Title	**Sufficient and Necessary LMI Conditions for Robust Stability of Rationally Time-Varying Uncertain Systems**
Author(s)	**Chesi, G**
Citation	**IEEE Transactions on Automatic Control, 2013, v. 58 n. 6, p. 1546-1551**
Issued Date	**2013**
URL	**http://hdl.handle.net/10722/184513**
Rights	**Creative Commons: Attribution 3.0 Hong Kong License**

Sufficient and Necessary LMI Conditions for Robust Stability of Rationally Time-Varying Uncertain Systems

Graziano Chesi, *Senior Member, IEEE*

Abstract—This technical note addresses robust stability of uncertain systems with rational dependence on unknown time-varying parameters constrained in a polytope. First, the technical note proves that a sufficient linear matrix inequality (LMI) condition that we previously proposed, based on homogeneous polynomial Lyapunov functions (HPLFs) and on the introduction of an extended version of Polya's theorem, is also necessary. Second, the technical note proposes a new sufficient and necessary LMI condition by exploiting properties of the simplex and sum-of-squares (SOS) parameter-dependent polynomials. Lastly, the technical note investigates relationships among these conditions and conditions based on the linear fractional representation (LFR). It is worth remarking that sufficient and necessary LMI conditions for this problem have not been proposed yet in the literature.

Index Terms—Lyapunov function, rational dependence, time-varying, uncertain system.

I. INTRODUCTION

Studying robust stability is a fundamental problem in systems with uncertainty that basically amounts to establishing whether an equilibrium point is stable for all the admissible values of the uncertainty. Several frameworks have been proposed for addressing this problem, in particular based on Lyapunov functions (LFs). For instance, in [1], [2] robust stability conditions are obtained based on the existence of parameter-dependent LFs for time-invariant uncertainty and common LFs for time-varying uncertainty. LMI conditions for robust stability are introduced in the pioneering book [3] and successively extended in numerous directions, in particular by exploiting SOS polynomials, see e.g. [4] which reviews relaxations based on the S-procedure and addresses the construction of asymptotically exact relaxations, and [5] which proposes a framework for robustness analysis of polytopic systems with time-invariant and time-varying uncertainties. See also [6] and references therein for pioneering studies in the field of SOS polynomials, and [7] for a survey of techniques based on SOS polynomials in control systems.

Concerning uncertain systems affected by time-varying uncertainty, existing works typically consider the case of systems linearly dependent on an uncertain vector constrained into a polytope. Pioneering works have investigated the existence of quadratic LFs, see e.g. [8]. In order to reduce the conservatism, methods based on nonquadratic LFs have been proposed. In particular, necessary and sufficient conditions for asymptotic stability of a class of differential inclusions are provided in [9]. The construction of piecewise quadratic LFs is proposed in [10] for two-term LFs by using the S-procedure, in [11] for nonlinear and hybrid systems providing connections with frequency domain methods such as the circle and Popov criteria, in [12] by combining genetic algorithms with LMIs for addressing robust control design, and in [13] by introducing the class of composite LFs for studying systems with input and state constraints. Conditions based on polyhedral LFs are

considered in [14] which provides a constructive approach for the case of complex matrices, in [15] which proves non-conservatism of this class for linear systems with time-varying uncertainties and input disturbances, and in [16] which provides a universal class of LFs in the form of powers of p norms by smoothing polyhedral LFs. The construction of HPLFs is addressed in [17] which provides sufficient conditions based on convex optimization, in [18] which proposes the use of SOS polynomials, and in [19] which proves that the conditions in [18] are not only sufficient but also necessary. Some of these methods have been extended to address the case of rational dependence on the uncertainty. For instance, [20] investigates the existence of quadratic LFs through the LFR, [21] extends this method to establish the existence of HPLFs through the LFR, and [22] investigates the existence of HPLFs through the introduction of an extended version of Polya's theorem.

This technical note addresses robust stability of uncertain systems with rational dependence on unknown time-varying parameters constrained in a polytope. First, the technical note proves that the sufficient LMI condition that we previously proposed in [22] is also necessary for a sufficiently large degree of the HPLF. Second, the technical note proposes a new sufficient and necessary LMI condition via HPLFs by exploiting properties of the simplex and SOS parameter-dependent polynomials. Lastly, the technical note investigates relationships among these conditions and conditions based on the LFR, in particular showing that the proposed conditions are not more conservative than the conditions in [20] (based on quadratic LFs) and [21] (based on HPLFs). Some numerical examples illustrate the proposed results. A preliminary version of this technical note (where necessity of the conditions is proved only for second-order systems) appeared in [23].

The technical note is organized as follows. Section II introduces the problem formulation and some preliminaries on the representation of polynomials. Section III provides the proposed results. Section IV presents some illustrative examples. Lastly, Section V concludes the technical note with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

Notation: $\mathbb{N}, \mathbb{R}$: natural and real number sets; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of A; $A > 0 (A \geq 0)$: symmetric positive definite (semidefinite) matrix A; $he(A) = A + A'$; $\nabla v(x)$: first derivative row vector of the function $v(x)$; $conv\{a, b, \ldots\}$: convex hull of vectors $a, b, \ldots$; $diag\{A, B, \ldots\}$: block diagonal matrix with blocks $A, B, \ldots$; $\star$: corresponding block in symmetric matrices.

