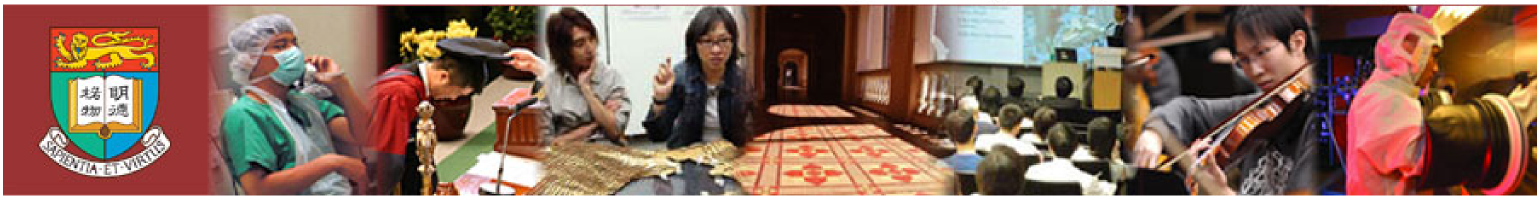

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Localized pulses for the quintic derivative nonlinear Schrödinger equation on a continuous-wave background

C. Rogers[1] and K. W. Chow[2,*]

[1]*Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems, School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia*

[2]*Department of Mechanical Engineering, University of Hong Kong, Pokfulam, Hong Kong*

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Quintic derivative nonlinear Schrödinger equations arise in various physical contexts, notably in the study of hydrodynamic wave packets and media with negative refractive index. A procedure to isolate propagating wave patterns in such nonlinear Schrödinger equations is proposed which is based on two integrals of motion. As an illustration of the method, a "gray" solitary pulse, a "dark" localized mode with nonzero minimum in intensity on a continuous-wave background is identified.

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Background and motivation. Derivative nonlinear Schrödinger (DNLS) equations occur frequently in applications, for instance, in models incorporating self-steepening effects in optical pulses [1,2]. Moreover, nonlinear Schrödinger (NLS) equations incorporating fifth order (quintic) nonlinearity are of fundamental interest in the physics of nonlinear optics [3,4]. It is natural to consider models in nonlinear dynamics combining these two features, and indeed such systems likewise have physical applications [5].

In hydrodynamics, packets of free surface waves are governed by the nonlinear Schrödinger equation to leading order. However, in the parameter regime $kh \approx 1.363$, where k is the wave number and h is the water depth, cubic nonlinearity weakens considerably, and higher order effects need to be restored. Appropriate rescaling then produces a quintic DNLS equation [6].

In the study of the behavior of metamaterials and of media with negative refractive index, the concepts of negative permittivity and permeability are important. Again, higher order effects must be incorporated in the nonlinear Schrödinger equation, if additional physical phenomena such as third order dispersion are to be described [7]. Wave packets of ultrashort electromagnetic pulses in left-handed materials can be studied by the slowly varying approximations, and a DNLS equation with quintic nonlinearity can also be derived in such contexts [8]. There the coefficients of the nonlinear terms can be related to the third order susceptibility of the medium.

The model. Here we consider the evolution of a wave envelope Ψ governed by a quintic derivative nonlinear Schrödinger equation.

$$i\Psi_t + \hat{\lambda}\Psi_{xx} + \hat{\mu}|\Psi|^2\Psi + i\hat{\alpha}|\Psi|^2\Psi_x + \hat{\nu}|\Psi|^4\Psi = 0. \quad (1)$$

The form $|\Psi|^2\Psi_x$ is chosen instead of $(|\Psi|^2\Psi)_x$, also commonly studied in the literature, purely for the convenience of the subsequent mathematical manipulations. The two forms of the DNLS equations can be related by a gauge transformation [9]. Physically, this term involving the parameter $\hat{\alpha}$ is usually associated with the "self-steepening" phenomena in optics [10]. In hydrodynamics, the coordinates t and x are typically slow time and spatial coordinate traveling with the group velocity. In the context of optical fiber physics, they denote distance and retarded time, respectively.

Modulation instabilities of continuous waves and localized solutions on a zero background have recently been investigated [8,11]. The aim here is to extend the analysis of localized modes to a configuration with nonvanishing boundary conditions.

Special exact, traveling wave solutions of the quintic DNLS equation have been obtained by an appropriate ansatz [12,13]. Here, the route to exact solutions is via two "integrals of motion." The resulting squared amplitude of the complex envelope satisfies a generic equation generally associated with elliptic functions, and thus a large variety of exact solutions can be generated. As an illustration of the procedure, a "gray" solitary wave on a continuous-wave background, a "dark" localized mode with a nonzero minimum in intensity, is derived.

