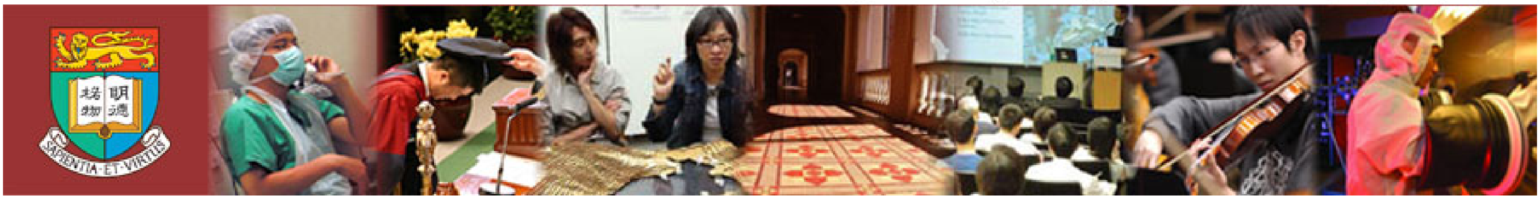

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Synchronization Conditions for Multiagent Systems With Intrinsic Nonlinear Dynamics

Dongkun Han and Graziano Chesi

***Abstract*—This brief studies local and global synchronization in multiagent systems with nonlinear dynamics with respect to equilibrium points and periodic orbits. For local synchronization, a method is proposed based on the transformation of the original system into an uncertain polytopic system and on the use of homogeneous polynomial Lyapunov functions. For global synchronization, another method is proposed based on the search for a suitable polynomial Lyapunov function. The proposed methods exploit linear matrix inequalities and have several advantages. In particular, the proposed methods require the solution of convex optimization problems. Also, the proposed methods exploit more complex Lyapunov functions than the quadratic Lyapunov functions typically considered in the literature and included in this brief as a special case.**

***Index Terms*—Lyapunov function, multiagent systems, sum of squares, synchronization.**

I. Introduction

Synchronization of complex networks has been attracting a considerable amount of academic interest, due to its broad applications in various scientific communities such as the World Wide Web, neural networks, wireless communication, and electrical power grid. Another hot topic is the consensus in multiagent systems, which shares common features with synchronization (see, for instance, [1]–[6]).

Lyapunov methods have been successfully applied to derive synchronization conditions. In particular, a synchronization problem is investigated in [4] by using edge-based adaptation laws via a Lyapunov theoretic approach. In [7], both local and global synchronizations in complex networks are investigated through generalized algebraic connectivity. In [8], some results are proposed under the assumption that the network is weighted balanced, and a distributed algorithm is proposed via nonsmooth analysis.

In this brief, local and global synchronization problems of multiagent systems with nonlinear dynamics are studied. For local synchronization, a method is proposed based on the transformation of the original system into an uncertain polytopic system and on the use of homogeneous polynomial Lyapunov functions (HPLFs). For global synchronization, another method is proposed based on the search for a suitable polynomial Lyapunov function (PLF). The proposed methods exploit linear matrix inequalities (LMIs) and have several advantages. In particular, the proposed methods require the solution of convex optimization problems. Also, the proposed methods exploit more complex Lyapunov functions than the quadratic Lyapunov functions (QLFs) typically considered in the literature and included in this brief as a special case.

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II. Preliminaries

A. Problem Formulation

The notations used throughout this brief are as follows: $\mathbb{N}, \mathbb{R}$: natural and real number sets; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; A': transpose of A; $A > 0$ $(A \geq 0)$: symmetric positive definite (semidefinite) matrix A; 0_n: origin of $\mathbb{R}^n$; I_n: $n \times n$ identity matrix; $A \bigotimes B$: Kronecker product of matrices A and B; $\mathrm{he}(A)$: $A + A'$, with $A \in \mathbb{R}^{n\times n}$; $\mathrm{co}\{X_1, \ldots. X_p\}$: convex hull of matrices $X_1, \ldots, X_p \in \mathbb{R}^{m\times n}$; and $X^{[i]}$: ith Kronecker power, i.e.,

$$X^{[i]} = \begin{cases} X \otimes X^{[i-1]} & \text{if } i > 1 \\ 1 & \text{if } i = 0. \end{cases}$$

Let $\mathcal{G} = (\mathcal{A}, \mathcal{E}, \mathcal{G})$ be a weighted and directed graph, where $\mathcal{A} = \{A_1, \ldots, A_\mathrm{N}\}$ is a finite nonempty set to describe the set of n nodes of a multiagent system, $\mathcal{E}$ is the set of directed edges belonging to $\mathcal{A} \times \mathcal{A}$, and G is an $N \times N$ weighted adjacency matrix. A directed edge from A_j to A_i is described by G_{ij}, which represents an information transmitting channel from the jth node to the ith node.