We consider the system

$$\begin{cases} \dot{x}(t) = A_{rat}(p(t))x(t) \\ p(t) \in \mathcal{P} \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $p(t) \in \mathbb{R}^q$ is the time-varying uncertain vector, $A_{rat} : \mathbb{R}^q \to \mathbb{R}^{n \times n}$ is a matrix rational function, and $\mathcal{P} \subset \mathbb{R}^q$ is a bounded convex polytope expressed as

$$\mathcal{P} = conv\left\{p^{(1)}, \ldots, p^{(r)}\right\} \tag{2}$$

where $p^{(1)}, \ldots, p^{(r)} \in \mathbb{R}^q$ are given vectors. The matrix rational function $A_{rat}(p)$ is expressed as

$$A_{rat}(p) = \frac{A(p)}{a(p)} \tag{3}$$

where $A : \mathbb{R}^q \to \mathbb{R}^{n \times n}$ is a matrix polynomial and $a : \mathbb{R}^q \to \mathbb{R}$ is a polynomial. Throughout the technical note we assume that $p(t)$ ensures the existence of the solution $x(t)$ of (1), and that

$$a(p) > 0 \quad \forall p \in \mathcal{P}. \qquad (4)$$

Let us observe that (4) ensures that $A_{rat}(p)$ is bounded for all admissible values of p, which is reasonable especially when $A_{rat}(p)$ represents a real system.

Problem. The problem that we consider in this technical note consists of establishing whether the origin is a robustly asymptotically stable equilibrium point of (1), i.e.

$$\begin{cases} \forall \varepsilon > 0 \exists \delta > 0 : \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0 \; \forall p(\cdot) \in \mathcal{P} \\ \lim_{t \to \infty} x(t) = 0_n \quad \forall x(0) \in \mathbb{R}^n \quad \forall p(\cdot) \in \mathcal{P}. \end{cases} \qquad (5)$$

B. Gram Matrix Method

For $x \in \mathbb{R}^n$, let $h(x)$ be a homogeneous polynomial of degree $2m$. Let $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ be a vector containing all monomials of degree equal to m, where

$$\sigma(n, m) = \frac{(n + m - 1)!}{(n-1)!m!}. \qquad (6)$$

Then, $h(x)$ can be written as

$$h(x) = x^{\{m\}'} \left(H + L(\alpha) \right) x^{\{m\}} \qquad (7)$$

where $H = H' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a symmetric matrix such that $h(x) = x^{\{m\}'} H x^{\{m\}}$, $L(\alpha) = L(\alpha)' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{ L = L' : \; x^{\{m\}'} L x^{\{m\}} = 0 \right\}, \qquad (8)$$

and $\alpha \in \mathbb{R}^{\omega(n,m)}$ is a free vector with

$$\omega(n, m) = \frac{1}{2} \sigma(n, m) (\sigma(n, m) + 1) - \sigma(n, 2m). \qquad (9)$$

The representation (7) is known as Gram matrix method and square matricial representation (SMR), and allows one to establish whether a polynomial is SOS via an LMI. Indeed, $h(x)$ is SOS if and only if there exists α such that $H + L(\alpha) \geq 0$. See, e.g., [7] for details.

III. ROBUST STABILITY ANALYSIS

A. Equivalent Models

First of all, let us observe that (1) can be equivalently represented with other models. A well-known one exploits the LFR, see e.g. [20]. With the LFR model, (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + \bar{B}y(t) \\ z(t) = \bar{C}x(t) + \bar{D}y(t) \\ y(t) = \bar{E}\left(p(t)\right) z(t) \\ \bar{E}(p) = \text{diag}\left(p_1 I_{s_1}, \ldots, p_q I_{s_q}\right) \\ p(t) \in \mathcal{P} \end{cases} \qquad (10)$$

where $x(t), p(t)$ and $\mathcal{P}$ are as in (1), $y(t), z(t) \in \mathbb{R}^d$ are auxiliary vectors with $d = s_1 + \cdots + s_q$ for some nonnegative integers $s_1, \ldots, s_q$, and $\bar{A} \in \mathbb{R}^{n \times n}$, $\bar{B} \in \mathbb{R}^{n \times d}$, $\bar{C} \in \mathbb{R}^{d \times n}$, and $\bar{D} \in \mathbb{R}^{d \times d}$ are appropriate matrices. Indeed, $A_{rat}(p)$ in (1) is related to the matrices in (10) by

$$A_{rat}(p) = \bar{A} + \bar{B}\left(I - \bar{E}(p)\bar{D}\right)^{-1} \bar{E}(p)\bar{C}. \qquad (11)$$

Consequently, the LFR is said well-posed if

$$\det\left(I - \bar{E}(p)\bar{D}\right) \neq 0 \quad \forall p \in \mathcal{P}. \qquad (12)$$

