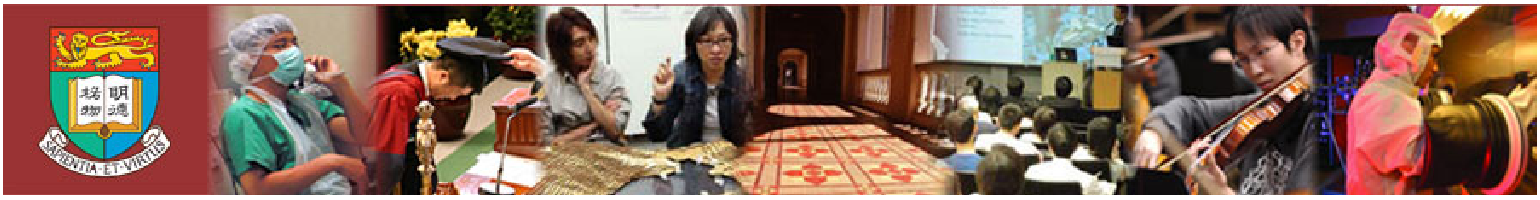

Title	Rational Lyapunov Functions for Estimating and Controlling the Robust Domain of Attraction
Author(s)	Chesi, G
Citation	Automatica, 2013, v. 49 n. 4, p. 1051-1057
Issued Date	2013
URL	http://hdl.handle.net/10722/184510
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Rational Lyapunov Functions for Estimating and Controlling the Robust Domain of Attraction

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Abstract

This paper addresses the estimation and control of the robust domain of attraction (RDA) of equilibrium points through rational Lyapunov functions (LFs) and sum of squares (SOS) techniques. Specifically, continuous-time uncertain polynomial systems are considered, where the uncertainty is represented by a vector that affects polynomially the system and is constrained into a semialgebraic set. The estimation problem consists of computing the largest estimate of the RDA (LERDA) provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LERDA. In particular, the paper shows that the computation of the best lower bound of the LERDA for chosen degrees of the SOS polynomials, which requires the solution of a nonconvex optimization problem with bilinear matrix inequalities (BMIs), can be reformulated as a quasi-convex optimization problem under some conditions. Moreover, the paper provides a necessary and sufficient condition for establishing tightness of this lower bound. Lastly, the paper discusses the search for optimal rational LFs using the proposed strategy.

1 Introduction

Studying the RDA of equilibrium points is a key problem in uncertain nonlinear systems. In fact, the RDA is the set of initial conditions for which the state of the system asymptotically converges to the equilibrium point under consideration for all admissible uncertainties. Hence, when dealing with uncertain nonlinear systems, it is not sufficient to establish that the desired equilibrium point is robustly locally asymptotically stable, but one has also to make sure that the initial condition lies inside the RDA.

It is well-known that studying the RDA is a nontrivial task. Indeed, for the case of uncertainty-free nonlinear systems, numerous methods have been proposed for computing inner estimates, see e.g. [1,2] where classic methods such as Zubov equation and La Salle theorem are discussed, and recent works such as [3, 4] based e.g. on the computation of reachable sets and logical composition of LFs. A common strategy of computing these estimates is based on linear matrix inequality (LMI) techniques and polynomial LFs, see e.g. [5–8] and references therein. Some of these methods have been extended to address controller synthesis for enlarging the domain of attraction, see e.g. [9], and the estimation of the RDA, see e.g. [10–12].

Clearly, it would be useful to enlarge the class of LFs that can be used with LMI techniques. Indeed, the use of maximal LFs was proposed in [13] for obtaining exact estimates of the domain of attraction. In particular, a procedure was developed, which allows one to approximate maximal LFs via rational functions. See also [14] where these functions are used in the case of uncertainty-free nonlinear systems, and [15] where a decomposition based on rational functions is introduced to model uncertainty in nonlinear systems.

This paper addresses the estimation and control of the RDA of equilibrium points through rational LFs and SOS techniques. Specifically, continuous-time uncertain polynomial systems are considered, where the uncertainty is represented by a vector that affects polynomially the system and is constrained into a semialgebraic set. The estimation problem consists of computing the LERDA provided by a given rational LF. The control problem consists of computing a polynomial static output controller of given degree for maximizing such a LERDA. In particular, the paper shows that the computation of the best lower bound of the LERDA for chosen degrees of the SOS polynomials, which requires the solution of a nonconvex optimization problem with BMIs, can be reformulated as a quasi-convex optimization problem under some conditions. Moreover, the paper provides a necessary and sufficient condition for establishing tightness of this lower bound. Lastly, the paper discusses the search for initial and optimal rational LFs using the proposed strategy. A numerical example suggesting that rational LFs outperform polynomial LFs of the same total degree is also provided. A preliminary version of this paper[1] appeared in [16].

[1]Without the necessary and sufficient condition for tightness of the lower bound and the search for optimal LFs.

2 Preliminaries

Notation: $\mathbb{R}$: space of real numbers; 0_n: $n \times 1$ null vector; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I: identity matrix (of size specified by the context); A': transpose of matrix A; $A > 0$, $A \geq 0$, $A = 0$: positive definite, positive semidefinite and null matrix A; $a > 0$, $a \geq 0$, $a = 0$: entrywise positive, entrywise nonnegative and null vector a; $\|a\|$: Euclidean norm of vector a; $A \otimes B$: Kronecker product of matrices A and B; $(...)'$: symmetric term, e.g. $(...)'A(B) = (B)'A(B)$; s.t.: subject to.

2.1 SOS Polynomials

A polynomial is said SOS if it is the sum of squares of polynomials. It turns out that establishing whether a polynomial is SOS amounts to checking feasibility of an LMI, see e.g. [17, 8] and references therein.

Indeed, let $p(x)$ be a polynomial of even degree with $x \in \mathbb{R}^n$. We can express $p(x)$ as

$$p(x) = m_p(x)' \left(P + L(\alpha)\right) m_p(x) \tag{1}$$

where $m_p(x)$ (called power vector) is a vector containing all the monomials of degree not greater than half the degree of $p(x)$, P is a symmetric matrix, $L(\alpha)$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{L = L' : \ m_p(x)' L m_p(x) = 0\right\}, \tag{2}$$

and α is a free vector. This representation is known as Gram matrix method and square matrix representation (SMR). The polynomial $p(x)$ is SOS if and only if there exists α such that $P + L(\alpha) \geq 0$.

