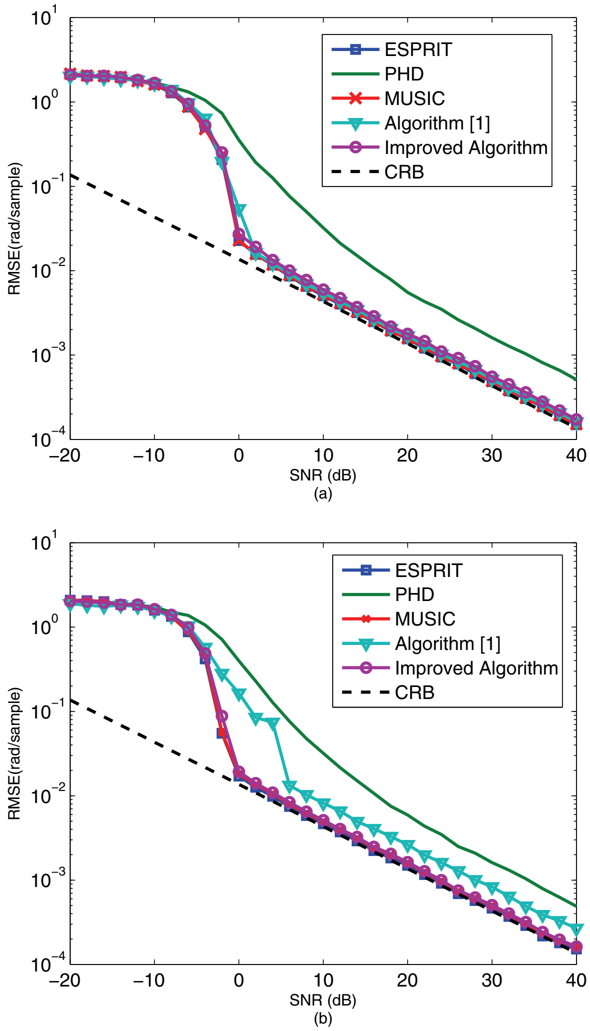


Fig. 2. Resultant RMSEs versus SNR, $N = 32$, $\omega_1 = 1$rad/sample and $\omega_2 = 2$rad/sample. (a) $M = 8$, (b) $M = 12$.

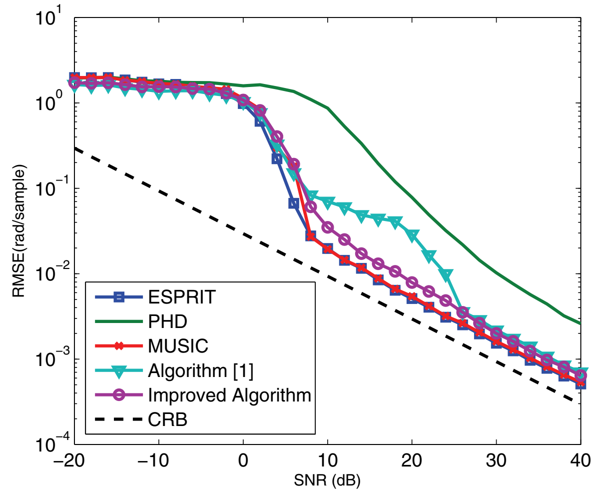


Fig. 3. Resultant RMSEs versus SNR, $N = 32$, $M = 8$, $\omega_1 = 1$rad/sample and $\omega_2 = 1.2$rad/sample.

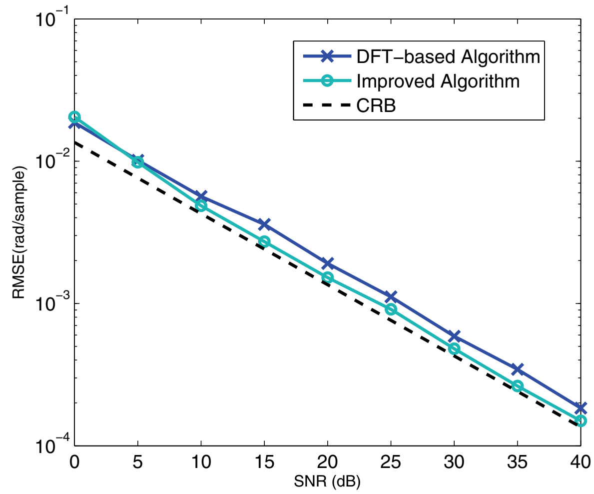


Fig. 4. Resultant RMSEs versus SNR, $N = 32$, $M = 8$, a single sinusoid with $\omega = 2$rad/sample.

required to determine the true one. Fig. 4 shows the resultant RMSEs versus SNR. It can be noticed that the DFT-based algorithm is outperformed by our proposed one. For the case of SNR $<$ 0dB, the RMSEs are not evaluated, because the DFT-based algorithm [6] is likely to fail at very low SNRs.

V. Conclusions

An improved algorithm for the frequency estimation of two sinusoids is developed based on eigendecomposition. Compared with the recently reported eigendecomposition-based algorithm [1], our proposed improved method can produce more estimates for each frequency to average out the numerical error. Therefore, the frequencies can be estimated with a higher accuracy. It is also found that the proposed algorithm achieves almost similar performance to MUSIC and ESPRIT at reduced complexity, though it has relatively lower ability to resolve close frequencies.

References

[1] Y. S. Chen, S. J. Shin, and B. Y. Liau, "An eigendecomposition-based algorithm for frequencies estimation of two sinusoids," *IEEE Commun. Lett.*, vol. 16, no. 8, pp. 1292–1295, Aug. 2012.

[2] V. F. Pisarenko, "The retrieval of harmonics from a covariance function," *Geophys. J. Roy. Atron. Soc.*, vol. 33, no. 3, pp. 347–366, Sept. 1973.

[3] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[4] R. Roy, P. Paulraj, and T. Kailath, "ESPRIT–a subspace rotation approach to estimation of parameters of cisoids in noise," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 5, pp. 1340–1342, Oct. 1986.

[5] J. Rissanen, "Modeling by shortest data description," *Automatica*, vol. 14, no. 5, pp. 465–471, Sept. 1978.

[6] Ç. Candan, "A method for fine resolution frequency estimation from three DFT samples," *IEEE Signal Process. Lett.*, vol. 18, no. 6, pp. 351–354, Jun. 2011.

[7] S. M. Kay, "Extensions for complex data and parameters," in *Fundamentals of Statistical Signal Processing: Estimation Theory.* Prentice-Hall, 1993.