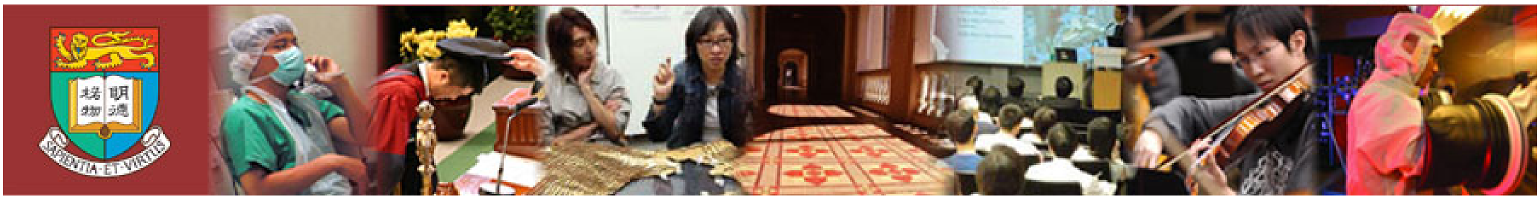

Title	The superposition of algebraic solitons for the modified Korteweg-de Vries equation
Author(s)	Chow, KW; Wu, CF
Citation	Communications in Nonlinear Science and Numerical Simulation, 2014, v. 19 n. 1, p. 49-52
Issued Date	2014
URL	http://hdl.handle.net/10722/185955
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The superposition of algebraic solitons for the modified Korteweg-de Vries equation

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PACS: 05.45.Yv; 02.30.Jr

CNSNS-D-13-00354

Re-submission date: June 2013

Communications in Nonlinear Science and Numerical Simulation:

Section of **Nonlinear Waves and Solitons**

Keywords: Algebraic solitons; Superposition principle

ABSTRACT

Many real nonlinear evolution equations exhibiting soliton properties display a special superposition principle, where an infinite array of equally spaced, identical solitons constitutes an exact periodic solution. This arrangement is studied for the modified Korteweg-de Vries equation with positive cubic nonlinearity, which possesses algebraic solitons with nonvanishing far field conditions. An infinite sum of equally spaced, identical algebraic pulses is evaluated in closed form, and leads to a complex valued solution of the nonlinear evolution equation.

1. Introduction

A remarkable property displayed by many classical real nonlinear evolution equations possessing soliton modes is a nonlinear superposition principle. More precisely, an infinite array of solitons placed at equal interval constitutes an *exact* periodic solution of the evolution equation. The speed of the periodic wave, however, is different from the component solitons due to nonlinear interaction. This phenomenon has been demonstrated for nonlinear lattices [1], the Korteweg-de Vries (KdV) [2, 3], the modified KdV (mKdV) [2, 3], the Benjamin-Ono [4], and the intermediate depth [5, 6] equations. For wave profiles with a discontinuity in the derivative, e.g. 'peakons' in the Camassa-Holm equation, a version of superposition principle is relevant too [7, 8] A variety of ingenious theoretical techniques has been employed in establishing these principles, ranging from direct algebraic rearrangement [2], summation by the Poisson formula [9], to elaborate identities in the theories of elliptic and theta functions [10]. Indeed such nonlinear superposition principles, together with particle-like properties and connections with algebraic geometry, constitute important hallmarks of solitons [11].

For partially integrable or even nonintegrable equations, some forms of superposition principle still exist, e.g. the regularized long wave [12], sixth order generalized Boussinesq [13] and convective fluid [14, 15] equations. For wave patterns in two or more spatial dimensions, this superposition of solitary pulses still applies, but the dynamics is more complicated [16, 17]. Other than the setting of evolution equation, this summation process for solitons is also relevant in geometry and other branches of mathematics and physics [18].

In general, the localized modes employed in the nonlinear superposition of all these previous studies are ones which decay to zero in the far field. It would be challenging to examine if a nonlinear superposition principle can be established starting with modes which do *not* decay in the far field. This question will be addressed for the modified Korteweg-de Vries equation with positive cubic nonlinearity (mKdV+). mKdV+ is chosen as it can be applied directly to many configurations of physical and engineering interests, e.g. two-layer fluids [19, 20]. Shallow water waves are important in oceanography, and various models, e.g. the Boussinesq, Bona-Chen and Rosenau-Kawahara equations, have been proposed [21, 22].