Parameter-dependent polynomials can be similarly expressed. Indeed, if $p(x, \theta)$ is a polynomial with coefficients depending polynomially on $\theta \in \mathbb{R}^{n_\theta}$, one can write

$$p(x, \theta) = (...)' \left(P + L(\alpha)\right) \left(m_{p\theta}(\theta) \otimes m_{px}(x)\right) \tag{3}$$

where $L(\alpha)$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \{L = L' : \ (...)' L \left(m_{p\theta}(\theta) \otimes m_{px}(x)\right) = 0\}. \tag{4}$$

As in the previous case, $p(x, \theta)$ is SOS if and only if there exists α such that $P + L(\alpha) \geq 0$.

2.2 Problem Formulation

Let us consider the system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f(x(t),\theta)+G(x(t),\theta)u(t) \\ y(t) & = & h(x(t),\theta) \\ x(0) & = & x_{init} \\ \theta & \in & \Theta \end{array}\right. \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_{init} \in \mathbb{R}^n$ is the initial condition, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the output, and $\theta \in \mathbb{R}^{n_\theta}$ is the uncertainty. The functions $f(x(t),\theta)$, $G(x(t),\theta)$ and $h(x(t),\theta)$ are polynomial. The uncertainty θ is constrained in the semialgebraic set

$$\Theta = \{\theta \in \mathbb{R}^{n_\theta} : \; a(\theta) \geq 0, \; b(\theta) = 0\} \tag{6}$$

where $a : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_a}$ and $b : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_b}$ are polynomial functions. In the sequel the dependence on the time t will be omitted for ease of notation unless specified otherwise.

We consider that the system is controlled via

$$u = k(y) \tag{7}$$

where $k : \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ is a polynomial function of chosen degree, either given or to determine. The controller $k(y)$ is constrained by

$$f(0_n,\theta)+G(0_n,\theta)k(h(0_n,\theta)) = 0 \;\; \forall \theta \in \mathbb{R}^{n_\theta} \tag{8}$$

which imposes that the origin is an equilibrium point for all the uncertainties, and by

$$k_{ij} \in [k_{ij}^-, k_{ij}^+] \;\; \forall i = 1,\ldots,n_u \; \forall j = 1,2,\ldots \tag{9}$$

for some $k_{ij}^-, k_{ij}^+ \in \mathbb{R}$, which imposes bounds on the coefficients k_{ij} of $k(y)$. We denote the set of polynomial functions $k(y)$ just defined with $\mathcal{K}$. Such a set is either a singleton (estimation problem) or a convex polytope (control problem).

The RDA of the origin is the set of initial conditions for which the state converges to the origin, i.e.

$$\mathcal{R} = \left\{ x_{init} \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = 0 \;\forall \theta \in \Theta \right\}. \tag{10}$$

We consider the estimation and control of the RDA via rational functions of the form

$$v(x) = \frac{v_{num}(x)}{v_{den}(x)} \tag{11}$$

where $v_{num}(x)$ and $v_{den}(x)$ are polynomials satisfying

$$\begin{cases} v_{num}(0_n) = 0, \; v_{num}(x) > 0 \; \forall x \in \Omega \setminus \{0_n\} \\ v_{den}(x) > 0 \; \forall x \in \Omega \end{cases} \tag{12}$$

and Ω is a subset of $\mathbb{R}^n$ including the origin. If Ω is unbounded we also suppose that $v(x)$ is radially unbounded. We denote the sublevel sets of $v(x)$ as

$$\mathcal{V}(c) = \{x \in \mathbb{R}^n : \; v(x) \leq c\} \tag{13}$$

where $c \in \mathbb{R}$. The function $v(x)$ is a LF for the origin if

$$\exists \delta > 0 : \; \dot{v}(x,\theta) < 0 \;\; \forall x \in \Omega \setminus \{0_n\}, \; \|x\| < \delta, \; \forall \theta \in \Theta. \tag{14}$$

The problems considered in this paper amount to computing (estimation problem) and to enlarging via controller design (control problem) the LERDA provided by $v(x)$, i.e. $\mathcal{V}(\gamma^*)$ where

$$\begin{array}{rl} \gamma^* = & \sup\limits_{c,k} \; c \\ \text{s.t.} & \begin{cases} \dot{v}(x,\theta) < 0 \; \forall x \in \mathcal{V}(c) \setminus \{0_n\} \; \forall \theta \in \Theta \\ k \in \mathcal{K} \\ \mathcal{V}(c) \subset \Omega. \end{cases} \end{array} \tag{15}$$

3 Proposed Results

3.1 Establishing Estimates

Theorem 1 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a rational function satisfying (11)–(12) and let $c \in \mathbb{R}$, $c > 0$, be such that $\mathcal{V}(c) \subset \Omega$. Suppose that there exist $k \in \mathcal{K}$ and polynomial functions $q : \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}$, $r : \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_a}$ and $s : \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_b}$ such that*

$$\left.\begin{array}{rcl} p(x,\theta) & > & 0 \\ q(x,\theta) & > & 0 \\ r(x,\theta) & \geq & 0 \end{array}\right\} \;\; \forall x \in \mathbb{R}_0^n \; \forall \theta \in \mathbb{R}^{n_\theta} \tag{16}$$

where

$$\begin{array}{rcl} p(x,\theta) & = & -w(x,\theta) - q(x,\theta)\left(c v_{den}(x) - v_{num}(x)\right) \\ & & -r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) \end{array} \tag{17}$$

and

$$\begin{array}{rcl} w(x,\theta) & = & \left(v_{den}(x)\nabla v_{num}(x) - v_{num}(x)\nabla v_{den}(x)\right)' \\ & & \cdot \left(f(x,\theta) + G(x,\theta) K b_k(h(x,\theta))\right). \end{array} \tag{18}$$

Then, $v(x)$ is a LF for the origin and $c < \gamma^*$.