It is useful to observe that, since $a(p)$ in (3) can be selected equal to $\det(I - \bar{E}(p)\bar{D})$, assumption (4) coincides with (12). For (10) the *LFR degree* is defined as $d_{LFR} = \max\{s_1, \ldots, s_q\}$. As p varies in $\mathcal{P}$, $\bar{E}(p)$ describes a polytope of matrices, whose vertices are given by

$$\bar{E}_i = \bar{E}\left(p^{(i)}\right) \quad \forall i = 1, \ldots, r. \qquad (13)$$

Another model for representing (1) consists of adopting a canonical set for the uncertain vectors, in particular the simplex. Indeed, (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = C_{rat}\left(s(t)\right) x(t) \\ s(t) \in \mathcal{S} \end{cases} \qquad (14)$$

where $\mathcal{S}$ is the simplex

$$\mathcal{S} = \left\{ s \in \mathbb{R}^r : \sum_{i=1}^r s_i = 1, \quad s_i \geq 0 \right\} \qquad (15)$$

$s = (s_1, \ldots, s_r)' \in \mathbb{R}^r$ is a new time-varying uncertain vector, and $C_{rat} : \mathbb{R}^r \to \mathbb{R}^{n \times n}$ is a matrix rational function that we express as

$$C_{rat}(s) = \frac{C(s)}{c(s)} \qquad (16)$$

where $C : \mathbb{R}^r \to \mathbb{R}^{n \times n}$ is a matrix homogeneous polynomial and $c : \mathbb{R}^r \to \mathbb{R}$ is a homogeneous polynomial. The original uncertain vector p in (1) is related to s by $p = \varphi(s)$ where

$$\varphi(s) = \sum_{i=1}^r s_i p^{(i)}. \qquad (17)$$

Thus, one has

$$\forall s \in \mathcal{S} \begin{cases} C(s) & = & A\left(\varphi(s)\right) \\ c(s) & = & a\left(\varphi(s)\right). \end{cases} \qquad (18)$$

B. Conditions for Robust Stability

One can establish robust stability of (1) by looking for a continuous function $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} & v(x) \text{ is positive definite} \\ \text{systems (1) and (10):} & \dot{v}(x, p) < 0 \; \forall x \in \mathbb{R}_0^n \; \forall p \in \mathcal{P} \\ \text{system (14):} & \dot{v}(x, s) < 0 \; \forall x \in \mathbb{R}_0^n \; \forall s \in \mathcal{S} \end{cases} \qquad (19)$$

where $\dot{v}(x, p)$ and $\dot{v}(x, s)$ are the time derivatives of $v(x)$ along the trajectories of the systems (1) and (10), respectively. If such a function $v(x)$ exists, then $v(x)$ is said a common LF.

Since the system is linear in the state, the candidate LF $v(x)$ can be chosen homogeneous in x. Let us consider the case where such a homogeneous function $v(x)$ is polynomial. We can write the candidate HPLF $v(x)$ according to Section II-B as

$$v(x) = x^{\{m\}'} V x^{\{m\}} \qquad (20)$$

where $m \in \mathbb{N}$ defines the degree of $v(x)$, which is equal to $2m$, and $V = V' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$.

A method for investigating robust stability of (1) through HPLFs was proposed in [5], [21] by exploiting the LFR model (10). Specifically, for $\bar{A} \in \mathbb{R}^{n \times n}$ let $\bar{A}^\sharp$ be the matrix defined by

$$\frac{dx^{\{m\}}}{dx} \bar{A}x = \bar{A}^\sharp x^{\{m\}}. \qquad (21)$$

The matrix $\bar{A}^{\sharp}$ is known as extended matrix of $\bar{A}$ and can be calculated via the formula

$$A^{\sharp} = (K'K)^{-1}K'\left(\sum_{i=0}^{m-1} I_{n^{m-1-i}} \otimes A \otimes I_{n^i}\right)K \qquad (22)$$

where $K \in \mathbb{R}^{n^m \times \sigma(n,m)}$ is the matrix satisfying

$$x^{[m]} = Kx^{\{m\}} \qquad (23)$$

and $x^{[m]}$ denotes the m-th Kronecker power of x. Define

$$R(V, \bar{G}, \bar{H}, \bar{E}) = R_1 + R_2$$
$$R_1 = \begin{pmatrix} \mathrm{he}(V\bar{A}^{\sharp}) & V(\bar{B}\bar{E})^{\sharp} \\ \star & 0 \end{pmatrix}$$
$$R_2 = \begin{pmatrix} \mathrm{he}(\bar{G}\bar{C}^{\sharp}) & \bar{G}(\bar{D}\bar{E})^{\sharp} - \bar{G} + (\bar{H}\bar{C}^{\sharp})' \\ \star & \mathrm{he}\left(\bar{H}(\bar{D}\bar{E})^{\sharp} - \bar{H}\right) \end{pmatrix}. \qquad (24)$$

Define also

$$\mathcal{L}_{LFR} = \{L = L' : f(x,z)'Lf(x,z) = 0\} \qquad (25)$$

where

$$f(x,z) = \begin{pmatrix} x^{\{m\}} \\ z \otimes x^{\{m-1\}} \end{pmatrix}. \qquad (26)$$

