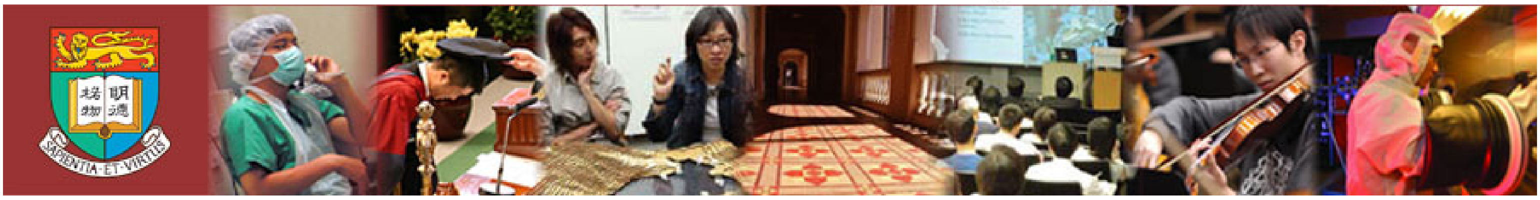

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DENSITY ESTIMATES ON COMPOSITE POLYNOMIALS

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Abstract

J.F. Ritt introduced the concepts of prime and composite polynomials and proved three fundamental theorems on factorizations (in the sense of compositions) of polynomials in 1922. In this paper, we shall give a density estimate on the set of composite polynomials.

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1. Introduction and Preliminaries

Let p be a non-linear polynomial in one complex variable. We say that p is *prime* if and only if there do not exist two complex polynomials q_1 and q_2 both with degree greater than one such that $p(z) = q_1(q_2(z))$. Otherwise, p is called *composite* or *decomposable.*

Clearly, for a given polynomial p, one can always factorize it as a composition of prime polynomials only and this factorization will be called a *prime factorization.* In 1922, J.F. Ritt [13] proved three fundamental results on the factorizations of complex polynomials. Since then many people have tried to give different proofs or generalizations of Ritt's theorems to certain classes of rational functions (see for example, [6], [8], [5], [16], [9],[2],[11] and [10]).

It is worth pointing out that the factorizations of entire or meromorphic functions have also been considered by many people. For a detailed discussion of this topic, we refer the reader to [7], [4] and [3]. One can also find a discussion on factorizations of infinite Blaschke products in [15].

The set of critical values of a polynomial plays an important role in determining if the polynomial is prime or not (see for example Theorem A below). By considering the number of distinct critical values of a polynomial, Beardon [1] showed that for each fixed positive integer n, the set of degree n composite polynomials lies in some hypersurface in $\mathbb{C}^{n+1}$ which implies that the set of composite polynomials is of measure zero and hence almost all polynomials are prime. In this paper, we shall give a density estimate on how small the set of degree n composite polynomials is. This kind of density estimation was first used by Smale in his work on the efficiency of Newton's method [14] . In fact, Smale found a density estimate on a set $\mathcal{V}_{\rho,n}$ of 'bad' polynomials of degree n which fail to arrive at an approximate zero when applying the Newton's method a certain fixed number of times. (A point z_0 is called an *approximate zero* of p if $z_0 \to z^*$, $p(z^*) = 0$ and $|p(z_n)/p(z_{n-1})| < \frac{1}{2}$ for all $n \in \mathbb{N}$, where $z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}$.)

Smale's Density Estimate ([14, Theorem 5.(1)])**.** *For any $R > \frac{1}{3}$,*

$$\frac{\mathrm{Vol}(\mathcal{V}_{\rho,n} \cap P(R))}{\mathrm{Vol}(P(R))} \leq 150(n+2)^{4/3}\rho^{2/3}, \tag{1.1}$$

where $P(R)$ denote the polycylinder of radius R. We call $\dfrac{\mathrm{Vol}(\mathcal{V}_{\rho,n}) \cap P(R)}{\mathrm{Vol}(P(R))}$ the density of $\mathcal{V}_{\rho,n}$.

2. The main result

Without loss of generality, we may assume that p is a normalized polynomial of degree $n \geq 2$, that is, $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$. Now w is a critical value of p if and only if $p'(z) = 0$ and $p(z) - w = 0$ have a common root if and only if the resultant $\text{Res}(p - w, p') = 0$. Denote $\text{Res}(p - w, p')$ by $\Phi(w)$. Clearly, $\Phi(w)$ is a polynomial in w of degree $n - 1$ and p has $n - 1$ critical values (may not be distinct). Now we state the theorem proved by Beardon [1].

Theorem A ([1, Theorem 3.2])**.** *If a polynomial p of degree $n \geq 2$ has more than $\lfloor \frac{n}{2} \rfloor$ distinct critical values (here $\lfloor x \rfloor$ is the integer part of a real number x), then it is prime. In particular, if p has $n - 1$ distinct critical values, then it is prime.*

If p is composite, then p has at most $n - 2$ distinct critical values by Theorem A, and this is equivalent to saying that $\Phi(w) = 0$ has a repeated root or equivalently,

$$\Psi(a_1, \cdots, a_{n-1}) := \text{Res}(\Phi, \Phi') = 0.$$

Let $\mathcal{W}_n = \{(a_1, \cdots, a_{n-1}) \in \mathbb{C}^{n-1} : \Psi(a_1, \cdots, a_{n-1}) = 0\}$. Then the set $\mathcal{C}_n := \{(a_1, \cdots, a_{n-1}) \in \mathbb{C}^{n-1} : z^n + a_{n-1}z^{n-1} + \cdots + a_1 z \text{ is composite}\}$ is contained in $\mathcal{W}_n$.