Proof. Suppose that (16) holds, and let $x \in \mathcal{V}(c) \setminus \{0_n\}$ and $\theta \in \Theta$. Then, the definition of Θ and the third inequality in (16) imply that

$$\left\{ \begin{array}{rcl} r(x,\theta)'a(\theta) & \geq & 0 \\ s(x,\theta)'b(\theta) & = & 0. \end{array} \right.$$

From this and the first inequality in (16) it follows that

$$\begin{array}{rcl} 0 & < & -w(x,\theta) - q(x,\theta)(cv_{den}(x) - v_{num}(x)) \\ & & -r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) \\ & \leq & -w(x,\theta) \end{array}$$

since $q(x,\theta) > 0$ from the second inequality in (16) and since $v(x) \leq c$. This implies that

$$0 > w(x,\theta) = v_{den}(x)^2 \dot{v}(x,\theta).$$

Hence, it follows that

$$\dot{v}(x,\theta) < 0$$

i.e. $v(x)$ is a LF for the origin and $c < \gamma^*$. $\square$

Theorem 1 provides the condition (16) for establishing whether the sub-level set $\mathcal{V}(c)$ of $v(x)$ is included in the RDA for an admissible controller. This condition is based on the introduction of the auxiliary polynomial functions $q(x,\theta)$, $r(x,\theta)$ and $s(x,\theta)$, which act as multipliers. Also, this condition does not require a priori knowledge of the fact whether $v(x)$ is a LF for the origin.

The condition (16) can be checked through an LMI feasibility test by exploiting SOS polynomials. Indeed, since any SOS polynomial is nonnegative, one has the following: (16) holds if there exist $k(y)$, $q(x,\theta)$, $r(x,\theta)$, $s(x,\theta)$ and ε such that

$$\left\{ \begin{array}{l} p(x,\theta) - \varepsilon\varphi_1(x,\theta) \text{ is SOS} \\ q(x,\theta) - \varepsilon\varphi_2(x,\theta) \text{ is SOS} \\ r_i(x,\theta) \text{ is SOS } \forall i = 1,\ldots,n_a \\ k \in \mathcal{K} \\ \varepsilon > 0 \end{array} \right. \tag{19}$$

where $\varphi_i(x,\theta)$, $i = 1,2$, are positive definite polynomials in x for all θ. For instance, one can simply choose $\varphi_i(x,\theta) = \|x\|^2$ if the linearized system of (5) is asymptotically stable, which is the standard case. If the linearized system is only marginally stable, one can replace $\varepsilon\varphi_i(x,\theta)$ with $\sum_{j,l} \varepsilon_{j,l} x_j^{2l}$ constrained by $\varepsilon_{j,l} \geq 0$ for all j,l and $\sum_l \varepsilon_{j,l} = \varepsilon$ for all j.

3.2 LERDA: from Nonconvex to Quasi-Convex BMI Problem

Theorem 1 can be exploited to estimate and control the LERDA, i.e. to solve problem (15). Indeed, from the LMI feasibility test (19) one can define a natural lower bound of γ^* as

$$\begin{array}{l} \hat{\gamma} = \sup\limits_{c,k,q,r,s,\varepsilon} \quad c \\ \text{s.t.} \quad \left\{ \begin{array}{l} \text{(19) holds} \\ \mathcal{V}(c) \subset \Omega \end{array} \right. \end{array} \tag{20}$$

where the constraint $\mathcal{V}(c) \subset \Omega$ can be ensured via a condition similar to (19) whenever Ω is a semialgebraic set. However, a difficulty arises: the computation of $\hat{\gamma}$ is not straightforward because the first constraint in (19) is a BMI: in fact, the Gram matrix of $p(x,\theta)$ is a bilinear function of c and the coefficients of $q(x,\theta)$, which are variables in (20). Unfortunately, optimization problems with BMIs are generally nonconvex.

Let us observe that one might think to overcome this difficulty by moving the multiplier $q(x,\theta)$ in front of $w(x,\theta)$, i.e. replacing $p(x,\theta)$ with

$$\begin{array}{rcl} \tilde{p}(x,\theta) & = & -q(x,\theta)w(x,\theta) - (cv_{den}(x) - v_{num}(x)) \\ & & -r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta). \end{array} \tag{21}$$

One of the drawbacks of this solution is that the degree of $\tilde{p}(x,\theta)$ is larger than that of $p(x,\theta)$, since the degree of $w(x,\theta)$ is larger than that of $cv_{den}(x)-v_{num}(x)$: this can easily lead to huge increases in the number of the LMI scalar variables in (19).

One way to tackle (20) is through a one-parameter sweep on c where the LMI feasibility test (19) is performed for fixed values of c. However, this solution has the drawback that the LMI feasibility test (19) has to be repeated numerous times.

Hereafter, we propose a method to solve (20) through a generalized eigenvalue problem (GEVP), which is a quasi-convex optimization problem. Specifically, for $\mu \in \mathbb{R}$ let us define the polynomials

$$\left\{ \begin{array}{rcl} p_1(x,\theta) & = & -w(x,\theta) + q(x,\theta)v_{num}(x) \\ & & -r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) \\ p_2(x,\theta) & = & q(x,\theta)\bar{v}(x) \\ \bar{v}(x) & = & v_{den}(x) + \mu v_{num}(x). \end{array} \right. \tag{22}$$

Let P_2 and Q be Gram matrices of $p_2(x,\theta)$ and $q(x,\theta)$ according to (3), and let $\bar{V}$ be a Gram matrix of $\bar{v}(x)$ according to (1). Let T be the matrix satisfying

$$m_{q\theta}(\theta) \otimes m_{qx}(x) \otimes m_v(x) = T \left(m_{p\theta}(\theta) \otimes m_{px}(x) \right) \tag{23}$$

where $m_{q\theta}(\theta) \otimes m_{qx}(x)$, $m_v(x)$ and $m_{p\theta}(\theta) \otimes m_{px}(x)$ are the power vectors for the Gram matrices Q, $\bar{V}$ and P_2. Observe that this matrix exists and is unique because $m_{p\theta}(\theta) \otimes m_{px}(x)$ contains all the monomials in $m_{q\theta}(\theta) \otimes m_{qx}(x) \otimes m_v(x)$ without repetitions.