In this brief, we investigate multiagent systems with directional information exchange described by

$$\dot{x}_i(t) = f\left(x_i(t)\right) - c\sum_{j=1}^{N} L_{ij}\Gamma x_j(t),\ i, j = 1, \ldots, N \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the state of the ith agent, N is the number of agents, c is the coupling weight, $f(x_i) \in \mathbb{R}^n$ is a nonlinear function, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix where $\gamma_i > 0$ stands for an agent communicating through the ith state, and L_{ij} is the ijth entry of the Laplacian matrix $L \in \mathbb{R}^{N\times N}$ given by $L_{ij} = -G_{ij}$ for all $i \neq j$ and by $L_{ii} = -\sum_{j=1, j\neq i}^{N} L_{ij}$.

We can rewrite the uncertain multiagent dynamical system (1) in compact form as

$$\dot{x}(t) = g\left(x(t)\right) - c(L \otimes \Gamma)x(t) \quad (2)$$

where $x(t) = (x_1(t)', \ldots, x_N(t)')'$, and $g(x(t)) = (f(x_1(t))', \ldots, f(x_N(t))')'$. Let $s(t) \in \mathbb{R}^n$ be a solution of an isolated node, i.e.,

$$\dot{s}(t) = f\left(s(t)\right). \quad (3)$$

Let us observe that $s(t)$ can be either an equilibrium point, a periodic orbit, or a chaotic orbit. Then, two synchronization problems are proposed as follows.

Problem 1: To establish if the multiagent dynamical system (2) achieves local synchronization, i.e., for any ϵ, there exist $\kappa(\epsilon)$ and $T > 0$ such that $\|x_i(0) - x_j(0)\| \le \kappa(\epsilon)$ implies $\|x_i(t) - x_j(t)\| \le \epsilon$ for all $t > T$ and $i, j = 1, \ldots, N$.

Problem 2: To establish if the multiagent dynamical system (2) achieves global synchronization, i.e., for any ϵ, there exists $T > 0$ such that $\|x_i(t) - x_j(t)\| \le \epsilon$ for all $t > T$ and $i, j = 1, \ldots, N$ (regardless of $\|x_i(0) - x_j(0)\|$).

B. Positive Polynomials via LMIs

A useful tool for establishing whether a polynomial is nonnegative consists of establishing whether it is a sum-of squares (SOS) polynomial, and this is equivalent to an LMI feasibility test, as explained in [9] and references therein.

Specifically, let $x \in \mathbb{R}^r$, and let $h(x)$ be a polynomial with all the monomials having degree between $2l$ and $2m$. Let $x^{\{l,m\}} \in \mathbb{R}^{\sigma(r,l,m)}$ be a vector containing all monomials of degree between l and m where

$$\sigma(r,l,m) = \frac{(r+m)!}{r!m!} - \frac{(r+l-1)!}{r!(l-1)!}. \quad (4)$$

Then, $h(x)$ can be written according to the square matrix representation (SMR) as

$$h(x) = x^{\{l,m\}'} \left(H + E(\delta)\right) x^{\{l,m\}} \quad (5)$$

where $H \in \mathbb{R}^{\sigma(r,m,l) \times \sigma(r,m,l)}$ is a symmetric matrix, and $E(\delta)$ is a linear parametrization of the linear subspace, i.e.,

$$\mathcal{E} = \left\{ E = E' \ : \ x^{\{l,m\}'} E x^{\{l,m\}} = 0 \right\}. \quad (6)$$

Observe that $E(\delta)$ can be simply generated with standard software since it is a parametrization of the solutions of a system of linear equations.