Theorem 1 ([5], [21]): Let $m \geq 1$ be an integer, and let $L(\alpha)$ be a linear parametrization of $\mathcal{L}_{LFR}$ in (25). The origin of (10) is robustly asymptotically stable if there exist a symmetric matrix V, matrices $\bar{G}$ and $\bar{H}$, and vectors $\alpha^{(1)}, \ldots, \alpha^{(r)}$ satisfying the following system of LMIs:

$$\begin{cases} V > 0 \\ R(V, \bar{G}, \bar{H}, \bar{E}_i) + L\left(\alpha^{(i)}\right) < 0 & \forall i = 1, \ldots, r. \end{cases} \qquad (27)$$

Moreover, if the LFR degree is $d_{LFR} = 1$, a less conservative condition is obtained by requiring that there exist matrices $\bar{G}_i$ and $\bar{H}_i$ (in place of $\bar{G}$ and $\bar{H}$) such that

$$\begin{cases} V > 0 \\ R(V, \bar{G}_i, \bar{H}_i, \bar{E}_i) + L\left(\alpha^{(i)}\right) < 0 & \forall i = 1, \ldots, r. \end{cases} \qquad (28)$$

Theorem 1 provides a condition for establishing whether there exists a HPLF $v(x)$ for (1) by exploiting its equivalent LFR model (10). This condition has the advantage that the negative definiteness of $\dot{v}(x,p)$ for all $p \in \mathcal{P}$ is investigated by checking only the vertices of $\mathcal{P}$. In fact, by exploiting (10), it follows that $\dot{v}(x,p)$ is a homogeneous polynomial in x and z that depends affine linearly on p. One has that $\dot{v}(x,p)$ can be written as $\dot{v}(x,p) = f(x,z)'(R_1 + R_2 + L(\alpha))f(x,z)$ where R_1 satisfies $\dot{v}(x,p) = f(x,z)'R_1 f(x,z)$, R_2 introduces some degrees of freedom as $f(x,z)'R_2 f(x,z) = 0$ given by the relationship between x and z in (10), and $L(\alpha)$ introduces other degrees of freedom as $f(x,z)'L(\alpha)f(x,z) = 0$ given by the presence of monomials in $f(x,z)$. Therefore, by imposing (27), (28), one imposes that $v(x)$ and $-\dot{v}(x,p)$ have positive definite Gram matrices at the vertices of $\mathcal{P}$, hence implying (19). Theorem 1 includes for $m = 1$ the conditions proposed in [20] based on quadratic LFs, see [5], [21].

Another method for investigating robust asymptotical stability of (1) through HPLFs was proposed in [22] together with an extension of Polya's theorem to the case of structured matrix polynomials. Specifically, for a nonnegative integer k define

$$G(s) = C(s)^{\sharp}\left(\sum_{i=1}^{r} s_i\right)^k \qquad (29)$$

and express $G(s)$ as

$$G(s) = (G_1, \ldots, G_l)\left(s^{\{\delta+k\}} \otimes I_{\sigma(n,m)}\right) \qquad (30)$$

where $G_1, \ldots, G_l \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$, $l = \sigma(r, \delta+k)$, and δ is the degree of $A(p)$ in (3).

Theorem 2 ([22]): Let $m \geq 1$ and $k \geq 0$ be integers, and let $L(\alpha)$ be a linear parametrization of $\mathcal{L}$ in (8). The origin of (1) is robustly asymptotically stable if there exist a symmetric matrix V and vectors $\alpha^{(1)}, \ldots, \alpha^{(l)}$ satisfying the following system of LMIs:

$$\begin{cases} V > 0 \\ \mathrm{he}(VG_i) + L\left(\alpha^{(i)}\right) < 0 & \forall i = 1, \ldots, l. \end{cases} \qquad (31)$$

Theorem 2 provides a condition for establishing whether there exists a HPLF $v(x)$ for (1) by exploiting its equivalent model (14). This condition is built by introducing an extension of Polya's theorem, which allows one to establish whether a homogeneous polynomial $h(s)$ is positive for all $s \in \mathcal{S}$ by requiring that $(\sum_{i=1}^{r} s_i)^k h(s)$ has positive coefficients for some integer k. The extension of Polya's theorem introduced in Theorem 2 investigates whether $\dot{v}(x,s)$ is negative definite for all $s \in \mathcal{S}$. This is done by requiring that the coefficients of $(\sum_{i=1}^{r} s_i)^k \dot{v}(x,s)$ with respect to s, which are homogeneous polynomials in x, have negative definite Gram matrices $\mathrm{he}(VG_i) + L(\alpha^{(i)})$, hence implying that such coefficients are negative definite. This has the advantage of providing a condition that is not only sufficient but also necessary as it will be shown in Theorem 3. The disadvantage is that, depending on the system under investigation, the computational burden required by this condition can be large.

The first contribution of the technical note is to prove that the condition of Theorem 2 is not only sufficient but also necessary as explained in the following result.

Theorem 3: The origin of (1) is robustly asymptotically stable if and only if there exist m and k such that the condition of Theorem 2 holds.