Lemma 1 *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a rational function satisfying* (11)–(12)*, and let $\mu \in \mathbb{R}$. Then, P_2 can be chosen as*

$$P_2 = T' \left(Q \otimes \bar{V} \right) T. \tag{24}$$

Moreover,

$$Q > 0 \quad \text{and} \quad \bar{V} > 0 \quad \Rightarrow \quad P_2 > 0. \tag{25}$$

Proof. Let us pre- and post-multiply P_2 in (24) by $\left(m_{p\theta}(\theta) \otimes m_{px}(x) \right)'$ and its transpose. We get:

$$\begin{array}{rcl} 0 & < & (...)' P_2 \left(m_{p\theta}(\theta) \otimes m_{px}(x) \right) \\ & = & (...)' \left(Q \otimes \bar{V} \right) \left(m_{q\theta}(\theta) \otimes m_{qx}(x) \otimes m_v(x) \right) \\ & = & (...)' Q \left(m_{q\theta}(\theta) \otimes m_{qx}(x) \right) m_v(x)' \bar{V} m_v(x) \\ & = & p_2(x, \theta) \end{array}$$

i.e. P_2 can be chosen as in (24). Then, (25) follows due to the fact that $Q \otimes \bar{V} > 0$ if $Q > 0$ and $\bar{V} > 0$ and due to the fact that T has full column rank. □

Lemma 1 provides a formula for P_2, in particular showing that this matrix can be chosen positive definite whenever $Q > 0$ and $\bar{V} > 0$. We will discuss these conditions after the following result.

Theorem 2 *Let us suppose that $\Omega = \mathbb{R}^n$, and let $v : \mathbb{R}^n \to \mathbb{R}$ be a rational function satisfying* (11)–(12)*. Then, for any $\mu \in \mathbb{R}$, $\mu > 0$, one has that*

$$\hat{\gamma} = -\frac{z^*}{1 + \mu z^*} \tag{26}$$

where z^* is the solution of

$$
\begin{array}{rl}
z^* = & \inf\limits_{z,k,q,r,s,\varepsilon} \; z \\
s.t. & \left\{ \begin{array}{l} zp_2(x,\theta) + p_1(x,\theta) - \varepsilon\varphi_1(x,\theta) \textit{ is SOS} \\ q(x,\theta) - \varepsilon\varphi_2(x,\theta) \textit{ is SOS} \\ r_i(x,\theta) \textit{ is SOS } \forall i = 1,\ldots,n_a \\ k \in \mathcal{K} \\ \varepsilon > 0 \\ 1 + \mu z > 0. \end{array} \right.
\end{array}
\tag{27}
$$

Proof. Let us consider $zp_2(x,\theta)+p_1(x,\theta)$ in the first constraint of (27). One has that

$$
\begin{array}{rl}
 & zp_2(x,\theta) + p_1(x,\theta) \\
= & zq(x,\theta)(v_{den}(x) + \mu v_{num}(x)) - w(x) \\
 & +q(x,\theta)v_{num}(x) - r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) \\
= & -w(x,\theta) - q(x,\theta)(-zv_{den}(x) - z\mu v_{num}(x) \\
 & -v_{num}(x)) - r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) \\
= & -w(x) - r(x,\theta)'a(\theta) - s(x,\theta)'b(\theta) - (1+\mu z)q(x,\theta) \\
 & \cdot\left(\dfrac{-z}{1+\mu z}v_{den}(x) - v_{num}(x)\right).
\end{array}
$$

Hence, the first inequality in (19) coincides with the first inequality in (27) whenever Q and c are replaced by

$$
\begin{array}{rcl}
Q & \to & Q(1+\mu z) \\
c & \to & \dfrac{-z}{1+\mu z}.
\end{array}
$$

Since $1+\mu z$ is positive, the constraints in (19) are equivalent to those in (27), and hence (26) holds. $\square$

Theorem 2 states that the solution of (20) can be found by solving the equivalent optimization problem (27). The advantage of this transformation is that (27) is a GEVP whenever the Gram matrices Q and $\bar{V}$ of $q(x,\theta)$ and $\bar{v}(x)$ are positive definite. In fact, in such a case, the bilinear part of the Gram matrix of the first constraint in (27) has the form zP_2, where the condition $P_2 > 0$ required by (27) to be a GEVP is ensured by Lemma 1. GEVPs have the nice property to belong to the class of quasi-convex optimization problems, which can be systematically solved since they are free of local minima, see e.g. [18].

Let us discuss the requirements on Q and $\bar{V}$ for (27) being a GEVP. First, one can simply ensure that $Q > 0$ by not including the unnecessary constant monomial in the power vector $m_{qx}(x)$ and by choosing $\varphi_2(x, \theta) = \|m_{q\theta}(\theta) \otimes m_{qx}(x)\|^2$: in fact, with this choice, the second constraint in (27) is equivalent to $Q \geq \varepsilon I$. Second, one can simply ensure that $\bar{V} > 0$ by choosing any $\mu > 0$ if $v_{num}(x)$ and $v_{den}(x)$ have positive definite Gram matrices (the Gram matrix of $v_{num}(x)$ is built with a power vector $m_{vnum}(x)$ that does not include the unnecessary constant monomial). If $v_{num}(x)$ or $v_{den}(x)$ have not positive definite Gram matrices, one can attempt to find $\mu > 0$ such that $\bar{V} > 0$ through an LMI feasibility test (since $\bar{V}$ is linear in μ), and, if such a μ does not exist, $\hat{\gamma}$ has to be found as explained under (20).

3.3 LERDA: Tightness of the Lower Bound

Once that $\hat{\gamma}$ has been found, a natural question arises: is this lower bound tight?