Representation (5) allows one to establish whether a polynomial is SOS via LMIs. Indeed, $h(x)$ is SOS if there exist polynomials $h_1(x), h_2(x), \ldots$ such that

$$h(x) = \sum_i h_i(x)^2 \quad (7)$$

and this condition holds if and only if there exists δ such that the following LMI holds:

$$H + E(\delta) \ge 0. \quad (8)$$

In the sequel of this brief, we assume that l and m are the largest and the smallest integers, respectively, such that the monomials of $h(x)$ have degree between $2l$ and $2m$.

III. Conditions for Local Synchronization

A. System Transformation

For local synchronization, we introduce the following assumption on $f(x_i)$.

Assumption 1: Function $f(x_i)$ is continuously differentiable in a neighborhood of the solution $s(t)$.

Remark 1: This assumption is very mild one as it just requires that the first derivative of the vector field is continuous in a neighborhood of the solution of interest.

Let us subtract (3) from (1). We get the following system:

$$\dot{y}_i(t) = f\left(x_i(t)\right) f\left(s(t)\right) c \sum_{j=1}^{N} L_{ij} \Gamma y_j(t) \quad (9)$$

where $y_i = x_i - s$, $i = 1, \ldots, N$. System (9) can be linearized around $s(t)$ as

$$\dot{y}(t) = \left(I_N \otimes Df\left(s(t)\right)\right) y(t) c (L \otimes \Gamma) y(t) \quad (10)$$

where $y(t) = (y_1(t)', \ldots, y_N(t)')'$, and $Df(s(t)) \in \mathbb{R}^{n \times n}$ is the Jacobian matrix of $f(x_i)$ evaluated for $x_i = s(t)$. Let us define $z_i = y_1 - y_i$, $i = 2, \ldots, N$, and let $z(t) = (z_2(t)', \ldots, z_N(t)')'$. We obtain a reduced system as

$$\begin{aligned} \dot{z}(t) &= A(t) z(t) \\ &= \left(I_{N-1} \otimes Df\left(s(t)\right) - c(\widetilde{L} \otimes \Gamma) \right) z(t) \end{aligned} \quad (11)$$

where

$$\widetilde{L} = \begin{pmatrix} L_{22} - L_{12} & \ldots & L_{2N} - L_{1N} \\ \vdots & \ddots & \vdots \\ L_{N2} - L_{12} & \ldots & L_{NN} - L_{1N} \end{pmatrix}.$$

The next result directly follows from the definition of local synchronization.

Lemma 1: Suppose that Assumption 1 holds. The local synchronization of system (2) can be achieved if system (11) is asymptotically stable.

The next step consists of transforming (11) into an uncertain polytopic system of the following form:

$$\begin{cases} \dot{z}(t) = & \widehat{A}\left(p(t)\right) z(t) \\ p(t) \in & \mathcal{P} \end{cases} \quad (12)$$

where $p(t) \in \mathbb{R}^q$ is an uncertain parameter vector, $\mathcal{P}$ is the simplex defined by

$$\mathcal{P} = \mathrm{co}\{p^{(1)}, \ldots, p^{(w)}\}$$

and $\widehat{A}(p(t))$ is given by

$$\widehat{A}\left(p(t)\right) = \widehat{A}_0 + \sum_{i=1}^{q} p_i(t) \widehat{A}_i$$

for some $\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_q \in \mathbb{R}^{k \times k}$. This can be done by choosing any bounds $b_{ij}, c_{ij} \in \mathbb{R}$ satisfying

$$b_{ij} \le A_{ij}(t) \le c_{ij} \forall t \ge 0$$

for all $i, j = 1, \ldots, k$. Observe that such bounds exist since $Df(s(t))$ is continuous. Then, parameter $p_l(t)$ is assigned to each entry of $A_{ij}(t)$ choosing

$$\begin{cases} \widehat{A}_{0,ij} = & b_{ij} \\ \widehat{A}_{l,ij} = & c_{ij} - b_{ij} \end{cases}$$

in order to ensure that the uncertain polytopic system includes (11). Clearly, for entries of $A_{ij}(t)$ that are linearly dependent, one can introduce one parameter $p_l(t)$ only.

B. Local Synchronization Conditions

Robust stability of (12) can be investigated by HPLFs, a nonconservative class of Lyapunov functions whose construction can be tackled through LMIs (see, for example, [10]).