Now we are going to obtain a density estimate on $\mathcal{C}_n$. Let P_n be the set of normalized polynomials of degree n, that is, $P_n = \{p : p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z, a_i \in \mathbb{C}\}$. Thus P_n can be identified with $\mathbb{C}^{n-1} = \{(a_1, \cdots, a_{n-1}) : a_i \in \mathbb{C}\}$. Let $P(R)$ be the polycylinder defined by $\{\mathbf{a} = (a_1, \cdots, a_{n-1}) \in P_n : |a_i| < R, i = 1, \cdots, n - 1\}$. To obtain the volume of $P(R)$, we consider the standard volume on $\mathbb{C}^{n-1} = \mathbb{R}^{2n-2}$ for P_n. Let

$a_j = x_j + iy_j$ and $D_j(R) = \{(x_j, y_j) \in \mathbb{R}^2 : x_j^2 + y_j^2 < R^2\}$. Then we have

$$\begin{aligned}\mathrm{Vol}(P(R)) &= \int_{P(R)} d\mathbf{a} = \int_{|a_{n-1}|<R} \cdots \int_{|a_1|<R} da_1 \cdots da_{n-1} \\ &= \left(\int_{|a_1|<R} da_1\right) \cdots \left(\int_{|a_{n-1}|<R} da_{n-1}\right) \\ &= \left(\int_{D_1(R)} dx_1 dy_1\right) \cdots \left(\int_{D_{n-1}(R)} dx_{n-1} dy_{n-1}\right) \\ &= (\pi R^2)^{n-1}.\end{aligned}$$

Let $\mathcal{S}$ be any subset of $\mathcal{W}_n$ and let ρ be any positive real number, define

$$U_\rho(\mathcal{S}) = \bigcup_{f_0 \in \mathcal{S}} U_\rho(f_0),$$

where $U_\rho(f_0) = \{f \in P_n : |f'(0) - f_0'(0)| < \rho, f''(z) = f_0''(z) \text{ for all } z\}$. Clearly, $U_\rho(\mathcal{C}_n) \subset U_\rho(\mathcal{W}_n)$. Now we can state our main result.

Theorem 1. *For any $R > \rho > 0$,*

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{\mathrm{Vol}(U_\rho(\mathcal{W}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{n(n-2)\rho^2}{R^2}. \tag{2.1}$$

Remark 1. *By comparing the exponents of ρ in (1.1) and (2.1), for a fixed positive integer n, the upper bound in the estimate in Theorem 1 is much smaller than the one in Smale's estimate for sufficiently small $\rho > 0$.*

Remark 2. *We shall see in Section 3 that the constant $\frac{n(n-2)}{R^2}$ in the estimate*

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{n(n-2)\rho^2}{R^2}$$

is far from being sharp because $\mathcal{C}_n$ is in general a small subset of $\mathcal{W}_n$.

To prove Theorem 1, we need the following lemma.

Lemma 1. *The subset $\mathcal{W}_n \subset P_n$ is a complex algebraic hypersurface defined by the polynomial equation $\Psi(a_1, \cdots, a_{n-1}) = 0$, where Ψ is a polynomial of degree $n(n-2)$ in a_1.*

Proof of Lemma 1. Let us recall the definition of the resultant. For any two polynomials $u(z) = u_m z^m + u_{m-1} z^{m-1} + \cdots + u_0$ and $v(z) = v_n z^n + \cdots + v_0$, the resultant $\mathrm{Res}(u(z), v(z))$ of u and v is defined to be the determinant of the following $(m+n) \times (m+n)$ matrix

$$\begin{bmatrix} u_m & u_{m-1} & \cdots & u_1 & u_0 & 0 & \cdots & 0 \\ 0 & u_m & \ddots & u_2 & u_1 & u_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_m & u_{m-1} & u_{m-2} & \cdots & u_0 \\ v_n & v_{n-1} & \cdots & \cdots & v_0 & 0 & \cdots & 0 \\ 0 & v_n & \ddots & \cdots & v_1 & v_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n & v_{n-1} & v_{n-2} & \cdots & v_0 \end{bmatrix}.$$

As

$$p(z) - w = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z - w$$

and

$$p'(z) = n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + 2a_2 z + a_1,$$

we can see that $\Phi(w) := \mathrm{Res}(p - w, p')$ is the determinant of the following $(2n-1) \times (2n-1)$ matrix

$$\begin{bmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 & -w & 0 & \cdots & 0 \\ 0 & 1 & \ddots & a_3 & a_2 & a_1 & -w & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & -w \\ n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & 0 & \cdots & \cdots & 0 \\ 0 & n & \ddots & 3a_3 & 2a_2 & a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n & (n-1)a_{n-1} & \cdots & \cdots & a_1 & 0 \\ 0 & 0 & \cdots & 0 & n & (n-1)a_{n-1} & \cdots & \cdots & a_1 \end{bmatrix}. \quad (2.2)$$

Clearly, $\Phi(w)$ is a polynomial in w of degree $n-1$ whose leading coefficient is $(-1)^{n-1}n^n$, i.e.,

$$\Phi(w) = \sum_{i=0}^{n-1} F_i(a_1, \cdots, a_{n-1})w^i,$$

where $F_{n-1}(a_1, \cdots, a_{n-1}) = (-1)^{n-1}n^n$.