Theorem 3 *Let us consider the estimation problem, and let us suppose that* $\Omega = \mathbb{R}^n$ *and* $0 < \hat{\gamma} < \infty$. *Let us define*

$$\mathcal{M} = \{(x, \theta) \in \mathbb{R}_0^n \times \mathbb{R}^{n_\theta} : \; p^*(x, \theta) = 0\} \tag{28}$$

where $p^*(x, \theta)$ *is either* $zp_2(x, \theta) + p_1(x, \theta)$*, evaluated for the optimal values of the variables in* (27)*, or* $p(x, \theta)$*, evaluated for the optimal values of the variables in* (20)*. Define also*

$$\mathcal{N} = \{(x, \theta) \in \mathcal{M} : \; \theta \in \Theta, \; v(x) = \hat{\gamma}, \; \dot{v}(x, \theta) = 0\}. \tag{29}$$

Then,

$$\hat{\gamma} = \gamma^* \quad \Longleftrightarrow \quad \mathcal{N} \neq \emptyset. \tag{30}$$

Proof. "$\Rightarrow$" Suppose that $\hat{\gamma} = \gamma^*$, and let (x^*, θ^*) be the tangent point between the surface $\dot{v}(x, \theta) = 0$ and the sublevel set $\mathcal{V}(\gamma^*)$, i.e.

$$\left\{ \begin{array}{rcl} \dot{v}(x^*, \theta^*) & = & 0 \\ v(x^*) & = & \gamma^* \\ \theta^* & \in & \Theta. \end{array} \right.$$

Such a point exists from the definition of γ^* in (15) since $\dot{v}(x, \theta)$ is continuous. Let us observe that

$$\left\{ \begin{array}{rcl} w(x^*, \theta^*) & = & 0 \\ \gamma^* v_{den}(x^*) - v_{num}(x^*) & = & 0 \end{array} \right.$$

Let us also observe that $p^*(x,\theta)$ and $r_i^*(x,\theta)$ are SOS for all $i=1,\ldots,n_a$ due to the constraints in (27) and (20), and hence they are nonnegative. Moreover, since $a(\theta^*)\geq 0$ and $b(\theta^*)=0$, it follows that

$$\begin{array}{rcl} 0 & \leq & p^*(x^*,\theta^*) \\ & = & -w(x^*,\theta^*)-\zeta q^*(x^*,\theta^*)(\gamma^* v_{den}(x^*)-v_{num}(x^*)) \\ & & -r^*(x^*,\theta^*)'a(\theta^*)-s(x^*,\theta^*)'b(\theta^*) \\ & \leq & -w(x^*,\theta^*)-\zeta q^*(x^*,\theta^*)(\gamma^* v_{den}(x^*)-v_{num}(x^*)) \\ & = & 0 \end{array}$$

where $\zeta=1+\mu z^*$ if $p^*(x,\theta)$ has been obtained from $zp_2(x,\theta)+p_1(x,\theta)$ or $\zeta=1$ if $p^*(x,\theta)$ has been obtained from $p(x,\theta)$. Hence, $p^*(x^*,\theta^*)=0$, i.e. $(x^*,\theta^*)\in\mathcal{M}$. Therefore, $(x^*,\theta^*)\in\mathcal{N}$, and $\mathcal{N}\neq\emptyset$.

"$\Leftarrow$" Suppose that $\mathcal{N}\neq\emptyset$ and let (x,θ) be a point of $\mathcal{N}$. It follows that $v(x)=\hat{\gamma}$ and $\dot{v}(x,\theta)=0$. Since $\hat{\gamma}$ is a lower bound of γ^*, this implies that (x,θ) is a tangent point between the surface $\dot{v}(x,\theta)=0$ and the sublevel set $\mathcal{V}(\hat{\gamma})$. Hence, $\hat{\gamma}=\gamma^*$. □

Theorem 3 provides a condition for establishing whether the lower bound $\hat{\gamma}$ is tight for the estimation problem. In particular, this occurs if $\mathcal{N}$ in (29) is nonempty, which can be found via trivial substitution from $\mathcal{M}$ in (28). Regarding the computation of $\mathcal{M}$, let us observe that

$$\mathcal{M}=\left\{(x,\theta)\in\mathbb{R}_0^n\times\mathbb{R}^{n_\theta}:\ m_{p\theta}(\theta)\otimes m_{px}(x)\in\ker(M)\right\} \tag{31}$$

where M is a positive semidefinite Gram matrix M of $p^*(x,\theta)$, which exists since $p^*(x,\theta)$ is SOS due to the constraints in (27) and (20). The construction of $\mathcal{M}$ in (31), which consists of looking for power vectors into a linear subspace, can be obtained through linear algebra operations in standard cases, see e.g. [8] and references therein.

3.4 Searching for the LF

First of all, an initial LF can be typically obtained by analyzing the linearized system of (5). Indeed, define the matrix

$$A(\theta)=\left.\frac{df(x,\theta)}{dx}\right|_{x=0_n} \tag{32}$$

and let P be a symmetric matrix satisfying (if any)

$$\left.\begin{array}{rcl} P & > & 0 \\ PA(\theta)+A(\theta)'P & < & 0 \end{array}\right\}\ \forall\theta\in\Theta. \tag{33}$$

The condition (33) is equivalent to the existence of a quadratic LF for the linearized system, and can be investigated through LMIs by exploiting SOS matrix polynomials, see e.g. [8]. Then,

$$v_0(x) = x' P x \tag{34}$$

is a quadratic LF for the origin, and a rational LF can be simply obtained as

$$v(x) = \frac{v_0(x) + v_1(x)}{v_{den}(x)} \tag{35}$$

for any polynomials $v_1(x)$ and $v_{den}(x)$ such that $v(x)$ satisfies (11)–(12) and

$$\left\{ \begin{array}{rcl} \nabla v_1(0_n) & = & 0 \\ \nabla^2 v_1(0_n) & = & 0. \end{array} \right. \tag{36}$$

Observe that such a LF is guaranteed to provide nonempty estimates of the RDA since (33) holds.

At this point, one can conduct a search for a LF providing less conservative estimates of the RDA. In the literature, two main criteria have been proposed in the case of polynomial LFs. The first criterion attempts to maximize the volume of the estimate, while the second criterion attempts to maximize the size of a set with simple shape (generally fixed) included in the estimate, see e.g. [8].

