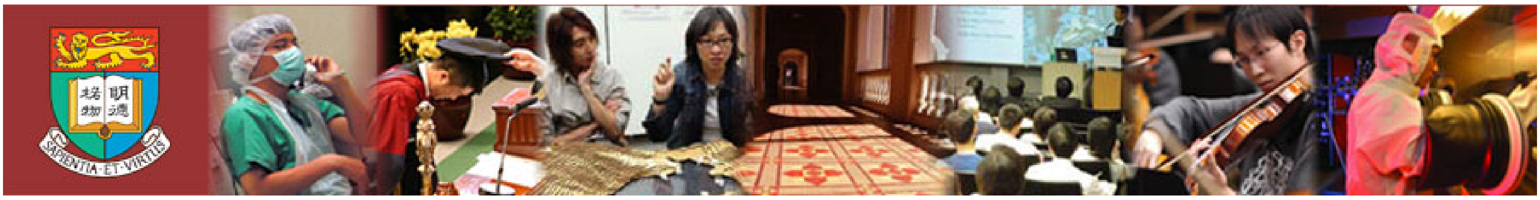

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A Gram-SOS Approach for Robust Stability Analysis of Discrete-Time Systems with Time-Varying Uncertainty

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Abstract— **This paper addresses the problem of establishing robust asymptotical stability of discrete-time systems affected by time-varying parametric uncertainty. Specifically, it is supposed that the coefficients of the system depend linearly on the uncertainty, and that the uncertainty is confined into a polytope. In the continuous-time case, the problem can be addressed by imposing that the system admits a common homogeneous polynomial Lyapunov function (HPLF) at the vertices of the polytope. Unfortunately, such a strategy cannot be used in the discrete-time case since the derivative of the HPLF is nonlinear in the uncertainty. The problem is addressed in this paper through linear matrix inequalities (LMIs) by proposing a novel method for establishing decrease of the HPLF. This method consists, firstly, of introducing a Gram matrix built with respect to the state and parametrized by an arbitrary vector function of the uncertainty, and secondly, of requiring that a transformation of the introduced Gram matrix is a sum of squares (SOS) of matrix polynomials. The proposed method provides a condition for robust asymptotical stability that is sufficient for any degree of the HPLF candidate and that includes quadratic robust stability as special case.**

I. INTRODUCTION

A fundamental and notoriously difficult problem in systems with uncertainty amounts to establishing whether a linear system affected by uncertain parameters is asymptotically stable for all the admissible values of the parameters. Various works have been proposed in the literature for addressing this problem, which can be classified according to different criteria, e.g. based on the nature of the system (such as continuous-time or discrete-time), type of uncertainty (such as time-invariant or time-varying), dependence of the coefficients of the system on the uncertainty (such as linear or rational), and shape of the set of admissible uncertainty (such as multi-interval or polytopic). See e.g. [1], [10] and references therein.

For continuous-time systems, numerous methods have been developed, typically focusing on systems where the coefficients depend linearly on the uncertainty and the uncertainty is confined into a polytope. These methods are generally based on the search for a suitable Lyapunov function that might prove robust asymptotical stability of the system, and the type of uncertainty characterizes the type of Lyapunov function that is searched for. Specifically, in the case of time-invariant uncertainty, pioneering methods have searched for a quadratic Lyapunov function, see e.g. [5], and more recent ones have proposed the use of parameter-dependent quadratic Lyapunov functions in order to reduce the conservatism, see e.g. [10] and references therein. Then, in the case of time-varying uncertainty, results obtained with quadratic Lyapunov functions have been improved by considering nonquadratic Lyapunov functions. See e.g. [14], [16], [19] where piecewise quadratic Lyapunov functions are searched for, [3], [4], [6] where the use of polyhedral and smoothed polyhedral Lyapunov functions is investigated, and [8], [10], [20] which address the construction of homogeneous polynomial Lyapunov functions (HPLFs).

For discrete-time systems, analogous methods have been developed in the case of time-invariant uncertainty, see e.g. [10], [13], [15], [17] and references therein. However, the case of time-varying uncertainty has been less investigated. Indeed, contrary to continuous-time systems where the time derivative of a (quadratic or nonquadratic) common Lyapunov function candidate is linear in the uncertainty and conditions can be derived by checking the vertices of the polytope, one has that the time difference of such a candidate is nonlinear in the uncertainty for discrete-time systems and checking the vertices does not suffice to ensure robust stability. Existing works include [2], [12], [18] where robust stability and robust stabilization are investigated through quadratic Lyapunov functions, set-induced Lyapunov functions, and parameter-dependent quadratic Lyapunov functions, respectively.