To find the coefficient of a_1^n in F_0, we consider the determinant of the matrix in (2.2). By subtracting the $(n-1+i)^{th}$ row from the i^{th} row for the determinant of the matrix in (2.2) $(i = 1, \cdots, n-1)$, we can see that $\Phi(w)$ is the determinant of the following $(2n-1) \times (2n-1)$ matrix

$$\begin{bmatrix}
1-n & (2-n)a_{n-1} & \cdots & -a_2 & 0 & -w & 0 & \cdots & 0 \\
0 & 1-n & \ddots & -2a_3 & -a_2 & 0 & -w & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1-n & (2-n)a_{n-1} & (3-n)a_{n-2} & (4-n)a_{n-3} & \cdots & -w \\
n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & 0 & \cdots & \cdots & 0 \\
0 & n & \ddots & 3a_3 & 2a_2 & a_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & n & (n-1)a_{n-1} & \cdots & \cdots & a_1 & 0 \\
0 & 0 & \cdots & 0 & n & (n-1)a_{n-1} & \cdots & \cdots & a_1
\end{bmatrix}.$$

It is easily seen that

$$F_0(a_1, a_2, \cdots, a_{n-1}) = (-1)^{n-1}(n-1)^{n-1}a_1^n + \sum_{i=0}^{n-1} G_i(a_2, \cdots, a_{n-1})a_1^i, \quad (2.3)$$

where G_i is a polynomial in the variables $a_2, \cdots, a_{n-1}$.

Now we show that for $i = 1, \cdots, n-2$, each term of F_i must involve some a_j for $j = 2, \cdots, n-1$. To prove this, consider

$$
\begin{aligned}
&\Phi|_{a_2=\cdots=a_{n-1}=0}(w)\\
&= \begin{vmatrix}
1-n & 0 & \cdots & 0 & 0 & -w & 0 & \cdots & 0\\
0 & 1-n & \ddots & 0 & 0 & 0 & -w & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 1-n & 0 & 0 & 0 & \cdots & -w\\
n & 0 & \cdots & 0 & a_1 & 0 & \cdots & \cdots & 0\\
0 & n & \ddots & 0 & 0 & a_1 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & n & 0 & \cdots & \cdots & a_1 & 0\\
0 & 0 & \cdots & 0 & n & 0 & \cdots & \cdots & a_1
\end{vmatrix}_{(2n-1)\times(2n-1)}\\
&= a_1 \begin{vmatrix} (1-n)I_{n-1} & -wJ_{n-1}\\ nI_{n-1} & a_1 I_{n-1}\end{vmatrix} + (-1)^{n-1} n \begin{vmatrix} (1-n)I_{n-1} & -wI_{n-1}\\ nI_{n-1} & a_1 J_{n-1}^T\end{vmatrix},
\end{aligned}
$$

where the last equality holds by expanding the last row of the above determinant and here I_m and J_m denote the $m \times m$ identity matrix and the $m \times m$ Jordan block with eigenvalues 0 respectively. Using the fact that $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - ACA^{-1}B)$ if $A, B, C, D \in \mathbb{C}^{m\times m}$ and A is invertible, we have

$$
\begin{aligned}
&\Phi|_{a_2=\cdots=a_{n-1}=0}(w)\\
&= a_1 \det[a_1(1-n)I_{n-1} + wnJ_{n-1}] + (-1)^{n-1} n \det[a_1(1-n)J_{n-1}^T + wnI_{n-1}]\\
&= (1-n)^{n-1}a_1^n + (-1)^{n-1}w^{n-1}n^n.
\end{aligned}
$$

In particular,

$$
F_j(a_1, 0, \cdots, 0) = 0, \text{ for } j = 1, \cdots n-2.
$$

Therefore, for $j = 1, \cdots, n-2$,

$$
F_j(a_1, a_2, \cdots, a_{n-1}) = \sum_{s_2+\cdots+s_{n-1}\geq 1} H^j_{s_2,\cdots,s_{n-1}}(a_1) a_2^{s_2} \cdots a_{n-1}^{s_{n-1}}, \tag{2.4}
$$

where $H^j_{s_2,\cdots,s_{n-1}}$ is a polynomial in the variable a_1.

As

$$\Phi(w) = F_{n-1}w^{n-1} + \cdots + F_0$$

and

$$\Phi'(w) = (n-1)F_{n-1}w^{n-2} + (n-2)F_{n-2}w^{n-3} + \cdots + F_1,$$

it follows that $\Psi(a_1, \cdots, a_{n-1})$ is the determinant of the following $(2n-3)\times(2n-3)$ matrix