Proof: Let the origin of (1) be robustly asymptotically stable. From [16] a LF for (1) can be chosen as $v_0(x) = \|Mx\|_{2p}^{2p}$ for some full column rank matrix M and some integer p. This implies that one can choose the LF $v(x) = v_0(x)^q$ where q is any integer such that $q \geq 1$. The time derivative of $v(x)$ is given by

$$\dot{v}(x,s) = qv_0(x)^{q-1}\dot{v}_0(x,s)$$
$$\dot{v}_0(x,s) = \nabla v(x)\frac{C(s)}{c(s)}x.$$

Since $v_0(x)$ is a HPLF, there exists $\varepsilon > 0$ such that

$$u(x,s) < 0 \quad \forall x \in \mathbb{R}_0^n \; \forall s \in \mathcal{S}$$
$$u(x,s) = qv_0(x)^{q-1}u_0(x,s)$$
$$u_0(x,s) = \dot{v}_0(x,s) + \varepsilon\|x\|^{2p}.$$

From [24] this implies that there exists a sufficiently large q (denoted as q_1) such that, for any fixed $s \in \mathcal{S}$, $-u(x,s)$ is positive definite and SOS in x. Let q be replaced by q_1, and define $m = pq$. Then, let us observe that one can write $v(x) = x^{\{m\}'}Vx^{\{m\}}$ where $V \geq 0$. Since $v(x) = \|Mx\|_{2p}^{2m}$, it follows that V can be chosen positive definite. Similarly, one can write $u(x,s) = x^{\{m\}'}U(s)x^{\{m\}}$ where $U(s) \leq 0$ for all $s \in \mathcal{S}$. Moreover, one can write $\dot{v}(x,s) = x^{\{m\}'}X(s)x^{\{m\}}$ for some $X(s)$. Since $u(x,s) - \dot{v}(x,s) = qv_0(x)^{q-1}\varepsilon\|x\|^{2p}$, it follows that $X(s)$ can be chosen negative definite for all $s \in \mathcal{S}$ (just observe

that $U(s) - X(s) = \tilde{U}(s)$ with $\tilde{U}(s) > 0$ for all $s \in \mathcal{S}$). Summarizing, one has that

$$\begin{cases} v(x) = x^{\{m\}'} V x^{\{m\}} \\ \dot{v}(x,s) = x^{\{m\}'} X(s) x^{\{m\}} \\ \quad V > 0 \\ \quad X(s) < 0 \quad \forall s \in \mathcal{S}. \end{cases}$$

Observe that $X(s)$ satisfies

$$c(s) X(s) = \mathrm{he}\left(V C(s)\right) + L\left(\phi(s)\right)$$

for some function $\phi : \mathbb{R}^r \rightarrow \mathbb{R}^{\omega(n,m)}$, where $L(\cdot)$ is a linear parametrization of the set $\mathcal{L}_{n,m}$ in (8). Since $\mathcal{S}$ is compact, $L(\cdot)$ continuous, and $c(s) > 0$ for all $s \in \mathcal{S}$, it follows that there exists a homogeneous polynomial $\psi : \mathbb{R}^r \rightarrow \mathbb{R}^{\omega(n,m)}$ such that

$$\mathrm{he}\left(V C(s)\right) + L\left(\psi(s)\right) < 0 \quad \forall s \in \mathcal{S}.$$

Let $Y(s)$ be a homogeneous matrix polynomial satisfying

$$Y(s) = \mathrm{he}\left(V C(s)\right) + L\left(\psi(s)\right) \quad \forall s \in \mathcal{S}.$$

It follows that $Y(s) < 0$ for all $s \in \mathcal{S}$ if and only if there exists an integer $k_0 \geq 0$ such that

$$Z(s) = \left(\sum_{i=1}^{r} s_i\right)^{k_0} Y(s)$$

has negative definite matrix coefficients, see e.g. [7]. Let us define $k = k_1 - \delta$, where k_1 is the degree of $Z(s)$. We have that the i-th matrix coefficient of $Z(s)$ is given by

$$Z_i = \mathrm{he}(V G_i) + L\left(\alpha^{(i)}\right)$$

where G_i is defined as in (30), and $\alpha^{(i)}$ is a suitable vector. Therefore, $Z_i < 0$ for all $i = 1, \ldots, l$, and hence the condition of Theorem 2 holds. $\quad\square$

The second contribution of the technical note is to propose a new condition for investigating robust stability of (1), whose sufficiency and necessity will be proved in Theorem 4 and Corollary 1, respectively. Specifically, let us define the notation

$$\mathrm{sq}(s) = \left(s_1^2, \ldots, s_r^2\right)' \tag{32}$$

and introduce the function

$$w(x,s) = x^{\{m\}'} \mathrm{he}\left(V G\left(\mathrm{sq}(s)\right)\right) x^{\{m\}} \tag{33}$$

where $G(s)$ is given by (29). We have that $w(x,s)$ is a homogeneous polynomial of degree $2m$ in x and $2(\delta + k)$ in s. We can express $w(x,s)$ as

$$w(x,s) = h(x,s)' W h(x,s) \tag{34}$$

where

$$h(x,s) = s^{\{\delta+k\}} \otimes x^{\{m\}} \tag{35}$$

and $W = W' \in \mathbb{R}^{l\sigma(n,m) \times l\sigma(n,m)}$. Define also the linear subspace

$$\mathcal{L}_{PD} = \left\{ L = L' : h(x,s)' L h(x,s) = 0 \right\} \tag{36}$$

whose dimension is given by

$$\nu(r, \delta + k, n, m) = \frac{1}{2} l\sigma(n,m) \left(l\sigma(n,m) + 1\right)$$
$$- \sigma\left(r, 2(\delta + k)\right) \sigma(n, 2m). \tag{37}$$