This paper addresses the problem of establishing robust asymptotical stability of discrete-time systems affected by time-varying structured uncertainty. Specifically, it is supposed that the coefficients of the system depend linearly on the uncertainty, and that the uncertainty is confined into a polytope. For this problem, a condition based on linear matrix inequalities (LMIs) is presented by proposing a novel method for establishing decrease of the HPLF. This method consists, firstly, of introducing a Gram matrix built with respect to the state and parametrized by an arbitrary vector function of the uncertainty, and secondly, of requiring that a transformation of the introduced Gram matrix is a sum of squares (SOS) of matrix polynomials. The proposed method provides a condition for robust asymptotical stability that is sufficient for any degree of the HPLF candidate and that includes quadratic robust stability as special case. A numerical example illustrates the proposed condition.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries about SOS polynomials. Section III describes the proposed method for establishing robust asymptotical stability. Section IV presents an illustrative example. Lastly, Section V concludes the paper with some final remarks.

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II. PRELIMINARIES

A. Problem Formulation

The notation used throughout the paper is as follows: $\mathbb{N}, \mathbb{R}$: natural and real number sets; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of A; $A > 0$, $A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix A; $\mathrm{he}(A) = A + A'$; $\mathrm{conv}\{a, b, \ldots\}$: convex hull of vectors $a, b, \ldots$; $\mathrm{diag}\{A, B, \ldots\}$: block diagonal matrix with blocks $A, B, \ldots$; $*$: corresponding block in symmetric matrices.

We consider the system

$$\left\{ \begin{array}{l} x(t+1) = A(s(t))x(t) \\ s(t) \in \mathcal{S} \\ t \geq 0 \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{N}$ is the discrete time, $s(t) \in \mathbb{R}^r$ is the time-varying uncertain vector, $A : \mathbb{R}^q \to \mathbb{R}^{n \times n}$ is a linear matrix function expresses as

$$A(s(t)) = \sum_{i=1}^{r} s_i(t) A_i \tag{2}$$

for some given matrices $A_1, \ldots, A_r \in \mathbb{R}^{n \times n}$, and $\mathcal{S}$ is the simplex of dimension r, i.e.

$$\mathcal{S} = \left\{ s \in \mathbb{R}^r : \ \sum_{i=1}^{r} s_i = 1, \ s_i \geq 0 \right\}. \tag{3}$$

Throughout the paper we assume that $p(t)$ ensures the existence of the solution $x(t)$ of the system (1).

Problem. The problem considered in this paper is to establish whether the origin of the system (1) is a robustly asymptotically stable equilibrium point, i.e.

$$\left\{ \begin{array}{l} \forall \varepsilon > 0 \ \exists \delta > 0 : \ \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \\ \qquad\qquad\qquad\qquad \forall t \geq 0 \ \forall s(\cdot) \in \mathcal{S} \\ \lim_{t \to \infty} x(t) = 0_n \ \forall x(0) \in \mathbb{R}^n \ \forall s(\cdot) \in \mathcal{S}. \end{array} \right. \tag{4}$$

B. SOS Polynomials

Here we briefly introduce some preliminaries about SOS polynomials and SOS matrix polynomials, see e.g. [7] and references therein for details.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $2m$. Then, $h(x)$ can be expressed as

$$h(x) = x^{\{m\}'} \left(H + L(\alpha) \right) x^{\{m\}} \tag{5}$$

where $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ is a vector whose entries are the monomials of degree m in x, where $\sigma(n, m)$ is the total number of such monomials given by

$$\sigma(n, m) = \frac{(n + m - 1)!}{(n - 1)! m!}, \tag{6}$$

$H \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a symmetric matrix such that

$$h(x) = x^{\{m\}'} H x^{\{m\}}, \tag{7}$$

$L : \mathbb{R}^{\omega(n,m)} \to \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a linear parametrization of the linear subspace

$$\mathcal{L}(n, m) = \{ L = L' : \ x^{\{m\}'} L x^{\{m\}} = 0 \}, \tag{8}$$

and $\alpha \in \mathbb{R}^{\omega(n,m)}$ is a free vector, where $\omega(n, m)$ is the dimension of $\mathcal{L}(n, m)$ given by

$$\omega(n, m) = \frac{1}{2} \sigma(n, m)(\sigma(n, m) + 1) - \sigma(n, 2m). \tag{9}$$