$$\begin{bmatrix} F_{n-1} & F_{n-2} & \cdots & F_2 & F_1 & F_0 & 0 & \cdots & 0 \\ 0 & F_{n-1} & \ddots & F_3 & F_2 & F_1 & F_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{n-1} & F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_0 \\ (n-1)F_{n-1} & (n-2)F_{n-2} & \cdots & 2F_2 & F_1 & 0 & \cdots & \cdots & 0 \\ 0 & (n-1)F_{n-1} & \ddots & 3F_3 & 2F_2 & F_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (n-1)F_{n-1} & (n-2)F_{n-2} & \cdots & \cdots & F_1 & 0 \\ 0 & 0 & \cdots & 0 & (n-1)F_{n-1} & (n-2)F_{n-2} & \cdots & \cdots & F_1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &\Psi(a_1, \cdots, a_{n-1}) \\ &= (n-1)^{n-1}F_{n-1}^{n-1}F_0^{n-2} \\ &\quad + \sum_{r_1+\cdots+r_{n-2}\geq 1} P_{r_1,\cdots,r_{n-2}}(F_0, F_{n-1})F_1^{r_1}\cdots F_{n-2}^{r_{n-2}}, \end{aligned} \tag{2.5}$$

where $P_{r_1,\cdots,r_{n-2}}$ is a polynomial in the variables F_0, F_{n-1}. By (2.3) and (2.4), the first term in (2.5) is

$$\begin{aligned} &(n-1)^{n-1}F_{n-1}^{n-1}F_0^{n-2} \\ &= (-1)^{n-1}(n-1)^{(n-1)^2}n^{n(n-1)}a_1^{n(n-2)} + \sum_{i=0}^{n(n-2)-1} T_i(a_2, \cdots, a_{n-1})a_1^i \end{aligned}$$

and the second term in (2.5) is

$$\begin{aligned} &\sum_{r_1+\cdots+r_{n-2}\geq 1} P_{r_1,\cdots,r_{n-2}}(F_0, F_{n-1})F_1^{r_1}\cdots F_{n-2}^{r_{n-2}} \\ &= \sum_{t_2+\cdots+t_{n-1}\geq 1} Q_{t_2,\cdots,t_{n-1}}(a_1)a_2^{t_2}\cdots a_{n-1}^{t_{n-1}}, \end{aligned}$$

where T_i is a polynomial in the variables $a_2, \cdots, a_{n-1}$ and $Q_{t_2,\cdots,t_{n-1}}$ is a polynomial in a_1 only. Then

$$\begin{aligned}
&\Psi(a_1, \cdots, a_{n-1}) \\
&= (-1)^{n-1}(n-1)^{(n-1)^2} n^{n(n-1)} a_1^{n(n-2)} + \sum_{i=0}^{n(n-2)-1} T_i(a_2, \cdots, a_{n-1}) a_1^i \\
&+ \sum_{t_2+\cdots+t_{n-1}\geq 1} Q_{t_2,\cdots,t_{n-1}}(a_1) a_2^{t_2} \cdots a_{n-1}^{t_{n-1}}. \qquad (2.6)
\end{aligned}$$

In particular, we can see that Ψ has degree at least $n(n-2)$ in a_1.

On the other hand, by expressing Ψ in term of the zeros $w_1, \cdots, w_{n-1}$ of $\Phi(w)$, that is,

$$\Psi(a_1, \cdots, a_{n-1}) = F_{n-1}^{2n-4} \prod_{i<j} (w_i - w_j)^2,$$

we can show that Ψ has degree at most $n(n-2)$ in a_1. This suffices to show that for any fixed $a_2, \cdots, a_{n-1}$,

$$\Psi_{a_2,\cdots,a_{n-1}}(a_1) = \Psi(a_1, \cdots, a_{n-1}) \leq O(|a_1|^{n(n-2)}).$$

To prove this, we need to use a theorem which gives an upper bound for the zeros of a polynomial in terms of the coefficients of the polynomial. To state this result, we need the following

Definition ([12, Definition 8.1.2])**.** *Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial of degree $n \geq 1$. Then the Cauchy bound of f, denoted by $\rho[f]$, is defined as the unique positive root of the equation $|c_0| + |c_1|x + \cdots |c_{n-1}|x^{n-1} = |c_n|x^n$ when f is not a monomial, and as zero otherwise (the uniqueness of the root was proved in [12, Lemma 8.1.1]).*

Theorem B ([12, Corollary 8.1.8])**.** *If $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, where $c_n \neq 0$, then*

$$\rho[f] \leq \max_{0\leq\nu\leq n-1} \left(n \left| \frac{c_\nu}{c_n} \right| \right)^{\frac{1}{n-\nu}}.$$

Remark 3. *Notice that all the zeros of the non-constant polynomial f lie in the closed disk with centre at the origin and radius $\rho[f]$ (see [12, Theorem 8.1.3]).*

Let r_i $(i = 1, \cdots, n-1)$ be the zeros (which may not be distinct) of p'. Applying Theorem B to $f(z) = p'(z) = nz^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + 2a_2 z + a_1$ for any fixed $a_2, \cdots, a_n$, then we have

$$|r_i| \leq \rho[p'] \leq \max_{0 \leq \nu \leq n-2} \left(\frac{(\nu+1)(n-1)}{n} |a_{\nu+1}| \right)^{\frac{1}{(n-1)-\nu}} \leq O(|a_1|^{\frac{1}{n-1}}).$$

Note that $p(r_i)$ is a critical value of p and therefore $\Phi(w) = 0$ if and only if $w = p(r_i)$. Hence

$$\Psi(a_1, \cdots, a_{n-1}) = F_{n-1}^{2n-4} \prod_{i<j} (p(r_i) - p(r_j))^2 = n^{2n(n-2)} \prod_{i<j} (p(r_i) - p(r_j))^2.$$

As

$$(p(r_i) - p(r_j))^2 \leq O(|a_1|^{\frac{2n}{n-1}})$$

and there are exactly $\frac{(n-1)(n-2)}{2}$ distinct pairs of $p(r_i) - p(r_j)$ for $i < j$, we have

$$\begin{aligned} \Psi_{a_2, \cdots, a_{n-1}}(a_1) &= \Psi(a_1, \cdots, a_{n-1}) \\ &= n^{2n(n-2)} \prod_{i<j} (p(r_i) - p(r_j))^2 \leq O(|a_1|^{n(n-2)}), \end{aligned} \quad (2.7)$$

for any fixed $a_2, \cdots, a_n$.