The structure of the paper can now be explained. The bilinear form of mKdV+ with nonzero boundary condition will first be reviewed. Exponentially decaying solitary pulses are obtained and algebraically decaying solitons are derived by taking a 'long wave' limit [23, 24] (Section 2). An infinite array of equally spaced solitons moving at a yet to be determined speed is formed, and summed mathematically to produce a rational expression of trigonometric functions. This solution, however, satisfies mKdV+ only for complex values for the far field condition (Section 3). A general superposition principle involving complex valued functions is established using contour integral techniques (Section 4). Conclusions and analogy with other evolution equations are discussed (Section 5).

2. Background

The mKdV+ equation

$$u_t + 6u^2u_x + u_{xxx} = 0 \tag{1}$$

admits solitons of the form (β real)

$$u = \beta \operatorname{sech}[\beta(x - \beta^2 t)],$$

and a periodic solution in terms of the Jacobi elliptic function dn as (r real)

$$u = r\,\mathrm{dn}[r(x - r^2(2 - k^2)t)] \qquad (k = \text{modulus of the Jacobi dn function}).$$

The speed of the solitary pulse is different from that of the periodic pattern except, of course, for the long limit (k tending to unity), where dn degenerates to sech.

This nonlinear superposition principle, namely, the periodic solution as an infinite array of solitary pulses with a velocity different from each individual soliton can be established via the Fourier series of dn [3]

$$\mathrm{dn}(x) = \frac{\sum \mathrm{sech}[n\pi / s] e^{2\pi i n x}}{\sum \mathrm{sech}[n\pi / s]}$$

$$= \frac{\sum \mathrm{sech}[\pi s (x - m)]}{\sum \mathrm{sech}[m\pi s]},$$

where the summation in these expressions extends over m, n from $-\infty$ to $+\infty$. The second equality is accomplished by the Poisson summation formula:

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} dt .$$

The focus of the present work is on using solitary pulses with nonzero boundary conditions in the far field as the building blocks. In that case, the appropriate bilinear transformation is [23, 24] (D is the Hirota operator)

$$u = u_0 + \left(\log \frac{G}{F} \right)_x ,$$

$$(D_t - 6u_0^2 D_x + D_x^3) G \cdot F = 0 ,$$

$$(D_x^2 + 2u_0 D_x) G \cdot F = 0 .$$

The 1-soliton is obtained by using

$$G = 1 + \exp[\alpha x + \alpha(6u_0^2 - \alpha^2)t + \eta^{(0)} + \psi_0],$$

$$F = 1 + \exp[\alpha x + \alpha(6u_0^2 - \alpha^2)t + \eta^{(0)} + \phi_0].$$

The phase constants are defined by

$$\exp(\phi_0) = 1 + \frac{\alpha}{2u_0},$$

$$\exp(\psi_0) = 1 - \frac{\alpha}{2u_0}.$$

The phase parameter $\eta^{(0)}$ originally arises from the freedom in choosing the origin for the spatial coordinate or time, but will now be exploited to assume arbitrary values, even complex ones. In particular, for $\exp(\eta^{(0)}) = -1$, $\alpha \to 0$ (long wave limit), one obtains the algebraic soliton going to u_0 in the far field:

$$u = u_0 - \frac{4u_0}{4u_0^2(x - 6u_0^2 t)^2 + 1}. \qquad (2)$$

As (2) is similar to the soliton of the Benjamin-Ono equation in terms of structure, one might be tempted to conclude that a nonlinear superposition should exist. We shall show in the next section that, however, such a summation process leads to a *complex valued* solution of mKdV+.

3. A complex valued solution of mKdV+

We now seek a solution of mKdV+ by a superposition of these algebraic solitons:

$$u = u_0 - \sum_{m=-\infty}^{\infty} \frac{4u_0}{4u_0^2(x - m\lambda - ct)^2 + 1}, \qquad (3)$$

where λ is the spacing between successive peaks of the sequence of solitary pulses. c is the phase speed of the pattern and will be different from that of the individual pulse due to nonlinear interactions.

To evaluate the infinite sum, we use the partial fraction decomposition of the hyperbolic cotangent,

$$\coth z = \sum_{n=-\infty}^{\infty} \frac{z}{z^2 + n^2\pi^2},$$

to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\zeta - n)^2 + s^2} = \pi\left\{\frac{\coth[\pi(s + i\zeta)] + \coth[\pi(s - i\zeta)]}{2s}\right\}. \qquad (4)$$

Further algebra shows that the sum (3) can actually be expressed explicitly as

$$u = u_0 - \frac{\alpha_0 \sinh[\alpha_0 / (2u_0)]}{\cosh[\alpha_0 / (2u_0)] - \cos[\alpha_0(x - ct)]}. \qquad (5)$$