The representation (5) is known as Gram matrix method and square matricial representation (SMR). This representation was introduced in [11] to establish whether a polynomial is SOS via an LMI feasibility test. Indeed, $h(x)$ is said SOS if there exist polynomials $h_1(x), \ldots, h_k(x)$ such that

$$h(x) = \sum_{i=1}^{k} h_i(x)^2 \tag{10}$$

and this condition holds if and only if there exists α satisfying the LMI

$$H + L(\alpha) \geq 0. \tag{11}$$

These definitions and results have been extended to the case of matrix polynomials. Specifically, let $H : \mathbb{R}^n \to \mathbb{R}^{d \times d}$ be a symmetric matrix homogeneous polynomial of degree $2m$. Then, $H(x)$ can be expressed as

$$H(x) = \left(x^{\{m\}} \otimes I \right)' (J + L(\alpha)) \left(x^{\{m\}} \otimes I \right) \tag{12}$$

where the identity matrix has size $d \times d$, $J \in \mathbb{R}^{d\sigma(n,m) \times d\sigma(n,m)}$ is a symmetric matrix such that

$$H(x) = \left(x^{\{m\}} \otimes I \right)' J \left(x^{\{m\}} \otimes I \right), \tag{13}$$

$L : \mathbb{R}^{\omega(n,m,d)} \to \mathbb{R}^{d\sigma(n,m) \times d\sigma(n,m)}$ is a linear parametrization of the linear subspace

$$\mathcal{L}(n, m, d) = \{ L = L' : \ \left(x^{\{m\}} \otimes I \right)' L \left(x^{\{m\}} \otimes I \right) = 0 \}, \tag{14}$$

and $\alpha \in \mathbb{R}^{\omega(n,m,d)}$ is a free vector, where $\omega(n, m, d)$ is the dimension of $\mathcal{L}(n, m, d)$ given by

$$\omega(n, m, d) = \frac{1}{2} d \left(\sigma(n, m) \left(d\sigma(n, m) + 1 \right) - (d + 1)\sigma(n, 2m) \right). \tag{15}$$

The representation (12) was introduced in [9] to establish whether a symmetric matrix polynomial is SOS via an LMI feasibility test. Indeed, $H(x)$ is said SOS if there exist matrix polynomials $H_1(x), \ldots, H_k(x)$ such that

$$H(x) = \sum_{i=1}^{k} H_i(x)' H_i(x) \tag{16}$$

and this condition holds if and only if there exists α satisfying the LMI

$$J + L(\alpha) \geq 0. \tag{17}$$

III. PROPOSED RESULTS

Let us search for a Lyapunov function proving that the origin of the system (1) is a robustly asymptotically stable equilibrium point. This can be done by searching for a function $v:\mathbb{R}^n\to\mathbb{R}$ such that

$$\left.\begin{array}{rcl} v(x) & > & 0 \\ \Delta v(x,s) & < & 0 \end{array}\right\} \quad \forall x\in\mathbb{R}_0^n \ \forall s\in\mathcal{S} \tag{18}$$

where

$$\Delta v(x,s) = v\left(A(s)x\right) - v(x). \tag{19}$$

If such a function exists, then $v(x)$ is a Lyapunov function for the origin of the system (1) common to all admissible uncertainties, i.e., a common Lyapunov function. In such a case, the origin of the system (1) is, hence, a robustly asymptotically stable equilibrium point.