In order to derive an LMI condition based on HPLFs for local synchronization of (1), let us introduce the following result.

Theorem 1: Suppose that Assumption 1 holds. The local synchronization of (1) can be achieved if there exists a continuously differentiable homogeneous function $v(z)$ such that

$$\forall z \neq 0 \quad \begin{cases} 0 < v(z) \\ 0 < -\varrho_i(z) \quad \forall i = 1, \dots, w \end{cases} \tag{13}$$

where

$$\varrho_i(z) = \dot{v}(z,p)|_{p=p^{(i)}}$$
$$\dot{v}(z,p) = \left(\frac{dv(z)}{dz}\right)' \left(\widehat{A}(p)z\right).$$

Such a $v(z)$ is a homogeneous Lyapunov function for (12).

Proof: Suppose that (13) holds. Let us observe that

$$\dot{v}(z,p) = \sum_{i=1}^{w} d_i(p)\varrho_i(z)$$

where $d_1(p), \dots, d_w(p) \in \mathbb{R}$ are such that

$$\begin{cases} \sum_{i=1}^{w} d_i(p)p^{(i)} = p \\ \sum_{i=1}^{w} d_i(p) = 1 \\ d_i(p) \geq 0 \quad \forall i = 1, \dots, w. \end{cases}$$

Hence, (13) implies that

$$\dot{v}(z,p) < 0 \forall z \neq 0$$

i.e., $v(z)$ is a Lyapunov function for (12) for all $p \in \mathcal{P}$, particularly a homogeneous Lyapunov function. Therefore, (12) is robustly asymptotically stable, and the local synchronization of (1) can be achieved. □

Let $v(z)$ be a homogeneous polynomial of degree $2m$. We can express $v(z)$ via the SMR as

$$v(z) = z^{\{m,m\}'} V z^{\{m,m\}}$$

where $V \in \mathbb{R}^{\sigma((N-1)n,m,m)\times\sigma((N-1)n,m,m)}$ is a symmetric matrix. In order to derive the LMI condition for local synchronization, let us introduce the following definition.

Definition 1: Let $\widehat{A}^{\#}$ be the matrix satisfying

$$\frac{dz^{\{m,m\}}}{dt} = \frac{\partial z^{\{m,m\}}}{\partial z}\widehat{A}z = \widehat{A}^{\#} z^{\{m,m\}}. \tag{14}$$

Then, $\widehat{A}^{\#}$ is called *extended matrix* of $\widehat{A}$.

Lemma 2: Let $z^{[m]}$ be the mth Kronecker power of z and K_m be the matrix satisfying $z^{[m]} = K_m z^{m,m}$ [11]. Then

$$\widehat{A}^{\#} = \left(K_m' K_m\right)^{-1} K_m' \left(\sum_{i=0}^{m-1} I_{m1_{-i}} \otimes \widehat{A} \otimes I_i\right) K_m.$$

Let us define

$$\widehat{A}_i = \widehat{A}(p^{(i)})$$

and let $\widetilde{A}_i^{\#}$ be the extended matrix of $\widetilde{A}_i$. The LMI condition for local synchronization is obtained as follows.

Theorem 2: Suppose that Assumption 1 holds. For any $m \geq 1$, let $E(\delta)$ be a linear parametrization of the linear subspace (6) with $l = m$ (see Section II-B for details). The local synchronization of (1) can be achieved if there exist a symmetric matrix V and $\delta^{(1)}, \dots, \delta^{(w)}$ such that

$$\begin{cases} 0 < V \\ 0 < -\mathrm{he}\left(V\widetilde{A}_i^{\#}\right) E(\delta^{(i)}) \quad \forall i = 1, \dots, w. \end{cases} \tag{15}$$

Proof: Suppose that (15) holds. Pre- and postmultiplying the first LMI in (15) by $z^{\{m,m\}'}$ and $z^{\{m,m\}}$, respectively, one has

$$0 < z^{\{m,m\}'} V z^{\{m,m\}} = v(z)$$

hence implying that $v(z)$ is positive definite since $z^{\{m,m\}'} z^{\{m,m\}} > 0$ for all $z \neq 0$. From (14) it follows that

$$\varrho_i(z) = z^{\{m,m\}'} he\left(V\widetilde{A}_i^{\#}\right) z^{\{m,m\}}$$

and hence, from the second LMI, one has $\varrho_i(z)$ that is negative definite. Hence, from Theorem 1, it follows that $v(z)$ is a HPLF for (12), and therefore, the local synchronization of (1) can be achieved. □

Let us observe that one can systematically establish if there exist a symmetric matrix V and $\delta^{(1)}, \dots, \delta^{(w)}$ such that (15) holds. In fact, this is an LMI condition, which amounts to solving a convex optimization problem (see [9] and references therein for details).