By (2.6) and (2.7), Ψ has degree $n(n-2)$ in a_1, more precisely, there exist polynomials $R_0, \cdots, R_{n(n-2)} \in \mathbb{C}[a_2, \cdots, a_{n-1}]$ such that

$$\Psi(a_1, \cdots, a_{n-1}) = \sum_{i=0}^{n(n-2)} R_i(a_2, \cdots, a_{n-1}) a_1^i,$$

where $R_{n(n-2)} \not\equiv 0$. Therefore $\mathcal{W}_n$ is the complex hypersurface defined by the polynomial equation $\Psi = 0$, where Ψ is of degree $n(n-2)$ in a_1. This completes the proof of Lemma 1. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $\chi : \mathbb{C}^{n-1} \to \{0,1\}$ be the characteristic function of $U_\rho(\mathcal{W}_n)$. By Lemma 1, we observe that for a generic $(a_2, \cdots, a_{n-1}) \in \mathbb{C}^{n-2}$, the intersection of $\mathcal{W}_n$ with the one dimensional coordinate plane $\{(z, a_2, \cdots, a_{n-1}) : z \in \mathbb{C}\}$ consists of at most $n(n-2)$ points. Hence we have

$$\begin{aligned}\left|\int_{|a_1|<R} \chi(a_1, a_2, \cdots, a_{n-1}) da_1\right| &\leq \left|\int_{|a_1|<\infty} \chi(a_1, a_2, \cdots, a_{n-1}) da_1\right| \\ &\leq n(n-2)\pi\rho^2.\end{aligned}$$

By the Fubini's theorem,

$$\begin{aligned}&\frac{\mathrm{Vol}(U_\rho(\mathcal{W}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \\ &= \frac{1}{(\pi R^2)^{n-1}} \int_{P(R)} \chi(\mathbf{a}) d\mathbf{a} \\ &= \frac{1}{(\pi R^2)^{n-1}} \int_{|a_2|, \cdots, |a_{n-1}|<R} \left[\int_{|a_1|<R} \chi(a_1, \cdots, a_{n-1}) da_1\right] da_2 \cdots da_{n-1} \\ &\leq \frac{1}{(\pi R^2)^{n-1}} \int_{|a_2|, \cdots, |a_{n-1}|<R} [n(n-2)\pi\rho^2] da_2 \cdots da_{n-1} \\ &= \frac{1}{(\pi R^2)^{n-1}} [n(n-2)\pi\rho^2](\pi R^2)^{n-2} = \frac{n(n-2)\rho^2}{R^2}.\end{aligned}$$

Since, $U_\rho(\mathcal{C}_n) \subset U_\rho(\mathcal{W}_n)$, we have

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{\mathrm{Vol}(U_\rho(\mathcal{W}_n) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{n(n-2)\rho^2}{R^2}.$$

□

3. Composite polynomials of small degrees

3.1. Degree 4 polynomials By considering composite polynomials of degree 4, we shall see that the density estimate of $\mathcal{C}_4$ in Theorem 1 is not sharp. In fact, by Theorem 1, we have

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_4) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{8\rho^2}{R^2}.$$

However, we actually have the following

Proposition 1.

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_4)\cap P(R))}{\mathrm{Vol}(P(R))} \le \frac{\rho^2}{R^2}.$$

To see this, note that if $p(z) = z^4 + a_3z^3 + a_2z^2 + a_1z$ is composite, then $\Psi(a_1, a_2, a_3) = 0$. By using mathematical software such as Mathematica, we obtain

$$\begin{aligned}\Psi(a_1, a_2, a_3) = &-4096(a_3^3 - 4a_3a_2 + 8a_1)^2 \\ &\times(108a_1^2 - 108a_3a_2a_1 + 27a_3^3a_1 + 32a_2^3 - 9a_3^2a_2^2)^3.\end{aligned}$$

From the proof of Theorem 1, we know that the upper bound $\frac{8\rho^2}{R^2}$ comes from the fact that $\Psi(a_1, a_2, a_3)$ is of degree 8 in a_1. We can get a much better bound $\frac{\rho^2}{R^2}$ by showing that $p(z)$ is composite if and only if $a_3^3 - 4a_3a_2 + 8a_1 = 0$.

To prove this, suppose that p is composite, then there exist some $A, B \in \mathbb{C}$ such that

$$z^4+a_3z^3+a_2z^2+a_1z = (z^2+Az)\circ(z^2+Bz) = z^4+2Bz^3+(A+B^2)z^2+ABz.$$

Comparing the coefficients, we have $a_3 = 2B, a_2 = A + B^2, a_1 = AB$. After eliminations of A and B, we have $a_3^3 - 4a_3a_2 + 8a_1 = 0$.