We consider a common Lyapunov function candidate $v(x)$ in the class of homogeneous polynomials, i.e. a common HPLF. We can express such a $v(x)$ as

$$v(x) = x^{\{m\}\prime} V x^{\{m\}} \tag{20}$$

where $m\in\mathbb{N}$ defines the degree of $v(x)$, which is equal to $2m$, and $V=V'\in\mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a symmetric matrix that contains the coefficients of $v(x)$ with respect to the basis defined by the chosen vector $x^{\{m\}}$ (i.e., V is a Gram matrix of $v(x)$). In order to determine $\Delta v(x,s)$ for $v(x)$ as in (20), let us introduce the matrix function $\Gamma_m:\mathbb{R}^{n\times n}\to\mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ as the matrix function satisfying the relationship

$$(Yx)^{\{m\}} = \Gamma_m(Y)x^{\{m\}} \tag{21}$$

where $Y\in\mathbb{R}^{n\times n}$. The matrix function $\Gamma_m(Y)$ can be computed with the formula

$$\Gamma_m(Y) = (K_m'K_m)^{-1}K_m'Y^{\otimes m}K_m \tag{22}$$

where $Y^{\otimes m}$ denotes the m-th Kronecker power of Y, i.e.

$$Y^{\otimes m} = \left\{\begin{array}{ll} Y^{\otimes m-1}\otimes Y & \text{if } m\ge 1 \\ 1 & \text{if } m=0. \end{array}\right. \tag{23}$$

and K_m is the matrix satisfying

$$x^{\otimes m} = K_m x^{\{m\}}. \tag{24}$$

It follows that

$$\Delta v(x,s) = x^{\{m\}\prime}\left(B_m(s)'VB_m(s) - V\right)x^{\{m\}} \tag{25}$$

where

$$B_m(s) = \Gamma_m(A(s)). \tag{26}$$

Let us observe that $B_m(s)$ is a homogeneous matrix polynomial of degree m in s since $A(s)$ is a linear matrix function.

The matrix function $B_m(s)'VB_m(s)-V$ is a Gram matrix of $\Delta v(x,s)$. It turns out that this Gram matrix is not unique. Indeed, let $L:\mathbb{R}^{\omega(n,m)}\to\mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ be a linear parametrization of the set $\mathcal{L}(n,m)$ in (8), and let $\beta:\mathbb{R}^r\to\mathbb{R}^{\omega(n,m)}$ be any vector function. It follows that

$$\Delta v(x,s) = x^{\{m\}\prime}\left(B_m(s)'VB_m(s) - V - L(\beta(s))\right)x^{\{m\}}. \tag{27}$$

This implies that $B_m(s)'VB_m(s)-V-L(\beta(s))$ is a Gram matrix of $\Delta v(x,s)$ for any choice of the vector function $\beta(s)$.

Let us introduce the notation

$$s^2 = \left(s_1^2,\ldots,s_r^2\right)' \tag{28}$$

and

$$\sqrt{s} = \left(\sqrt{s_1},\ldots,\sqrt{s_r}\right)'. \tag{29}$$

Hereafter we choose $\beta(s)$ so that $\beta(s^2)$ is a vector homogeneous polynomial of degree $2m_\beta$, i.e.

$$\beta(s^2) = \sum_{i_1+\ldots+i_r=2m_\beta i_1\ge 0,\ldots,i_r\ge 0} c_{i_1\cdots i_r}s_1^{i_1}\cdots s_r^{i_r} \tag{30}$$

where $c_{i_1\cdots i_r}\in\mathbb{R}^{\omega(n,m)}$. In other words, $\beta(s)$ is a vector homogeneous polynomial of degree $2m_\beta$ in the irrational variable $\sqrt{s}$.

Let us define the integer

$$w = 2\sigma(n,m) \tag{31}$$

and the linear function

$$o(s) = \sum_{i=1}^r s_i. \tag{32}$$

Let us define the function $W=W':\mathbb{R}^r\to\mathbb{R}^{w\times w}$ as

$$W(s) = \begin{pmatrix} o(s)^aV + o(s)^{a-m_\beta}L(\beta(s)) & o(s)^{a-m}B_m(s)'V \\ * & o(s)^aV \end{pmatrix} \tag{33}$$

where

$$a = \max\{m, m_\beta\}. \tag{34}$$

Let us observe that $W(s)$ is a symmetric matrix function of size $w\times w$. Also, $W(s)$ satisfies the homogeneousness property

$$W(\delta s) = \delta^a W(s) \quad \forall\delta\ge 0\ \forall s\in\mathcal{S}. \tag{35}$$

For simplicity, we will assume in the sequel that $m_\beta=m$, i.e. the degree of $\beta(s^2)$ in s is equal to the degree of $v(x)$ in m.