Theorem 4: Let $m \geq 1$ and $k \geq 0$ be integers, and let $L(\alpha)$ be a linear parametrization of $\mathcal{L}_{PD}$ in (36). The origin of (1) is robustly asymptotically stable if there exist a symmetric matrix V and a vector α satisfying the following system of LMIs:

$$\begin{cases} V > 0 \\ W + L(\alpha) < 0. \end{cases} \tag{38}$$

Proof: Suppose that (38) holds. Pre- and post-multiplying the first LMI by $x^{\{m\}'}$ and $x^{\{m\}'}$, respectively, we obtain that

$$0 < x^{\{m\}'} V x^{\{m\}} = v(x)$$

which implies that $v(x)$ is positive definite since $x^{\{m\}} \neq 0$ for all $x \neq 0$. Then, pre- and post-multiplying the first LMI by $h(x,s)'$ and $h(x,s)$, respectively, we obtain that

$$0 > h(x,s)' \left(W + L(\alpha)\right) h(x,s) = w(x,s)$$

which implies that $w(x,s) < 0$ for all $x, s \neq 0$ since in such a case $h(x,s) \neq 0$. Let us define

$$\hat{w}(x,s) = w\left(x, \mathrm{sqr}(s)\right)$$

where

$$\mathrm{sqr}(s) = \left(\sqrt{s_1}, \ldots, \sqrt{s_r}\right)'$$

First, observe that $\hat{w}(x,s)$ is a homogeneous polynomial since $w(x,s)$ is a homogeneous polynomial whose monomials have even degrees in all entries of s. Then, observe that $w(x,s) < 0$ for all $x, s \neq 0$ if and only if

$$\hat{w}(x,s) < 0 \quad \forall x \neq 0 \, \forall s \in \mathcal{S}.$$

Lastly, observe that

$$\hat{w}(x,s) = c(s) \dot{v}(x,s) \left(\sum_{i=1}^{r} s_i\right)^{k}$$

which implies that $w(x,s) < 0$ for all $x, s \neq 0$ if and only if

$$\dot{v}(x,s) < 0 \quad \forall x \neq 0 \, \forall s \in \mathcal{S}$$

and hence the theorem holds. $\quad\square$

Theorem 4 provides a condition for establishing whether there exists a HPLF $v(x)$ for (1) by exploiting its equivalent model (14). This condition investigates negative definiteness of $\dot{v}(x,s)$ for all $s \in \mathcal{S}$, firstly, by investigating such definiteness for $\left(\sum_{i=1}^{r} s_i\right)^k \dot{v}(x,s)$. Then, the constraint $s \in \mathcal{S}$ is eliminated through a transformation on s, specifically by replacing s with $\mathrm{sq}(s)$. Lastly, the negative definiteness of the obtained homogeneous polynomial is investigated by requiring that its Gram matrix $W + L(\alpha)$ (a Gram matrix for parameter-dependent polynomials built according to the vector of monomials $h(x,s)$ in (34)) is negative definite. Similarly to Theorem 2, this has the advantage of providing a condition that is not only sufficient but also necessary as it will be shown in Corollary 1.

The third contribution of the technical note is to show an equivalence result for the conditions of Theorems 2 and 4.

TABLE I
EXAMPLE 1: LOWER BOUNDS OF ζ^* PROVIDED BY THEOREMS 2 (A) AND 4 (B)

$m \backslash k$	0	1	...	9	$m \backslash k$	0
1	2.000	2.703	...	3.873	1	4.256
2	2.000	2.968	...	4.892	2	5.550
3	2.000	3.000	...	5.547	3	6.233
	(A)				(B)	

TABLE II
EXAMPLE 2: LOWER BOUNDS OF ζ^* PROVIDED BY THEOREMS 2 (A) AND 4 (B)

$m \backslash k$	0	1	...	9	$m \backslash k$	0
1	1.096	1.290	...	1.357	1	1.357
2	1.101	1.652	...	3.705	2	9.018
3	1.101	1.652	...	4.095	3	$> 10^5$
	(A)				(B)	

Theorem 5: Let $m \geq 1$ be an integer. Then, the condition of Theorem 2 holds for some integer $k = k_1$ if and only if the condition of Theorem 4 holds for some integer $k = k_2$ (with $k_2 \leq k_1$).

Proof: Suppose that the condition of Theorem 2 holds for some integer $k = k_1$. Then, observe that the matrix W can be obtained as

$$W = \text{diag}(W_1, \ldots, W_l)$$

where $W_i = \text{he}(V G_i) + L(\alpha^{(i)})$. Hence, the condition of Theorem 4 holds with $\alpha = 0$ for some integer $k = k_2$, in particular $k_2 = k_1$.