Conversely, suppose that $a_3^3 - 4a_3a_2 + 8a_1 = 0$. Then

$$\begin{aligned}(z^2 + (a_2 - \frac{a_3^2}{4})z)\circ(z^2 + \frac{a_3}{2}z) &= z^4 + a_3z^3 + a_2z^2 + (a_2 - \frac{a_3^2}{4})(\frac{a_3}{2})z \\ &= z^4 + a_3z^3 + a_2z^2 + a_1z.\end{aligned}$$

So p is composite. Hence $\mathcal{C}_4 = \{(a_1, a_2, a_3) \in \mathbb{C}^3 : a_3^3 - 4a_3a_2 + 8a_1 = 0\}$.

Proof of Proposition 1. Note that $a_3^3 - 4a_3a_2 + 8a_1$ is of degree 1 in a_1, hence for any $(a_2, a_3) \in \mathbb{C}^2$, the intersection of $\mathcal{C}_4$ with the one dimensional coordinate plane $\{(z, a_2, a_3) : z \in \mathbb{C}\}$ consists of exactly one point. Hence we have

$$\left|\int_{|a_1|<R}\chi(a_1,a_2,a_3)da_1\right| \le \left|\int_{|a_1|<\infty}\chi(a_1,a_2,a_3)da_1\right| = \pi\rho^2.$$

It then follows from the proof of Theorem 1 that

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_4)\cap P(R))}{\mathrm{Vol}(P(R))} \le \frac{\rho^2}{R^2}.$$

This proves Proposition 1. □

Remark 4. *As* $\mathcal{C}_4 \subset \mathcal{W}_4$, $a_3^3-4a_3a_2+8a_1=0$ *implies that* $\Psi(a_1,a_2,a_3)=0$. *So there should be certain relation between* $a_3^3-4a_3a_2+8a_1$ *and* Ψ. *In fact, recall that*

$$\begin{aligned}\Psi(a_1,a_2,a_3) &= -4096(a_3^3-4a_3a_2+8a_1)^2\\ &\quad\times(108a_1^2-108a_3a_2a_1+27a_3^3a_1+32a_2^3-9a_3^2a_2^2)^3.\end{aligned}$$

It follows that $a_3^3-4a_3a_2+8a_1$ *is a factor of* Ψ *with multiplicity* 2.

3.2. Degree 6 polynomials By considering composite polynomials of degree 6, we also see that the density estimate of $\mathcal{C}_6$ in Theorem 1 is not sharp. In fact, by Theorem 1, we have

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_6)\cap P(R))}{\mathrm{Vol}(P(R))} \le \frac{24\rho^2}{R^2}.$$

However, we actually have the following

Proposition 2.

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_6)\cap P(R))}{\mathrm{Vol}(P(R))} \le \frac{2\rho^2}{R^2}.$$

To see this, note that if $p(z)=z^6+a_5z^5+a_4z^4+a_3z^3+a_2z^2+a_1z$ is composite, then $\Psi(a_1,a_2,a_3,a_4,a_5)=0$. By using mathematical software such as Mathematica, Ψ can be factorized to the following form:

$$\Psi(a_1,a_2,a_3,a_4,a_5) = C[q(a_1,a_2,a_3,a_4,a_5)]^3[r(a_1,a_2,a_3,a_4,a_5)]^2,$$

for some constant C and some polynomials $q, r \in \mathbb{C}[a_1, a_2, a_3, a_4, a_5]$ such that q has degree 4 in a_1 and r has degree 6 in a_1. From the proof of Theorem 1, we know that the upper bound $\frac{24\rho^2}{R^2}$ comes from the fact that $\Psi(a_1, a_2, a_3)$ is of degree 24 in a_1. We can get a much better bound $\frac{2\rho^2}{R^2}$ by showing the following

Lemma 2. *The polynomial* $p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$ *is composite if and only if*

$$\begin{cases} 5a_5^3 + 27a_3 - 18a_5 a_4 = 0 \\ a_5^5 - 3a_5^3 a_4 + 27a_5 a_2 - 81a_1 = 0 \end{cases} \tag{3.1}$$

or

$$\begin{cases} a_5^5 - 8a_5^3 a_4 + 8a_5^2 a_3 + 16a_5 a_4^2 - 32a_4 a_3 + 64a_1 = 0 \\ 5a_5^4 - 24a_5^2 a_4 + 32a_5 a_3 + 16a_4^2 - 64a_2 = 0 \end{cases} . \tag{3.2}$$

Proof of Lemma 2. Suppose that $p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$ is composite, without loss of generality, we only need to consider the following two different kinds of factorizations:

1. $z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z = (z^3 + Az^2 + Bz) \circ (z^2 + Cz)$;
2. $z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z = (z^2 + Az) \circ (z^3 + Bz^2 + Cz)$.

For case 1,

$$\begin{aligned} &z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z \\ =\ &z^6 + 3Cz^5 + (3C^2 + A)z^4 + (C^3 + 2AC)z^3 + (AC^2 + B)z^2 + BCz. \end{aligned}$$

Comparing the coefficients, we have

$$\begin{cases} a_5 = 3C \\ a_4 = 3C^2 + A \\ a_3 = C^3 + 2AC \\ a_2 = AC^2 + B \\ a_1 = BC \end{cases} .$$

After eliminations, we obtain two equations $5a_5^3 + 27a_3 - 18a_5a_4 = 0$ and $a_5^5 - 3a_5^3a_4 + 27a_5a_2 - 81a_1 = 0$.