In the present framework, the first criterion can be formulated as

$$\begin{array}{rcl} \tau^* & = & \sup\limits_{v} \ \tau(\mathcal{V}(\gamma^*)) \\ & & \text{s.t.} \ \ v(x) \text{ satisfies (11)–(12)} \end{array} \tag{37}$$

where $\tau(\mathcal{V}(\gamma^*))$ denotes an approximation of the volume of the sublevel set $\mathcal{V}(\gamma^*)$, while the second criterion can be formulated as

$$\begin{array}{rcl} \rho^* & = & \sup\limits_{v,\rho} \ \rho \\ & & \text{s.t.} \ \left\{ \begin{array}{l} v(x) \text{ satisfies (11)–(12)} \\ \mathcal{B}(\rho) \subseteq \mathcal{V}(\gamma^*) \end{array} \right. \end{array} \tag{38}$$

where $\mathcal{B}(\rho)$ is the sublevel set

$$\mathcal{B}(\rho) = \{x \in \mathbb{R}^n : \ b(x) \leq \rho\} \tag{39}$$

and $b(x)$ is a chosen positive definite polynomial (e.g., $\mathcal{B}(\rho)$ is a sphere by choosing $b(x) = \|x\|^2$). In these criteria, the conditions (11), (12) and $\mathcal{B}(\rho) \subseteq \mathcal{V}(\gamma^*)$ can be ensured through SOS polynomials analogously to (19).

Unfortunately, both criteria unavoidably lead to nonconvex optimization problems, as it happens also in the case of quadratic or polynomial LFs [8]. This is due to the product of the LF $v(x)$ with the multiplier $q(x,\theta)$ in the first constraint of (16): indeed, such a product makes this constraint a BMI. These nonconvex optimization problems can be solved locally in various ways, e.g. by alternatively fixing the LF and the multiplier as done in Example 1 in the next section. Let us observe that Theorem 2 can be used when the LF is fixed in order to estimate γ^*.

4 Examples

In these examples the degree in x and the degree in θ of the multipliers $q(x,\theta)$ and $r(x,\theta)$ are chosen equal to the respective maximum values for which the degrees of $p(x,\theta)$ are equal to those of $w(x,\theta)$ times a quadratic polynomial in x. The polynomials $\varphi_1(x,\theta)$ and $\varphi_2(x,\theta)$ are chosen equal to $\|m_{p\theta}(\theta) \otimes m_{px}(x)\|^2$ and $\|m_{q\theta}(\theta) \otimes m_{qx}(x)\|^2$, respectively, where the power vectors $m_{px}(x)$ and $m_{qx}(x)$ do not include the unnecessary constant monomial.

4.1 Example 1

Let us consider the system described by

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -(1+2\theta)x_1 + x_2 - \theta x_1 x_2^2 \\ \dot{x}_2 & = & -3x_1 - 2x_2 + \theta x_1^2 + (2\theta - 1)x_1^3 \\ \theta & \in & [0,1]. \end{array} \right.$$

We consider the problem of determining the LERDA of the origin provided by the LF

$$v(x) = \frac{(x_1^2 + x_2^2)(x_1^2 + x_1 x_2 + x_2^2 + 1)}{1 + x_1 + x_1^2 + 2x_2^2},$$

i.e. computing γ^* in (15). This LF is simply built according to (32)–(36).

Hence, we compute the lower bound $\hat{\gamma}$ in (26). It is easy to see that $v_{num}(x)$ and $v_{den}(x)$ are SOS and have positive definite Gram matrices, and we can choose any $\mu > 0$ to achieve $\bar{V} > 0$ according to the discussion under Theorem 2, in particular we use $\mu = 1$. We find $\hat{\gamma} = 1.145$. Next, we investigate the tightness of the found lower bound. We compute the set $\mathcal{N}$ in (29), hence finding

$$\mathcal{N} = \left\{ (x,\theta) : \; x = (1.135, -1.362)', \; \theta = 0 \right\}$$

and from Theorem 3 we conclude that $\hat{\gamma}$ is tight since $\mathcal{N}$ is nonempty, i.e. $\hat{\gamma} = \gamma^*$. Figure 1 shows the curve $\dot{v}(x, \theta) = 0$ for some admissible values of θ, the boundary of the LERDA, and the x-part of the point in $\mathcal{N}$.

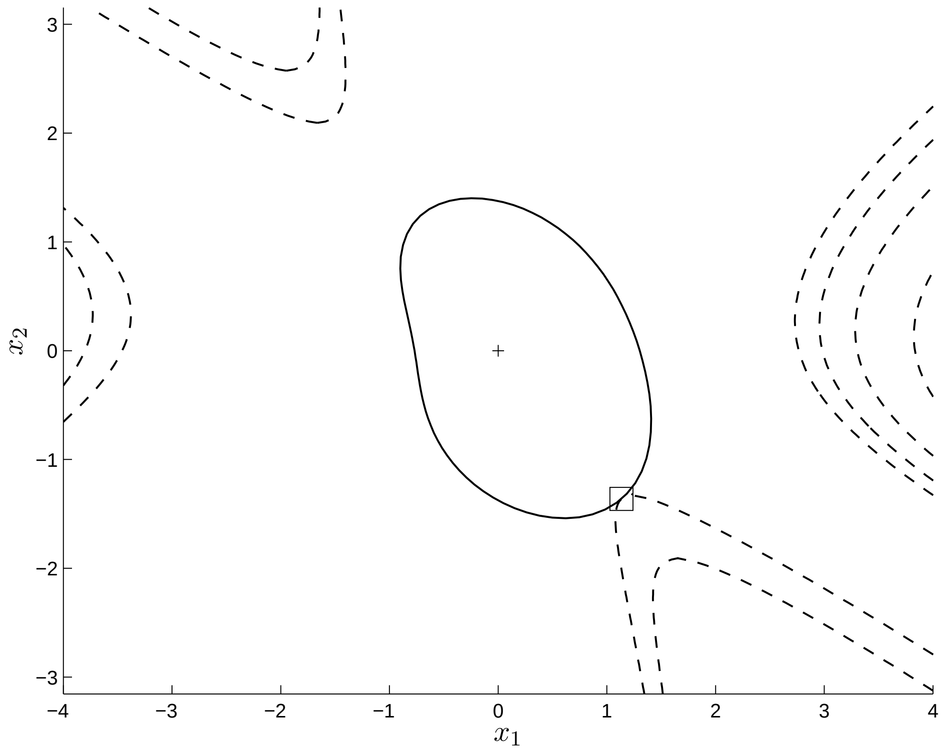

Figure 1: Example 1. Curve $\dot{v}(x, \theta) = 0$ for some admissible θ (dashed), boundary of the LERDA $\mathcal{V}(\gamma^*)$ (solid line), and x-part of the point in $\mathcal{N}$ ("□" mark).