IV. Conditions for Global Synchronization

In order to investigate the global synchronization of (1), let us rewrite (9) as

$$\dot{y}(t) = \psi\left(y(t), s(t)\right) - c(L \otimes \Gamma)y(t) \tag{16}$$

where $y(t) = (y_1(t)', \dots, y_N(t)')'$, $\psi(y(t), s(t)) = (\psi(y_1(t), s(t))', \dots, \psi(y_N(t), s(t))')'$, and

$$\psi\left(y_i(t), s(t)\right) = f\left(y_i(t) + s(t)\right) - f\left(s(t)\right), \; i = 1, \dots, N. \tag{17}$$

Let us introduce the following assumption on $f(x)$.

Assumption 2: Function $f(x_i)$ is polynomial.

Remark 2: Various existing approaches for global synchronization such as [4], [6], and [7] assume the QUAD condition (or one-side Lipschits condition). However, the QUAD condition is not satisfied for simple nonlinearities such as quadratic and cubic functions. Instead, Assumption 2 includes such nonlinearities and also includes important systems such as Lorenz and Hamiltonian systems. Moreover, continuous functions can be approximated arbitrarily well by polynomial ones, which means that Assumption 2 is indeed mild.

The following result directly follows from [12].

Lemma 3: Let $\sigma = (\sigma_1, \dots, \sigma_N)'$ with $\sigma_i > 0$, $i = 1, \dots, N$, and $\sum_{i=1}^{N} \sigma_{i=1}$. The global synchronization of (1) can be achieved if there exists a matrix, i.e.,

$$M = (I_N - 1_N\sigma') \otimes I_n \tag{18}$$

such that

$$\lim_{t\to\infty} \|My(t)| = 0. \tag{19}$$

For ease of description, let us first consider the case where $s(t)$ is constant. We have the following result.

Theorem 3: Suppose that Assumption 2 holds. The global synchronization of (1) can be achieved if there exist $\varepsilon \in \mathbb{R}$, a continuously differentiable function $v(y)$, and two functions $u_1(y)$ and $u_2(y)$ such that

$$\begin{cases} 0 \leq \varphi_i(y) \quad \forall y \forall i = \quad 1, \ldots, 4 \\ 0 < \varepsilon \end{cases} \tag{20}$$

where

$$\begin{aligned} \varphi_1(y) &= u_1(y) - \varepsilon \\ \varphi_2(y) &= u_2(y) - \varepsilon \\ \varphi_3(y) &= v(y) - u_1(y)\|My\|^2 \\ \varphi_4(y) &= -\dot{v}(y) - u_2(y)\|My\|^2 \end{aligned} \tag{21}$$

$$\dot{v}(y) = \left(\frac{dv(y)}{dy}\right)' (\psi(y,s) - c(L \otimes \Gamma)y). \tag{22}$$

Proof: Suppose that (20) holds. From the first inequality for $i = 3$, we get

$$v(y) \geq u_1(y)\|My\|^2$$

and since $u_1(y)$ is positive from the first inequality for $i = 1$

$$v(y) > 0 \; \forall y : \; My \neq 0.$$

Similarly, for $i = 4$, we obtain that

$$\dot{v}(y) < 0 \quad \forall y : \; My \neq 0.$$

Hence, $v(y)$ is positive, and its time derivative is negative whenever $My \neq 0$. This implies that (19) holds, and therefore, global synchronization of (1) can be achieved. □

Theorem 3 provides a condition for global synchronization of (1) based on the idea of searching for a Lyapunov function $v(y)$ proving (19). Let us observe that the role of term My in the definition of $\varphi_3(y)$ and $\varphi_4(y)$ is to require that $v(y)$ and $-\dot{v}(y)$ are positive whenever the synchronization is not achieved, since this implies that $v(y)$ will decrease until My vanishes.