To determine the proper speed c in terms of u_0, one substitutes the sum expression back into the original mKdV+. The surprise is that only complex valued u_0 will be feasible:

$$u_0 = -iA, \sin\left(\frac{\sqrt{L}}{A}\right) = \frac{\sqrt{L}}{\sqrt{A^2 + L}},\ L = -\frac{c}{4} + \frac{3u_0^2}{2},\ \alpha_0 = 2\sqrt{L}. \qquad (6)$$

In other words, the function $u = p + iq$ of (5) would be a complex valued solution of mKdV+, where real p, q will satisfy

$p_t + 6[(p^2 - q^2)p_x - 2pqq_x] + p_{xxx} = 0,$

$q_t + 6[(p^2 - q^2)q_x + 2qpp_x] + q_{xxx} = 0.$

4. Complex Analysis

This whole mechanism can also be verified by proving the formula

$$\begin{aligned} u(z) &= u_0 - \sum_{n=-\infty}^{\infty} \frac{1/u_0}{(z - z_0 - nT)(z - z_0 - nT - i/u_0)} \\ &= u_0 + \frac{(2\pi/T)(\sigma/2)}{\cos[2\pi(z - z_0 - i/(2u_0))/T] - \delta/2}, \end{aligned} \tag{7}$$

by contour integrals, where the complex valued parameter $T \in C$ is the period,

$\sigma/2 = -\sinh[\pi/(u_0 T)]$,

$\delta/2 = \cosh[\pi/(u_0 T)]$, and

$\cos[i\pi/(Tu_0)] = iTu_0/\sqrt{\pi^2 - u_0^2 T^2}$.

Let

$$U(\xi) = u_0 - \frac{1}{u_0} \sum_{n=-\infty}^{\infty} \frac{1}{(\xi - nT)^2 + 1/(4u_0^{\,2})},$$

$$g(\xi) = \frac{1}{[(\xi - \omega T)^2 + 1/(4u_0^{\,2})](\exp(i2\pi\omega) - 1)},$$

$$I = \int_{|\omega|=R} \frac{1}{[(\xi - \omega T)^2 + 1/(4u_0^{\,2})]} \frac{d\omega}{\exp(i2\pi\omega) - 1}.$$

By the residue theorem and the fact that

$I = O(\frac{1}{R}) \to 0$ as $R \to +\infty, |R - n| > \varepsilon_0 \in (0, \frac{1}{2})$, for all $n \in \mathbf{Z}$, we have

$$0 = \sum_{n=-\infty}^{\infty} \frac{1}{(\xi - nT)^2 + 1/(4u_0^{\ 2})} + 2\pi i \ \mathrm{Res}(g, \frac{\xi + i/(2u_0)}{T}) + 2\pi i \ \mathrm{Res}(g, \frac{\xi - i/(2u_0)}{T})$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{(\xi - nT)^2 + 1/(4u_0^{\ 2})} + \frac{2\pi u_0}{T}\frac{1}{e^A - 1} - \frac{2\pi u_0}{T}\frac{1}{e^B - 1},$$

where $A = [2\pi i(\xi + i/2u_0)]/T, B = [2\pi i(\xi - i/2u_0)]/T$, which reduces to (7) on simplification.

5. Conclusions

Many real integrable nonlinear evolution equations exhibit an elegant superposition principle, where an infinite array of equally spaced, identical solitons constitutes an exact periodic solution. The critical point is that the velocity of the periodic pattern is different from each individual soliton, due to nonlinear interactions. The present analysis, however, shows that this property may not hold, or at least not in a straightforward way, for modes which do not vanish in the far field.

More precisely, an infinite array of equally spaced algebraic solitons of mKdV+, with each component solitary pulse satisfying a nonzero far field condition, will only yield a *complex valued* solution of the partial differential equation. This phenomenon arises from summing a sequence of such algebraic pulses moving at a yet to be determined speed. The closed form for the sum is obtained either by contour integration or partial fraction decomposition of elliptic functions, and can only satisfy the nonlinear evolution equation if complex values of the far field conditions are allowed. The challenge now is to investigate whether such a phenomenon can be generalized to other evolution equations possessing solitons which do not vanish in the far field. From a more general perspective, elegant properties of solitons, e.g. these nonlinear superposition principles, might be destroyed by small structural changes of the evolution equation, and it would be valuable to study how perturbation and variational methods can be applied under these circumstances [25, 26].

Acknowledgements

Dr. T. W. Ng of the Mathematics Department of the University of Hong Kong has given generous help. Partial financial support has been provided by postgraduate studentship and the Incentive Award Scheme of the University of Hong Kong.

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