Conversely, suppose that $5a_5^3 + 27a_3 - 18a_5a_4 = 0$ and $a_5^5 - 3a_5^3a_4 + 27a_5a_2 - 81a_1 = 0$. Then

$$\begin{aligned}
&(z^3 + (a_4 - \frac{a_5^2}{3})z^2 + (a_2 - \frac{a_5^2a_4}{9} + \frac{a_5^4}{27})z) \circ (z^2 + \frac{a_5}{3}z) \\
&= z^6 + a_5z^5 + a_4z^4 + (-\frac{5a_5^3}{27} + \frac{2a_5a_4}{3})z^3 + a_2z^2 + (\frac{a_5^5}{81} - \frac{a_5^3a_4}{27} + \frac{a_5a_2}{3})z \\
&= z^6 + a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z.
\end{aligned}$$

So p is composite.

For case 2,

$$\begin{aligned}
&z^6 + a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z \\
&= z^6 + 2Bz^5 + (2C + B^2)z^4 + (A + 2BC)z^3 + (C^2 + AB)z^2 + ACz.
\end{aligned}$$

Comparing the coefficients, we have

$$\begin{cases}
a_5 = 2B \\
a_4 = 2C + B^2 \\
a_3 = A + 2BC \\
a_2 = C^2 + AB \\
a_1 = AC
\end{cases}.$$

After eliminations, we obtain two equations $a_5^5 - 8a_5^3a_4 + 8a_5^2a_3 + 16a_5a_4^2 - 32a_4a_3 + 64a_1 = 0$ and $5a_5^4 - 24a_5^2a_4 + 32a_5a_3 + 16a_4^2 - 64a_2 = 0$.

Conversely, suppose that $a_5^5 - 8a_5^3a_4 + 8a_5^2a_3 + 16a_5a_4^2 - 32a_4a_3 + 64a_1 = 0$ and $5a_5^4 - 24a_5^2a_4 + 32a_5a_3 + 16a_4^2 - 64a_2 = 0$. Then

$$\begin{aligned}
&(z^2 + (a_3 - \frac{a_5a_4}{2} + \frac{a_5^3}{8})z) \circ (z^3 + \frac{a_5}{2}z^2 + (\frac{a_4}{2} - \frac{a_5^2}{8})z) \\
&= z^6 + a_5z^5 + a_4z^4 + a_3z^3 + (\frac{5a_5^4}{64} - \frac{3a_5^2a_4}{8} + \frac{a_4^2}{4} + \frac{a_5a_3}{2})z^2 \\
&\quad +(-\frac{a_5^5}{64} + \frac{a_5^3a_4}{8} - \frac{a_5a_4^2}{4} - \frac{a_5^2a_3}{8} + \frac{a_4a_3}{2})z \\
&= z^6 + a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z.
\end{aligned}$$

So p is composite. $\square$

Let

$$\begin{aligned}
R_1^{(1)}(a_1,a_2,a_3,a_4,a_5) &= 5a_5^3+27a_3-18a_5a_4,\\
R_2^{(1)}(a_1,a_2,a_3,a_4,a_5) &= a_5^5-3a_5^3a_4+27a_5a_2-81a_1,\\
R_1^{(2)}(a_1,a_2,a_3,a_4,a_5) &= a_5^5-8a_5^3a_4+8a_5^2a_3+16a_5a_4^2-32a_4a_3+64a_1,\\
R_2^{(2)}(a_1,a_2,a_3,a_4,a_5) &= 5a_5^4-24a_5^2a_4+32a_5a_3+16a_4^2-64a_2.
\end{aligned}$$

By Lemma 2,

$$\begin{aligned}
\mathcal{C}_6 &= \{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5 : R_1^{(1)}=R_2^{(1)}=0 \text{ or } R_1^{(2)}=R_2^{(2)}=0\}\\
&= \{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5 : R_1^{(1)}=R_2^{(1)}=0\}\\
&\quad \cup\{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5 : R_1^{(2)}=R_2^{(2)}=0\}\\
&= \mathcal{C}_6^{(1)}\cup\mathcal{C}_6^{(2)},
\end{aligned}$$

where $\mathcal{C}_6^{(1)} = \{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5 : R_1^{(1)}=R_2^{(1)}=0\}$ and $\mathcal{C}_6^{(2)} = \{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{C}^5 : R_1^{(2)}=R_2^{(2)}=0\}$.

Now we look at some examples of composite polynomials of degree 6.

1. Let $p(z)=z^6+2z^4+z^2$. It is easily seen that $(0,1,0,2,0)\in\mathbb{C}^5$ satisfies both (3.1) and (3.2). Hence $(0,1,0,2,0)\in\mathcal{C}_6^{(1)}\cap\mathcal{C}_6^{(2)}\subset\mathcal{C}_6$. Therefore, there are two different kinds of factorizations for p:

$$z^6+2z^4+z^2=(z^3+2z^2+z)\circ z^2=z^2\circ(z^3+z).$$

2. Let $p(z)=z^6+z^4+z^2$. It is easy to check that $(0,1,0,1,0)\in\mathbb{C}^5$ satisfies (3.1), but not (3.2). Hence $(0,1,0,1,0)\in\mathcal{C}_6^{(1)}\setminus\mathcal{C}_6^{(2)}\subset\mathcal{C}_6$. Therefore,

$$z^6+z^4+z^2=(z^3+z^2+z)\circ z^2$$

and $z^6+z^4+z^2$ cannot be written in the form $(z^2+Az)\circ(z^3+Bz^2+Cz)$.