Lastly, we compare the estimates provided by rational LFs and polynomial LFs. We consider the search for optimal LFs enlarging the estimate of the RDA according to the criterion (38) with the simple choice $b(x) = \|x\|^2$. For the case of rational LFs, we search over the LFs with numerator and denominator of degree 4 and 2, while, for the case of polynomial LFs, we search over the LFs of degree 6. The BMI problem (38) is solved with the initialization $v(x) = (x_1^2 + x_2^2)(x_1^2 + x_1x_2 + x_2^2 + 1)$ in both cases, which is the numerator of the LF previously considered, and which is a special case of the considered rational LFs and polynomial LFs.

The found solutions of the BMI problem are as follows: with rational

LFs, the lower bound of ρ^* is $\hat{\rho}_r = 31.346$ provided by

$$\begin{array}{l} v_r(x) = (18.301x_1^2 + 3.321x_1x_2 + 3.198x_2^2 - 0.204x_1^3 \\ -0.032x_1^2x_2 - 0.031x_1x_2^2 - 0.002x_2^3 + 0.017x_1^4 - 0.039x_1^3x_2 \\ +0.024x_1^2x_2^2 + 0.002x_1x_2^3 + 0.003x_2^4)/(113.297 - 1.262x_1 \\ +0.294x_2 + 0.123x_1^2 + 0.515x_1x_2 + 0.570x_2^2) \end{array}$$

and, with polynomial LFs, the lower bound of ρ^* is $\hat{\rho}_p = 19.842$ provided by

$$\begin{array}{l} v_p(x) = 38.242x_1^2 + 0.486x_1x_2 + 3.786x_2^2 - 0.004x_1^3 \\ -0.012x_1^2x_2 - 0.004x_1x_2^2 - 0.000x_2^3 - 2.272x_1^4 + 0.001x_1^3x_2 \\ -0.472x_1^2x_2^2 - 0.033x_1x_2^3 - 0.043x_2^4 + 0.000x_1^5 + 0.001x_1^4x_2 \\ +0.000x_1^3x_2^2 + 0.000x_1^2x_2^3 - 0.000x_1x_2^4 - 0.000x_2^5 + 0.046x_1^6 \\ -0.002x_1^5x_2 + 0.015x_1^4x_2^2 + 0.002x_1^3x_2^3 + 0.003x_1^2x_2^4 \\ +0.001x_1x_2^5 + 0.000x_2^6. \end{array}$$

Figure 2 shows the estimates $\mathcal{V}(\gamma^*)$ provided by $v_r(x)$ (first largest closed curve), $v_p(x)$ (second largest closed curve) and $v(x)$ (fourth largest closed curve). In order to show the importance of the denominator in $v_r(x)$, this figure also shows the estimate $\mathcal{V}(\gamma^*)$ provided by the numerator of $v_r(x)$ (third largest closed curve).

Although the found solutions might be affected by the presence of local optima in the BMI problems, they suggest that rational LFs can provide larger estimates than polynomial LFs. This is especially true by observing that the considered polynomial LFs have more degrees of freedom than the considered rational LFs. In fact, in the considered polynomial LFs there are 24 free coefficients (locally quadratic polynomial of degree 6 normalized up to a scale factor), while in the considered rational LFs there are only 16 free coefficients (the numerator is a locally quadratic polynomial of degree 4 and the denominator is a polynomial of degree 2, both normalized up to a scale factor).

4.2 Example 2

Let us consider the system described by

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 + \theta x_2^3 \\ \dot{x}_2 & = & -x_1 - x_2 + 2(1-\theta^2)x_1^2 + x_1 u \\ y & = & x_1 - x_2 \\ \theta & \in & [-1, 1]. \end{array} \right.$$

We consider the design of a polynomial output controller for enlarging the RDA of the origin by using the LF