The following theorem provides a condition for establishing robust stability of the origin of the system (1) in terms of an LMI feasibility test.

Theorem 1: Let $m\in\mathbb{N}$, $m\ge 1$. Suppose that there exist a symmetric matrix $V=V'\in\mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$, a vector function $\beta:\mathbb{R}^r\to\mathbb{R}^{\omega(n,m)}$ as in (30) and a scalar $c\in\mathbb{R}$ satisfying the LMI feasibility test

$$\left\{\begin{array}{l} W(s^2) - c\|s\|^{2m}I_w \text{ is SOS} \\ c>0. \end{array}\right. \tag{36}$$

Then, the origin of the system (1) is a robustly asymptotically stable equilibrium point.

Proof. First of all, let us observe that $W(s^2)$ is a symmetric matrix homogeneous polynomial of degree $2m$. This means that $W(s^2)-c\|s\|^{2m}I_w$ is a symmetric matrix homogeneous polynomial as well, and, hence, the SOS condition in (36) can be defined.

Next, let us suppose that (36) holds, and let us show that (36) implies that $W(s)$ is positive definite over the simplex, i.e.

$$W(s) > 0 \quad \forall s \in \mathcal{S}.$$

For definition of SOS matrix polynomial in Section II-B, (36) implies that

$$W(s^2) - c\|s\|^{2m} I_w \geq 0 \quad \forall s \in \mathbb{R}^r.$$

Since $c > 0$, one obtains

$$W(s^2) > 0 \quad \forall s \in \mathbb{R}_0^r.$$

The positive definiteness of $W(s^2)$ over $\mathbb{R}_0^r$ implies the positive definiteness of $W(s)$ over the simplex. Indeed, suppose for contradiction that there exists $\bar{s} \in \mathcal{S}$ such that $W(\bar{s}) \not> 0$. Since $\bar{s}_i \in [0,1]$ for all $i = 1, \ldots, r$, we can define the vector

$$\hat{s} = \sqrt{\bar{s}}.$$

We have that

$$\begin{cases} \hat{s} \neq 0 \\ \hat{s}^2 = \bar{s} \end{cases}$$

which implies that there exists $\hat{s} \in \mathbb{R}_0^r$ such that $W(s^2) \not> 0$. But this contradicts the positive definiteness of $W(s^2)$ over $\mathbb{R}_0^r$.

In order to complete the proof, let us show that the positive definiteness of $W(s)$ over the simplex implies that the origin of the system (1) is a robustly asymptotically stable equilibrium point. From the definition of $W(s)$ and Schur lemma, it follows that the positive definiteness of $W(s)$ over the simplex is equivalent to the condition

$$\begin{cases} o(s)^a V > 0 \\ o(s)^a V + o(s)^{a-m_\beta} L(\beta(s)) \\ \quad - o(s)^{2(a-m)} B_m(s)' V \left(o(s)^a V\right)^{-1} V B_m(s) > 0 \end{cases}$$

for all $s \in \mathcal{S}$. Since $o(s) = 1$ for all $s \in \mathcal{S}$, this condition can be rewritten as

$$\begin{cases} V > 0 \\ D(s) < 0 \quad \forall s \in \mathcal{S} \end{cases}$$

where

$$D(s) = B_m(s)' V B_m(s) - V - L(\beta(s)).$$

Let x be any vector in $\mathbb{R}_0^n$, and let us pre- and post-multiply these inequalities by $x^{\{m\}'}$ and $x^{\{m\}}$, respectively. From the first inequality we obtain

$$x^{\{m\}'} V x^{\{m\}} > x^{\{m\}'} 0 x^{\{m\}} \quad \forall x \in \mathbb{R}_0^n$$

i.e.

$$v(x) > 0 \quad \forall x \in \mathbb{R}_0^n.$$

By observing that

$$\begin{aligned} x^{\{m\}'} B_m(s)' V B_m(s) x^{\{m\}} &= A(s) x^{\{m\}'} V A(s) x^{\{m\}} \\ &= v\left(A(s)x\right) \end{aligned}$$

and that

$$x^{\{m\}'} L(\beta(s)) x^{\{m\}} = 0,$$

from the second inequality we obtain

$$x^{\{m\}'} D(s) x^{\{m\}} < x^{\{m\}'} 0 x^{\{m\}} \quad \forall x \in \mathbb{R}_0^n \; \forall s \in \mathcal{S}$$

i.e.