3. Let $p(z) = z^6 + 2z^4 + z^3 + z^2 + z$. It is easy to verify that $(1, 1, 1, 2, 0) \in \mathbb{C}^5$ satisfies (3.2), but not (3.1). Hence $(1, 1, 1, 2, 0) \in \mathcal{C}_6^{(2)} \setminus \mathcal{C}_6^{(1)} \subset \mathcal{C}_6$. Therefore,

$$z^6 + 2z^4 + z^3 + z^2 + z = (z^2 + z) \circ (z^3 + z)$$

and $z^6 + 2z^4 + z^3 + z^2 + z$ cannot be written in the form $(z^3 + Az^2 + Bz) \circ (z^2 + Cz)$.

From the above examples, we have $\mathcal{C}_6 = \mathcal{C}_6^{(1)} \cup \mathcal{C}_6^{(2)}$, where $\mathcal{C}_6^{(1)} \cap \mathcal{C}_6^{(2)} \neq \emptyset$, $\mathcal{C}_6^{(1)} \setminus \mathcal{C}_6^{(2)} \neq \emptyset$ and $\mathcal{C}_6^{(2)} \setminus \mathcal{C}_6^{(1)} \neq \emptyset$.

Proof of Proposition 2. It follows from (3.1) and (3.2) in Lemma 2 that for any $(a_2, a_3, a_4, a_5) \in \mathbb{C}^4$, there exists at most one $a_1' \in \mathbb{C}$ such that

$$(a_1', a_2, a_3, a_4, a_5) \in \mathcal{C}_6^{(1)},$$

and similarly there exists at most one $a_1'' \in \mathbb{C}$ such that

$$(a_1'', a_2, a_3, a_4, a_5) \in \mathcal{C}_6^{(2)}.$$

Therefore, the intersection of $\mathcal{C}_6$ with the one dimensional coordinate plane

$$\{(z, a_2, a_3, a_4, a_5) : z \in \mathbb{C}\}$$

consists of at most two points. Hence we have

$$\left| \int_{|a_1|<R} \chi(a_1, a_2, a_3, a_4, a_5) da_1 \right| \leq \left| \int_{|a_1|<\infty} \chi(a_1, a_2, a_3, a_4, a_5) da_1 \right| \leq 2\pi\rho^2.$$

It then follows from the proof of Theorem 1 that

$$\frac{\mathrm{Vol}(U_\rho(\mathcal{C}_6) \cap P(R))}{\mathrm{Vol}(P(R))} \leq \frac{2\rho^2}{R^2}.$$

This proves Proposition 2. □

Remark 5. *As $\mathcal{C}_6 \subset \mathcal{W}_6$, $R_1^{(1)} = R_2^{(1)} = 0$ or $R_1^{(2)} = R_2^{(2)} = 0$ implies that $\Psi = 0$. So there should be certain relation between $R_1^{(1)}, R_2^{(1)}, R_1^{(2)}, R_2^{(2)}$ and Ψ. Now we discuss such relation. Recall that Ψ can be factorized to the following form:*

$$\Psi(a_1, a_2, a_3, a_4, a_5) = C[q(a_1, a_2, a_3, a_4, a_5)]^3[r(a_1, a_2, a_3, a_4, a_5)]^2,$$

for some constant C and some polynomials $q, r \in \mathbb{C}[a_1, a_2, a_3, a_4, a_5]$. Let $I^{(1)}$ and $I^{(2)}$ be the ideals $< R_1^{(1)}, R_2^{(1)} >$ and $< R_1^{(2)}, R_2^{(2)} >$ generated by $R_1^{(1)}, R_2^{(1)}$ and $R_1^{(2)}, R_2^{(2)}$ respectively. Using mathematical software such as Mathematica, we find a Groebner basis $G^{(1)}$ for $I^{(1)}$ and a Groebner basis $G^{(2)}$ for $I^{(2)}$. When dividing r by $G^{(1)}$ and $G^{(2)}$ respectively, both the remainders are zero. Hence $r \in I^{(1)}$ and $r \in I^{(2)}$ or equivalently,

$$r = r_1^{(1)} R_1^{(1)} + r_2^{(1)} R_2^{(1)} = r_1^{(2)} R_1^{(2)} + r_2^{(2)} R_2^{(2)}$$

for some $r_1^{(1)}, r_2^{(1)}, r_1^{(2)}, r_2^{(2)} \in \mathbb{C}[a_1, a_2, a_3, a_4, a_5]$. Therefore

$$\Psi(a_1, a_2, a_3, a_4, a_5) = Cq^3(r_1^{(1)} R_1^{(1)} + r_2^{(1)} R_2^{(1)})^2 = Cq^3(r_1^{(2)} R_1^{(2)} + r_2^{(2)} R_2^{(2)})^2.$$

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