$$v(x) = \frac{4x_1^2 + 2x_1x_2 + 2x_2^2 + x_1^4 + x_2^4}{1 + x_1^2 + x_2^2}$$

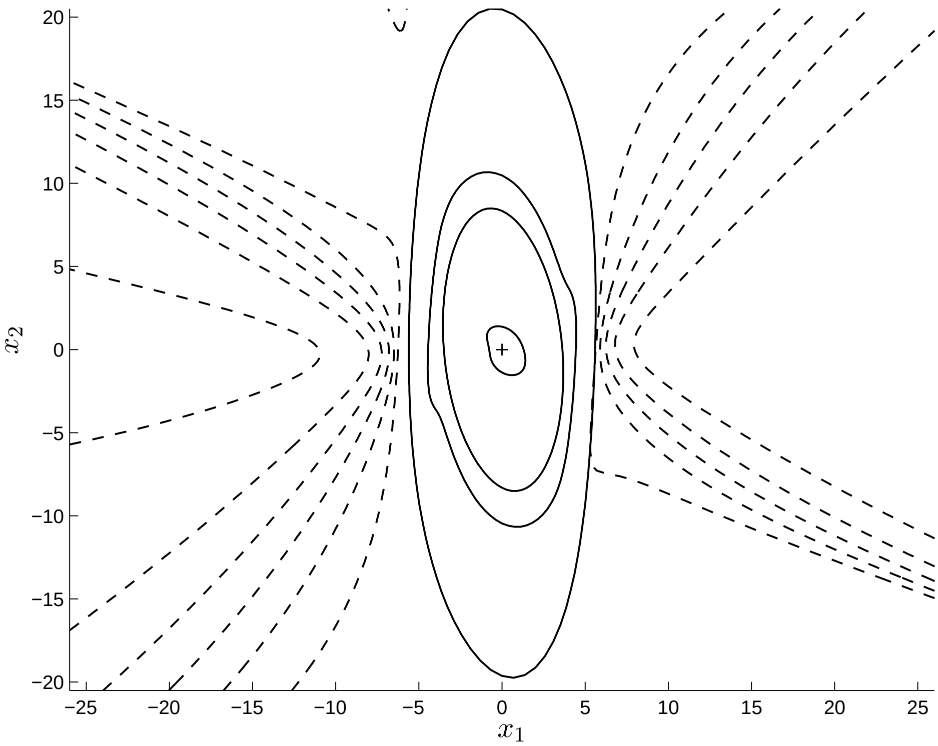

Figure 2: Example 1. Curve $\dot{v}_1(x,\theta) = 0$ for some admissible θ (dashed), boundary of the LERDA $\mathcal{V}(\gamma^*)$ for $v_r(x)$, $v_p(x)$, the numerator of $v_r(x)$, and $v(x)$ (solid line, from the largest to the smallest).

(as in Example 1, the LF is simply found according to (32)–(36)). The control structure is chosen as

$$\begin{cases} u = k(y) \\ k(y) = k_{11} + k_{12}y + k_{13}y^2 \\ k_{11}, k_{12}, k_{13} \in [-1, 1] \end{cases}$$

where k_{11}, k_{12}, k_{13} are the coefficients to determine. Observe that, in the closed-loop system, the origin is an equilibrium point for all possible values of k_{11}, k_{12}, k_{13}.

We compute the lower bound $\hat{\gamma}$ in (26) (as in Example 1, any $\mu > 0$ can be chosen, and we use $\mu = 1$). Firstly, we consider $k(y)$ constant (i.e., $k_{12} = k_{13} = 0$), finding

$$\hat{\gamma} = 0.960, \quad k(y) = -0.718.$$

For $k(y)$ linear (i.e. $k_{13} = 0$) we obtain

$$\hat{\gamma} = 1.058, \quad k(y) = -0.778 - 0.151y.$$

Lastly, we suppose that $k(y)$ is quadratic, finding

$$\hat{\gamma} = 1.392, \quad k(y) = -1.000 - 0.961y + 0.805y^2.$$

Next, we investigate the tightness of the lower bound for the quadratic controller. The set $\mathcal{N}$ in (29) is given by

$$\mathcal{N} = \left\{(x, \theta) : \ x = (0.494, 0.494)', \ \theta = 0.241\right\}$$

and hence from Theorem 3 we conclude that $\hat{\gamma}$ is tight. Figure 3 shows the boundaries of the LERDA provided by the three found controllers, the curve $\dot{v}(x, \theta) = 0$ corresponding to the quadratic controller for some admissible values of θ, and the x-part of the point in $\mathcal{N}$.

5 Conclusion

It has been shown that the best lower bound of the LERDA for chosen degrees of the SOS polynomials, which requires the solution of a nonconvex optimization problem with BMIs, can be obtained by solving a quasi-convex optimization problem under some conditions. Moreover, a necessary and sufficient condition for establishing tightness of this lower bound has been provided. As discussed and shown in the paper, the proposed results can readily be exploited in the search for optimal LFs.

Acknowledgement

The author would like to thank the Editors and the Reviewers for their useful comments.

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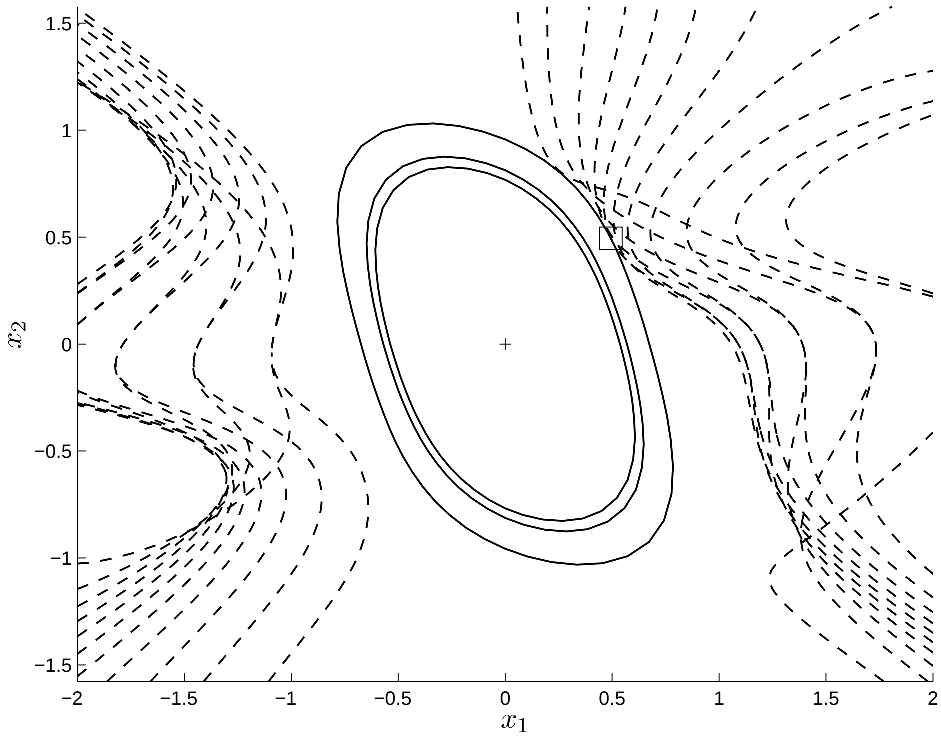

Figure 3: Example 2. Boundaries of the LERDA provided by the three found controllers (solid line), curve $\dot{v}(x, \theta) = 0$ for the quadratic controller for some admissible θ (dashed), and x-part of the point in $\mathcal{N}$ ("□" mark).

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