$$\Delta v(x,s) < 0 \quad \forall x \in \mathbb{R}_0^n \; \forall s \in \mathcal{S}.$$

Consequently, $v(x)$ is a HPLF for the origin of the system (1) common to all admissible uncertainties, and therefore the theorem holds. □

Theorem 1 provides a condition for establishing whether the origin of the system (1) is a robustly asymptotically stable equilibrium point. This condition requires to check the existence of a symmetric matrix V, a vector function $\beta(s)$ as in (30) and a scalar c satisfying the condition (36). Let us observe that this condition is indeed an LMI feasibility test since $W(s^2)$ is a symmetric matrix homogeneous polynomial depending linearly on V and $\beta(s^2)$, and since the SOS condition for a symmetric matrix polynomial is equivalent to an LMI according to Section II-B. The condition provided by Theorem 1 is sufficient for any chosen integer m, which defines the degree of the HPLF candidate (equal to $2m$).

Let us observe that the case $m = 1$ corresponds to a quadratic Lyapunov function $v(x)$ since $x^{\{1\}} = x$ in (20). In such a case, the vector function $\beta(s)$ is not needed in the condition provided by Theorem 1 because for $m = 1$ one has

$$\mathcal{L} = \emptyset \tag{37}$$

and, hence, $L(\beta(s)) = 0$. This means that $W(s)$ does not depend on $\beta(s)$ for $m = 1$.

It is useful to clarify whether the condition provided by Theorem 1 covers the case of quadratic robust stability. In particular, the origin of the system (1) is said quadratically robustly asymptotically stable if (18) holds with a quadratic function $v(x)$. By expressing such a $v(x)$ as

$$v(x) = x' V x \tag{38}$$

where $V = V' \in \mathbb{R}^{n \times n}$, it follows that (18) with a quadratic function $v(x)$ is equivalent to the condition

$$\begin{cases} V > 0 \\ A(s)' V A(s) - V < 0 \end{cases} \quad \forall s \in \mathcal{S}. \tag{39}$$

The next result states that the condition provided by Theorem 1 with $m = 1$ is sufficient and necessary for quadratic robust stability.

Theorem 2: The origin of the system (1) is a quadratically robustly asymptotically stable equilibrium point if and only if the condition provided by Theorem 1 is satisfied with $m = 1$.

Proof. "⇐" Suppose that the origin of the system (1) is a quadratically robustly asymptotically stable equilibrium point. Hence, there exists V satisfying (39), and (39) can be rewritten as

$$\begin{pmatrix} V & A(s)'V \\ * & V \end{pmatrix} > 0 \quad \forall s \in \mathcal{S}.$$

Since $o(s)=1$ over $\mathcal{S}$, this implies that

$$\left(\begin{array}{cc} o(s)V & A(s)'V \\ * & o(s)V \end{array}\right) > 0 \quad \forall s \in \mathcal{S}.$$

Let us observe that the left hand side of the previous condition coincides with $W(s)$ for $m=1$ since, in such a case, one has that

$$\left\{\begin{array}{rcl} B_1(s) & = & A(s) \\ L(\beta(s)) & = & 0. \end{array}\right.$$

Since $W(s)$ is linear in s in this case, it follows that $W(s)$ can be expressed as

$$W(s)=\sum_{i=1}^{r} s_i W(s^{(i)})$$

where $s^{(1)},\ldots,s^{(r)} \in \mathbb{R}^r$ are the vertices of $\mathcal{S}$. Hence, the positive definiteness of $W(s)$ over $\mathcal{S}$ is equivalent to

$$W(s^{(i)}) > 0 \quad \forall i=1,\ldots,r.$$

This implies that $W(s^{(i)})$ admits a Cholesky factorization

$$W(s^{(i)}) = W_i' W_i$$

for some real matrix W_i, and, hence,

$$W(s^2)=\sum_{i=1}^{r} (s_i W_i)'(s_i W_i)$$

i.e. $W(s^2)$ is SOS. Moreover, since $W(s^{(i)})$ is positive definite for all $i=1,\ldots,r$, it follows that there exists c satisfying (36), in particular such a c can be chosen according to