In order to check the condition of Theorem 3 via LMIs, we consider the case where $v(y)$, $u_1(y)$, and $u_2(y)$ are polynomials. Clearly, $v(y)$ has no constant and linear monomials if it has to satisfy (20). Hence, let us parameterize $v(y)$, $u_1(y)$, and $u_2(y)$ as

$$\begin{aligned} v(y) &= w_0' y^{\{2,2m_0\}} \\ u_i(y) &= w_i' y^{\{0,2m_i\}}, \quad i = 1, 2 \end{aligned} \tag{23}$$

where, for all $i = 0, 1, 2$, m_i is an integer, and w_i is a vector of suitable size. Let us express $\varphi_i(y)$, $i = 1, \ldots, 4$, via the SMR as

$$\varphi_i(y) = y^{\{l_i,m_i\}'} \left(\Phi_i(\varepsilon, w) + E_i(\delta_i)\right) y^{\{l_i,m_i\}} \tag{24}$$

where $w = (w_0', w_1', w_2')'$.

Theorem 4: Suppose that Assumption 2 holds. The global synchronization of (1) can be achieved if there exist ε, w, and δ_i, $i = 1, \ldots, 4$, such that

$$\begin{cases} 0 \leq \Phi_i(\varepsilon, w) + E_i(\delta_i) \quad \forall i = \quad 1, \ldots, 4 \\ 0 < \varepsilon. \end{cases} \tag{25}$$

Proof: Suppose that (25). Pre- and postmultiplying the first LMI in (25) by $y^{\{l_i,m_i\}'}$ and $y^{\{l_i,m_i\}}$, respectively, one gets

$$\begin{aligned} 0 &\leq y^{\{l_i,m_i\}'} \left(\Phi_i(\varepsilon, w) + E_i(\delta_i)\right) y^{\{l_i,m_i\}} \\ &= \varphi_i(y) \quad \forall y \, \forall i = 1, \ldots, 4. \end{aligned}$$

Consequently, (20) holds, and from Theorem 3, we conclude that the global synchronization of (1) can be achieved. □

Theorem 4 provides the sought LMI condition for global synchronization of (1). This condition can be directly extended to the case where $s(t)$ is either a periodic orbit or a chaotic orbit by introducing an uncertain polytopic system, as done in Section III-A and by repeating the LMI condition in Theorem 4 at the vertices of the polytope. The details are omitted for conciseness.

V. Numerical Examples

In this section, two examples are provided to illustrate the proposed approach. These examples are deliberately simple for ease of description and due to space limits. The computations are done in MATLAB with the toolbox SeDuMi [13].

A. Example for Local Synchronization

Let us consider a two-agent system where each agent has a second-order dynamic. Model (1) is described by a nonlinear function $f(x)$ given by

$$f(x_i) = \begin{pmatrix} x_{i1} - x_{i2} - x_{i1}\left(x_{i1}^2 + x_{i2}^2\right) \\ x_{i1} + x_{i2} - x_{i2}\left(x_{i1}^2 + x_{i2}^2\right) \end{pmatrix}$$

where $x_i = (x_{i1}, x_{i2})'$, $i = 1, 2$. The linear part of (1) is described by the following constants:

$$c = 1, \; \Gamma = I_2, \; G = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Equation (3) holds, with

$$s(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad s(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Let us consider the problem of establishing local consensus for the second solution of $s(t)$, i.e., for the periodic orbit. Matrix $A(t)$ in (11) is given by

$$A(t) = \begin{pmatrix} -2 - 3\cos^2 t - \sin^2 t & -1 - 2\cos t \sin t \\ 1 - 2\cos t \sin t & -2 - \cos^2 t - 3\sin^2 t \end{pmatrix}.$$

As described in Section III-A, one can embed (11) into an uncertain polytopic system. In particular, by choosing $p_1 = \cos^2 t$ and $p_2 = \cos t \sin t$, it follows that $\widehat{A}(p)$ in (12) is given by

$$\widehat{A}(p) = \begin{pmatrix} -3 - 2p_1 & -1 - 2p_2 \\ 1 - 2p_2 & -5 + 2p_1 \end{pmatrix}.$$

Let us observe that $p_1 \in 0, 1$ and $p_2 \in [-0.5, 0.5]$, and hence, polytope $\mathcal{P}$ is given by

$$\mathcal{P} = \text{co}\left\{ \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \right\}.$$

We find that the LMI condition (15) holds and, hence, local synchronization can be achieved according to Theorem 2. In particular, an HPLF for this case is given by $v(z) = z_{21}^4 + z_{21}^2 z_{22}^2 + z_{22}^4$.