Analysis and the two integrals of motion. Here, propagating patterns of the type

$$\Psi = [\phi(x - ct) + i\psi(x - ct)]\exp[i(kx - \Omega t)] \quad (2)$$

are sought. Introduction of the representation, Eq. (2), into Eq. (1) generates two coupled nonlinear ordinary differential equations for the real (ϕ) and imaginary (ψ) parts of the envelope. The coupled nonlinear system turns out to admit a key pair of "integrals of motion," namely,

$$\dot{\phi}\psi - \dot{\psi}\phi = J + \left(k - \frac{c}{2\hat{\lambda}}\right)\sum + \frac{\hat{\alpha}}{4\hat{\lambda}}\sum{}^2 \quad (3)$$

and

$$\dot{\phi}^2 + \dot{\psi}^2 = 2\mathcal{H} + \left(k^2 - \frac{\Omega}{\hat{\lambda}}\right)\sum + \left(\frac{\hat{\alpha}k - \hat{\mu}}{2\hat{\lambda}}\right)\sum{}^2 - \frac{\hat{\nu}}{3\hat{\lambda}}\sum{}^3, \quad (4)$$

where

$$\sum = \phi^2 + \psi^2, \quad (5)$$

with J and $\mathcal{H}$ being two integration constants. The constant $\mathcal{H}$ corresponds to the Hamiltonian invariant. *The dot denotes the*

*Corresponding author: kwchow@hku.hk.

derivative with respect to the propagating wave phase variable $x - ct$.

Isolation of these two integrals of motion is a crucial step in the solution procedure. Thus, on use of the identity

$$(\phi^2 + \psi^2)(\dot\phi^2 + \dot\psi^2) - (\phi\dot\phi + \psi\dot\psi)^2 = (\phi\dot\psi - \psi\dot\phi)^2, \quad (6)$$

it is readily shown that the evolution of squared amplitude Σ is given by

$$\dot{\sum}^2 = \sum\left\{8\mathcal{H} + 4\left(k^2 - \frac{\Omega}{\hat\lambda}\right)\sum + 2\left(\frac{\hat\alpha k - \hat\mu}{\hat\lambda}\right)\sum^2 - \left(\frac{4\hat\nu}{3\hat\lambda}\right)\sum^3\right\} - 4\left[J + \left(k - \frac{c}{2\hat\lambda}\right)\sum + \frac{\hat\alpha}{4\hat\lambda}\sum^2\right]^2. \quad (7)$$

This equation admits a diversity of exact solutions determined by the physical parameters and the invariants J and $\mathcal{H}$. In general, the solutions involve elliptic functions.

To recover Ψ, it is convenient to introduce the auxiliary variables Δ and Θ given by

$$\Delta = \frac{\phi}{\psi}, \quad \Theta = \tan^{-1}\Delta = \tan^{-1}\left(\frac{\phi}{\psi}\right), \quad (8)$$

whence

$$\Theta = \int^{x-ct}\frac{J + \left(k - \frac{c}{2\hat\lambda}\right)\sum + \left(\frac{\hat\alpha}{4\hat\lambda}\right)\sum^2}{\sum}d\xi, \quad (9)$$

where ξ is a dummy variable of integration. The corresponding class of exact solutions of the DNLS equations [Eq. (1)] is then given by

$$\Psi = \sum^{1/2}\exp[-i\Theta + i(kx - \Omega t)], \quad (10)$$

where Σ [Eq. (5)], Θ are determined by Eqs. (7) and (9), respectively. Localized solutions obtained earlier in the literature can be readily recovered through the present mechanism.

In the long-wave limit, elliptic functions degenerate to hyperbolic functions. Here, to illustrate the procedure, we consider such a case with nonzero background, namely,

$$\sum = a - b\,\mathrm{sech}^2[r(x - ct)], \quad a > b > 0, \quad r > 0, \quad (11)$$

which satisfies

$$\dot{\sum}^2 = \frac{4r^2}{b}\left[-a^2(a-b) + a(3a-2b)\sum - (3a-b)\sum^2 + \sum^3\right]. \quad (12)$$

Physically, Eq. (11) describes a gray pulse with a and b measuring the background intensity and the local minimum, respectively. On alignment of the expressions in Eqs. (7) and (12), one arrives at defining equations for the angular frequency Ω and phase speed c, where the width of the pulse r, wave number k, and background and intensity defect parameters a,b are given by

$$\frac{\hat\alpha c}{2\hat\lambda^2} - \frac{\hat\mu}{\hat\lambda} = \frac{2r^2}{b}, \quad \frac{kc}{\hat\lambda} - \frac{c^2}{4\hat\lambda^2} - \frac{\Omega}{\hat\lambda} - \frac{\hat\alpha J}{2\hat\lambda} = -\frac{r^2(3a-b)}{b},$$
$$J^2 = \frac{a^2r^2(a-b)}{b}. \quad (13)$$

It is noted that, the Hamiltonian $\mathcal{H}$ can be expressed in terms of the physical parameters subject to the constraint:

$$\hat\nu = -\frac{3\hat\alpha^2}{16\hat\lambda}. \quad (14)$$

Analytically, there are four free parameters (a,b,k,r), if $\hat\lambda,\hat\mu,\hat\nu,\hat\alpha$ are known.