$$0 < c < \min_{i=1,\ldots,r} \lambda_{min}\left(W(s^{(i)})\right).$$

"⇒" Suppose that there exists V and c satisfying the condition provided by Theorem 1 with $m=1$ (in this case, $L(\beta(s))=0$ and hence $\beta(s)$ is not needed). Since $W(s^2)$ is a symmetric matrix homogeneous polynomial of degree 2 in s, without products among the entries of s, (36) implies that $W(s^2)$ can be expressed as in the equation above. Moreover, since $c>0$ in (36), the matrices W_i have full rank, hence implying that $W_i'W_i$ is positive definite. The proof is completed by evaluating $W(s)$ at the vertices of $\mathcal{S}$ and observing that, as done in the previous part of the proof, the positive definiteness of $W(s)$ at the vertices of $\mathcal{S}$ is equivalent in this case to (39). □

Theorem 2 states that the condition provided by Theorem 1 can be used to check quadratic robust stability by choosing $m=1$, moreover this condition is not only sufficient but also necessary in such a case.

IV. ILLUSTRATIVE EXAMPLE

Let us consider the system

$$x(t+1)=\left(\begin{array}{cc} 0 & 1 \\ -0.8 & p(t) \end{array}\right) x(t)$$

where $x(t) \in \mathbb{R}^2$ is the state vector and $p(t) \in \mathbb{R}$ is the time-varying uncertain scalar confined into the interval

$$\mathcal{P}=[0,\zeta].$$

The problem consists of determining the largest value of ζ, denoted by ζ^*, such that the system is robustly asymptotically stable for all time-varying $p(t)$ in $\mathcal{P}$.

This system can be written as in (1) with $r=2$ and

$$A_1=\left(\begin{array}{cc} 0 & 1 \\ -0.8 & 0 \end{array}\right), \quad A_2=\left(\begin{array}{cc} 0 & 1 \\ -0.8 & \zeta \end{array}\right)$$

by expressing $p(t)$ as $p(t)=\zeta s_2(t)$.

In order to estimate ζ^*, let us use the condition provided by Theorem 1. For any chosen value of m this condition allows one to establish a lower bound of ζ^*.

For $m=1$ (HPLF of degree 2) we have $B_1(s)=A(s)$. By using the condition provided by Theorem 1 we find the lower bound $\zeta_1=0.397$ of ζ^*. The HPLF ensuring ζ_1 is given by

$$v(x)=4.444x_1^2-1.104x_2x_1+5.556x_2^2.$$

Figure 1 shows a level set of $v(x)$ (inner curve). Let us observe that the vector function $\beta(s)$ is not needed for $m=1$ since the set $\mathcal{L}(2,1)$ in (8) is empty and, hence, $L(\beta(s))=0$. Let us also observe that ζ_1 is the lower bound of ζ^* ensured by quadratic stability according to Theorem 2.

For $m=2$ (HPLF of degree 4) we have

$$B_2(s)=\left(\begin{array}{ccc} 0 & 0 & o(s)^2 \\ 0 & -0.8o(s)^2 & \zeta(s_1s_2+s_2^2) \\ 0.64o(s)^2 & -1.6\zeta s_2 o(s) & \zeta^2 s_2^2 \end{array}\right).$$

By using the condition provided by Theorem 1 we find the lower bound $\zeta_2=0.471$ of ζ^*. The HPLF ensuring ζ_2 is given by

$$\begin{array}{rcl} v(x) & = & 2.619x_1^4+1.198x_1^3x_2+6.907x_2^2x_1^2-4.788x_2^3x_1 \\ & & +4.442x_2^4. \end{array}$$

Figure 1 shows a level set of $v(x)$ (central curve). Figure 2 shows the vector function $\beta(s)$ (consisting of one entry in this case) ensuring ζ_2.