Then, suppose that the condition of Theorem 4 holds for some integer $k = k_2$. This implies that $-w(x, s)$ is positive definite and SOS. Consequently, one has that

$$\begin{cases} \dot{v}(x, s) = x^{\{m\}'} U(s) x^{\{m\}} \\ U(s) < 0 \quad \forall s \in \mathcal{S}. \end{cases}$$

By proceeding as in the proof of Theorem 3, this implies that the condition of Theorem 2 holds for some integer $k = k_1$. $\square$

As a result, the condition of Theorem 4 enjoys the same nonconservatism result of Theorem 2 according to the following corollary.

Corollary 1: The origin of (1) is robustly asymptotically stable if and only if there exist m and k such that the condition of Theorem 4 holds.

Proof: Direct consequence of Theorems 3 and 5. $\square$

Lastly, the following corollary states that the conditions of Theorems 2 and 4 are not more conservative than the conditions of Theorem 1, and hence not more conservative than the conditions in [20] which are recovered in Theorem 1 for $m = 1$.

Corollary 2: Let $m \geq 1$ be an integer, and suppose that at least one of the conditions of Theorem 1 holds. Then, there exists k such that the conditions of Theorems 2 and 4 hold.

Proof: Direct consequence of Theorem 4 in [22] and Theorem 5 in the technical note. $\square$

Although both Theorems 2 and 4 provide sufficient and necessary conditions, their computational burden can be significantly different depending on the system under investigation. For instance, Section IV shows some cases where, for the same degree of the HPLF, Theorem

4 allows one to ensure robust asymptotical stability with $k = 0$ while Theorem 2 fails even for $k = 9$ (the computational time for the latter is significantly larger than that of the former).

IV. ILLUSTRATIVE EXAMPLES

A. Example 1

Let us consider the uncertain system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ \frac{-1 + p(t) - p(t)^2}{1 + p} & -1 \end{pmatrix} x(t)$$

where $p(t)$ is an uncertain time-varying parameter constrained according to $p(t) \in [0, \zeta]$. The problem consists of determining the maximum ζ, denoted by ζ^*, for which the origin is robustly asymptotically stable. Let us express the system as in (14). We have that $n = 2$, $r = 2$, and $\delta = 2$. The matrix homogeneous polynomial $C_{rat}(s)$ can be expressed as in (16) with

$$C(s) = \begin{pmatrix} 0 & (1 + \zeta)s_1 + s_2 \\ -(1 - \zeta)s_1 - s_2 - \zeta^2 s_1^2 & -(1 + \zeta)s_1 - s_2 \end{pmatrix}$$

and

$$c(s) = (1 + \zeta)s_1 + s_2.$$

Table I shows the lower bounds of ζ^* provided by Theorems 2 and 4. These lower bounds are found through a line search over ζ performed via a bisection algorithm.

B. Example 2

Here we consider the uncertain system (see the equation at the bottom of the page). where $p(t)$ is an uncertain time-varying parameter constrained according to $p(t) \in [0, \zeta]$. The problem consists of determining the maximum ζ, denoted by ζ^*, for which the origin is robustly asymptotically stable. In this case we have $n = 3$, $r = 2$, and $\delta = 2$. Table II shows the lower bounds provided by Theorems 2 and 4.

V. CONCLUSION

This technical note has proved that a sufficient LMI condition that we previously proposed for robust stability of uncertain systems with rational dependence on unknown time-varying parameters constrained in a polytope, through the introduction of an extended version of Polya's theorem, is also necessary. Second, the technical note has proposed a new sufficient and necessary LMI condition by exploiting properties of the simplex and SOS parameter-dependent polynomials. Lastly, the technical note has investigated relationships among these conditions and conditions based on the LFR.

REFERENCES

[1] B. R. Barmish, *New Tools for Robustness of Linear Systems.* New York: McMillan, 1994.

$$\dot{x}(t) = \begin{pmatrix} -9 + 11p(t) - 6p(t)^2 & 11 - 21p(t) + 13p(t)^2 & -3 + 2p(t) \\ -4 + 6p(t) - 3p(t)^2 & 5 - 11p(t) + 5p(t)^2 & -1 + p(t) \\ 8 - 12p(t) + 6p(t)^2 & -11 + 22p(t) - 12p(t)^2 & 2 - 2p(t) \end{pmatrix} x(t)$$

[2] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*, ser. Systems and Control: Foundations and Applications. Boston, MA: Birkhauser, 2008.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[4] C. W. Scherer, "LMI relaxations in robust control," *Eur. J. Control*, vol. 12, no. 1, pp. 3–29, 2006.

[5] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*. New York: Springer, 2009.

[6] B. Reznick, "Some concrete aspects of Hilbert's 17th problem," *Contemp. Math.*, vol. 253, pp. 251–272, 2000.

[7] G. Chesi, "LMI techniques for optimization over polynomials in control: A survey," *IEEE Trans. Autom. Control*, vol. 55, no. 11, pp. 2500–2510, Nov. 2010.

[8] K. Zhou and P. P. Khargonekar, "Stability robustness bounds for linear state-space models with structured uncertainties," *IEEE Trans. Autom. Control*, vol. 32, pp. 621–623, 1987.

[9] A. P. Molchanov and E. S. Piatnitskiy, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Syst. Control Lett.*, vol. 13, no. 1, pp. 59–64, 1989.