The phase, as defined by Eqs. (8) and (9), can also be evaluated explicitly in the present case as

$$\Theta = \frac{J}{ra}\left\{\frac{1}{2}\log\left|\frac{1 + \tanh r(x-ct)}{1 - \tanh r(x-ct)}\right| + \sqrt{\frac{b}{a-b}}\tan^{-1}\left[\sqrt{\frac{b}{a-b}}\tanh r(x-ct)\right]\right\} + \left(k - \frac{c}{2\hat\lambda} + \frac{\hat\alpha a}{4\hat\lambda}\right)(x-ct) - \frac{\hat\alpha b\tanh r(x-ct)}{4r\hat\lambda}. \quad (15)$$

Equation (10), together with the relations (11), (13), (14), and (15), determine a gray solitary pulse for the quintic DNLS equation (1). A typical pattern is illustrated in Fig. 1.

Modulation instability. Finally, a remark on modulation instability is in order, as there is a continuous-wave background. Starting from Eq. (1) directly, one continuous wave is given by (A_0 real)

$$\Psi = A_0\exp\left[i\left(\hat\mu A_0^2 + \hat\nu A_0^4\right)t\right].$$

Standard modulation instability analysis [14,15] involving modes of the form $\exp[i(Kx - \zeta t)]$ leads to a dispersion relation

$$\left(\zeta - \hat\alpha K A_0^2\right)^2 = \hat\lambda K^2\left(\hat\lambda K^2 - 2\hat\mu A_0^2 - 4\hat\nu A_0^4\right).$$

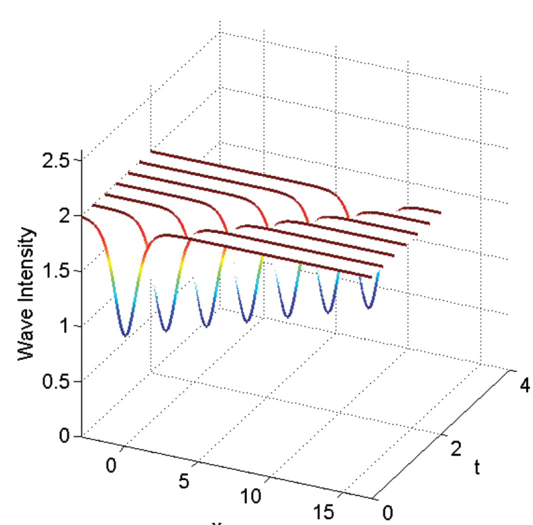

FIG. 1. (Color online) The motion of the "wave intensity" $|\Psi|^2$ of the wave envelope Ψ for $\hat\lambda = \hat\mu = \hat\nu = \hat\alpha = 1$, $a = 2$, $b = 1$, $k = 1$, $r = 1$, and $c = 6$.

For the special quintic nonlinearity given by Eq. (14), this simplifies to

$$\left(\zeta - \hat{\alpha} K A_0^2\right)^2 = \hat{\lambda} K^2 \left(\hat{\lambda} K^2 - 2\hat{\mu} A_0^2 + \frac{3\hat{\alpha}^2 A_0^4}{4\hat{\lambda}}\right).$$

Hence, in the language of optical physics, in the normal dispersion regime ($\hat{\lambda} < 0$), the plane wave is always stable (ζ being real) if the cubic nonlinearity $\hat{\mu}$ is positive. In the anomalous dispersion regime ($\hat{\lambda} > 0$), stability (ζ real) can be accomplished in the present case of negative quintic nonlinearity [Eq. (14)], provided that the amplitude A_0 is sufficiently large, namely,

$$A_0^2 > \frac{8\hat{\mu}\hat{\lambda}}{3\hat{\alpha}^2}.$$

Conclusion. A representation for traveling wave patterns for a quintic derivative nonlinear Schrödinger model admits two key integrals of motion. The squared modulus of the complex envelope is thereby shown to satisfy a nonlinear equation, which can in general be solved in terms of elliptic and hyperbolic functions. "Bright" and dark pulses described previously in recent literature are readily retrieved as special cases of the formalism presented here. A gray solitary pulse (dark modes with nonzero minimum intensity) on a continuous-wave background is derived as an illustration of the formalism. Regimes for the modulation instability of the plane wave background are investigated. In particular, this continuous wave is stable in the normal dispersion regime if the cubic nonlinearity is positive. In the anomalous dispersion regime, stability regime is identified. Hence such localized modes should be observable in practice, and will be valuable in future studies of hydrodynamic wave packets, physics of metamaterials, and media with negative refractive index.

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