For completeness, we report in Fig. 1 some simulations. In particular, the first subfigure shows the trajectory of $x(t)$ for the initial condition $x(0) = (1, 2, -1, -2)'$, while the second

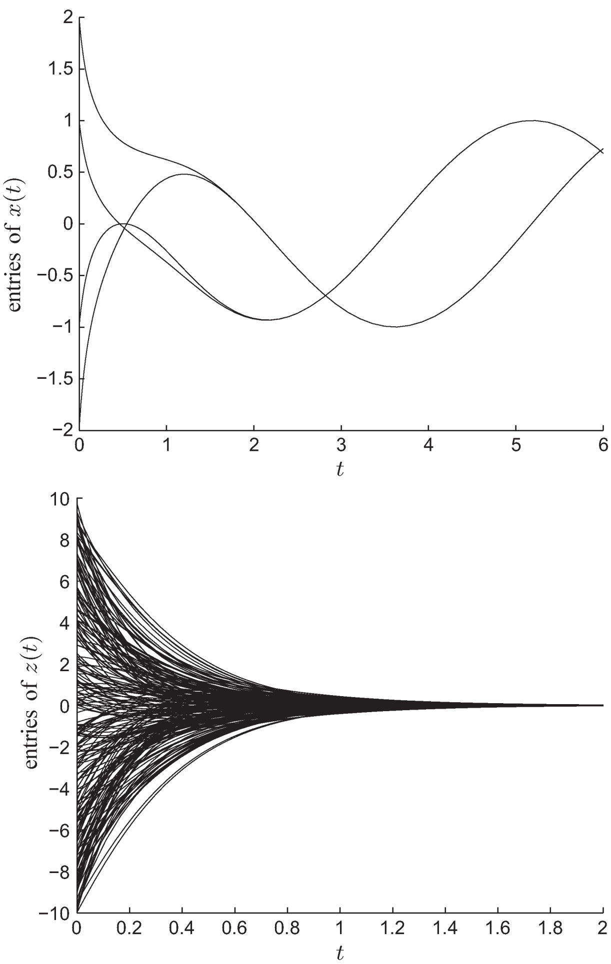

Fig. 1. Example for local synchronization.

subfigure shows 100 trajectories for $z(t)$ with initial conditions randomly chosen in $[-10, 10]^4$.

B. Example for Global Synchronization

Let us consider (1) with

$$f(x_i) = \begin{pmatrix} -x_{i2} \\ -x_{i1} - x_{i1}^3 - x_{i2} \end{pmatrix}$$

where $x_i = (x_{i1}, x_{i2})'$, $i = 1, 2$, and

$$c = 1$$

$$\Gamma = I_2$$

$$G = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}.$$

Equation (3) holds with $s(t) = (0, 0)'$. We consider the problem of establishing global consensus for this solution. To this end, we check the LMI condition (25) with auxiliary polynomials $u_i(y)$ with a degree of 2. We find that this condition cannot be satisfied using QLFs $v(y)$. Instead, the condition is feasible with Lyapunov functions with a degree of 4, in particular, the condition holds with $\varepsilon = 0.5$, $u_i(y) = 1 + y_i' y_i$, and $v(y) = y_1' y_1 + y_2' y_2 + (y_1' y_1)^2 + (y_2' y_2)^2 - y_{11}^2 y_{21}^2 - y_{12}^2 y_{22}^2$. Hence, from Theorem 4, global synchronization can be achieved.

VI. CONCLUSION

We have investigated local and global synchronizations in multiagent systems with nonlinear dynamics. For local synchronization, a method has been proposed based on the transformation into an uncertain polytopic system and on the use of HPLFs, while for global synchronization, another method has been proposed based on the search for a suitable PLF. Future work can consider the extension of these methods to switching systems following the frameworks introduced in [14]–[17].

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