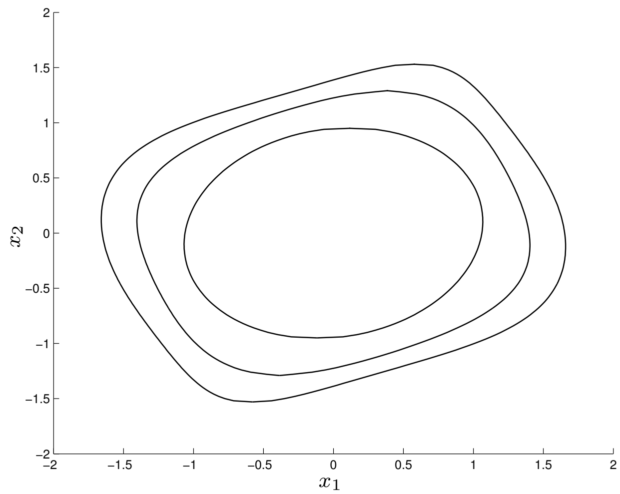
For $m=3$ (HPLF of degree 6) we find the lower bound $\zeta_3=0.523$ of ζ^*. The HPLF ensuring ζ_3 is given by

$$\begin{array}{rcl} v(x) & = & 0.982x_1^6+0.632x_1^5x_2+4.162x_1^4x_2^2+0.142x_2^3x_1^3 \\ & & +6.457x_2^4x_1^2-6.059x_2^5x_1+2.802x_2^6. \end{array}$$

Figure 1 shows a level set of $v(x)$ (outer curve). Figure 3 shows the vector function $\beta(s)$ (consisting of three entries in this case) ensuring ζ_3.

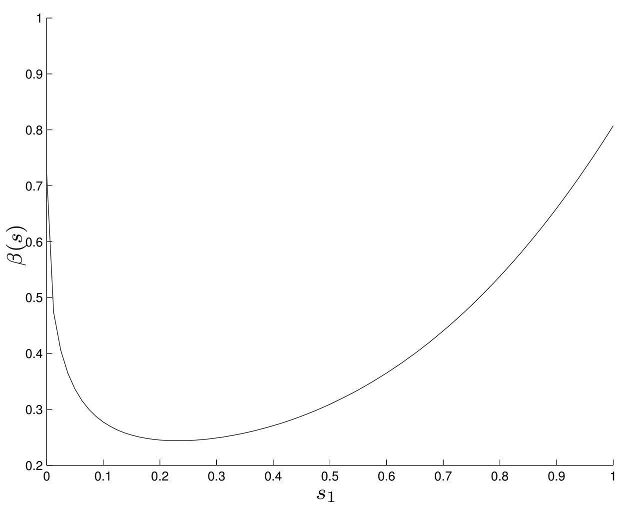
V. CONCLUSIONS

This paper has proposed a novel method based on HPLFs and LMIs for establishing robust asymptotical stability of discrete-time systems depending linearly on a time-varying uncertain vector constrained into a polytope. This method consists, firstly, of introducing a Gram matrix built with respect to the state and parametrized by a free function of the uncertainty, and secondly, of requiring that a transformation of the introduced Gram matrix is a SOS matrix polynomial. It has been shown that the proposed method provides a condition for robust asymptotical stability that is sufficient for any degree of the HPLF candidate and that includes quadratic robust stability as special case.



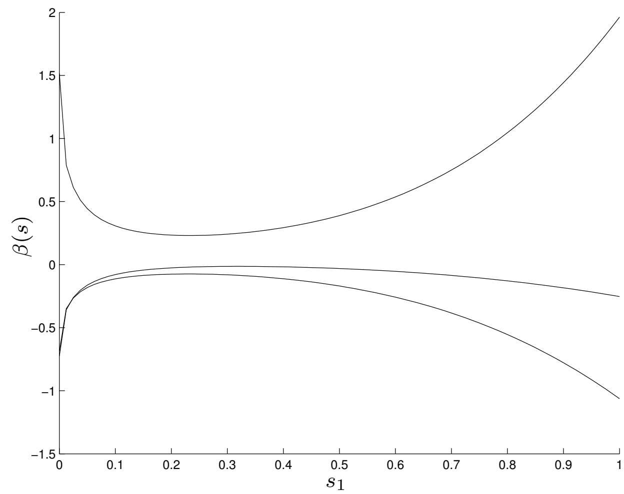
(a)

Fig. 1. Example 1: a level set of the HPLF found for $m = 1$ (inner curve), $m = 2$ (central curve), and $m = 3$ (outer curve).



(a)

Fig. 2. Example 1: vector function $\beta(s)$ found for $m = 2$.



(a)

Fig. 3. Example 1: vector function $\beta(s)$ found for $m = 3$.

VI. ACKNOWLEDGEMENTS

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