[10] L. Xie, S. Shishkin, and M. Fu, "Piecewise Lyapunov functions for robust stability of linear time-varying systems," *Syst. Control Lett.*, vol. 31, pp. 165–171, 1997.

[11] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Autom. Control*, vol. 43, no. 4, pp. 555–559, Apr. 1998.

[12] H. L. S. Almeida, A. Bhaya, D. M. Falcao, and E. Kaszkurewicz, "A team algorithm for robust stability analysis and control design of uncertain time-varying linear systems using piecewise quadratic Lyapunov functions," *Int. J. Robust and Nonlin. Control*, vol. 11, pp. 357–371, 2001.

[13] T. Hu and Z. Lin, "Composite quadratic Lyapunov functions for constrained control systems," *IEEE Trans. Autom. Control*, vol. 48, no. 3, pp. 440–450, Mar. 2003.

[14] R. K. Brayton and C. H. Tong, "Stability of dynamical systems: A constructive approach," *IEEE Trans. Circuits Syst.*, vol. CAS-26, pp. 224–234, 1979.

[15] F. Blanchini, "Nonquadratic Lyapunov functions for robust control," *Automatica*, vol. 31, pp. 451–461, 1995.

[16] F. Blanchini and S. Miani, "A new class of universal lyapunov functions for the control of uncertain linear systems," *IEEE Trans. Autom. Control*, vol. 44, no. 3, pp. 641–647, Mar. 1999.

[17] A. L. Zelentsovsky, "Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems," *IEEE Trans. Autom. Control*, vol. 39, no. 1, pp. 135–138, Jan. 1994.

[18] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "Homogeneous Lyapunov functions for systems with structured uncertainties," *Automatica*, vol. 39, no. 6, pp. 1027–1035, 2003.

[19] G. Chesi, "LMI conditions for time-varying uncertain systems can be non-conservative," *Automatica*, vol. 47, no. 3, pp. 621–624, 2011.

[20] F. Wang and V. Balakrishnan, "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems," *IEEE Trans. Autom. Control*, vol. 47, no. 5, pp. 720–734, May 2002.

[21] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "Robust analysis of LFR systems through homogeneous polynomial Lyapunov functions," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1211–1216, Jul. 2004.

[22] G. Chesi, "Robust stability of time-varying uncertain systems with rational dependence on the uncertainty," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2353–2357, Oct. 2010.

[23] G. Chesi, "A new condition and equivalence results for robust stability analysis of rationally time-varying uncertain linear systems," in *Proc. IEEE Multi-Conf. Syst. and Control*, Denver, CO, 2011, pp. 234–239.

[24] C. Scheiderer, "Positivity and sums of squares: A guide to some recent results," in *Emerging Applications of Algebraic Geometry, IMA Volumes in Mathematics and its Applications*, M. Putinar and S. Sullivant, Eds. New York: Springer, 2009, vol. 149, pp. 271–324.

Optimal Linear Filters for Discrete-Time Systems With Randomly Delayed and Lost Measurements With/Without Time Stamps

Shuli Sun, *Senior Member, IEEE*

Abstract—A novel model is developed to describe possible random delays and losses of measurements transmitted from a sensor to a filter by a group of Bernoulli distributed random variables. Based on the new developed model, an optimal linear filter dependent on the probabilities is presented in the linear minimum variance sense by the innovation analysis approach when packets are not time-stamped. The solution to the optimal linear filter is given in terms of a Riccati difference equation and a Lyapunov difference equation. A sufficient condition for the existence of the steady-state filter is given. At last, the optimal filter is given by Kalman filter when packets are time-stamped.

Index Terms—Optimal linear filter, packet dropout, random delay, steady-state filter.

I. INTRODUCTION

In recent years, the estimation problems for networked control systems and sensor networks have attracted a lot of attention due to the wide applications in sensor positioning, signal processing and control [1], [2]. In networked systems, random delays and packet dropouts almost exist in data transmission by unreliable communications. So, the research on estimation problems over networks is significant [3], [4].

In networked systems, the phenomena of random delays and packet dropouts can be described by stochastic parameters [5]–[18]. For systems with one-step random delay, Ray *et al.* [5] present a full-order linear filter in the least-mean-square sense. However, it is suboptimal by fixing it as the Kalman-like form. Yaz *et al.* [6] design a suboptimal filter by treating a colored noise as a white noise. Based on the covariance information approach, a recursive least-square linear estimator is solved [7]. Moreover, the robust filter is also studied in [8]. For systems with multiple random delays, two filters dependent on time stamps and probabilities are designed, respectively [9]. Recently, a steady-state H_2 filter for systems with one-step random delay or packet dropouts is proposed based on a unified stochastic parameterized model by the linear matrix inequality [10]. For a system with possible infinite packet dropouts, the optimal linear estimators in the linear minimum variance sense are presented by the innovation analysis approach [11], and the full- and reduced-order linear estimators are also designed by completing square [12]. Moreover, the filter for systems with bounded consecutive packet dropouts is also developed [13]. However, the multiple random delays are not taken into account in [10]–[13]. So far, the results above are focused on random delays or packet dropouts, respectively. In [14], [15], the optimal linear estimators are presented for systems with both random delays and packet dropouts, however,