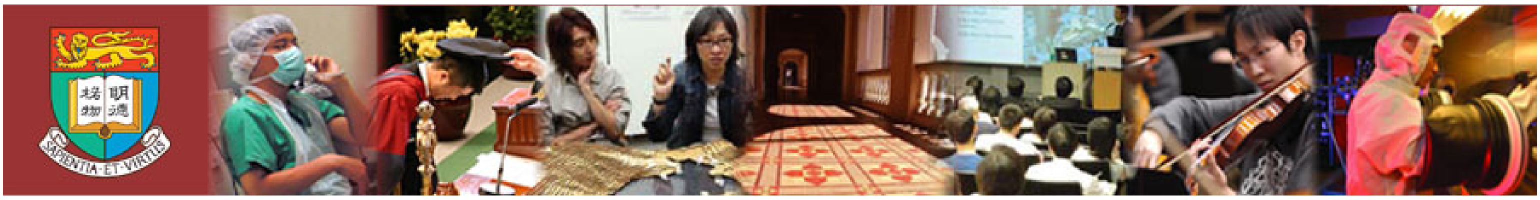

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Robust Consensus for a Class of Uncertain Multi-Agent Dynamical Systems

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***Abstract*—This paper investigates robust consensus for a class of uncertain multi-agent dynamical systems. Specifically, it is supposed that the system is described by a weighted adjacency matrix whose entries are polynomial functions of an uncertain vector constrained in a semi-algebraic set. For this uncertain topology, we provide necessary and sufficient conditions for ensuring robust first-order consensus and robust second-order consensus, in both cases of positive and non-positive weighted adjacency matrices. Moreover, we show how these conditions can be investigated through convex programming by using standard software. Some numerical examples illustrate the proposed results.**

***Index Terms*—Convex programming, multi-agent system, robust consensus, uncertain system.**

I. Introduction

THE model of multi-agent dynamical systems has been widely applied in the research of sensor networks, neural networks, and biological networks [1]–[5]. In particular, in recent years, interests are intensively cast on networked control and coordinated behavior in multi-agent systems [6]–[12]. Achieving consensus is a key problem in this area and as a growing number of applications of multi-agent system emerges, the research on consensus gains an essential importance on various areas such as complex dynamical network, filter design for multiple sensors, synchronization, formation, and rendezvous.

Traditional research topics focus on the deterministic system to establish static model, while a growing number of research focus attention on the uncertainties of multi-agent system according to the unexpected link failure, communication delay, interaction limit, and noise interference in the system [13]–[15]. A simple but compelling mathematic description of a group of autonomous agents is the Vicsek model, where possible changing of the nearest neighbor sets over time is an inherent property. This model is applied to the interaction with directional information exchange, hence introducing a more general model where each edge of a weighting matrix has a positive weighting factor.

In this paper, we investigate robust consensus for uncertain multi-agent dynamical systems. In particular, it is supposed that the weighted adjacency matrix of the closed-loop system is affected by uncertain parameters, reflecting for instance missing information on the control gains. Each entry of the weighted adjacency matrix is allowed to be a generic polynomial function of an uncertain vector constrained in a semi-algebraic set. This framework includes typical cases such as affine linear dependence of the system coefficients on an uncertain vector constrained in a polytope. For this uncertain topology, we provide necessary and sufficient conditions for ensuring robust first-order consensus and robust second-order consensus, in both cases of positive and nonpositive weighted adjacency matrices. These conditions are obtained in general by exploiting the uncertain Laplacian matrices of the system and by introducing parameter-dependent Lyapunov functions for a suitably transformed system. Moreover, we show how these conditions can be investigated through convex programming by using standard software. Some numerical examples illustrate the proposed results.

This paper is organized as follows. Section II provides the problem formulation and some preliminaries. Section III describes the proposed conditions for robust first-order consensus and robust second-order consensus. Section IV illustrates the proposed results with some numerical examples. Finally, Section V concludes the paper with some final remarks.

II. Preliminaries

A. Problem Formulation

Notation:

$\mathbb{N}, \mathbb{R}$	natural and real number sets;
A'	transpose of A;
$A > 0$ $(A \geq 0)$	symmetric positive definite (semidefinite) matrix A;
0_n	origin of $\mathbb{R}^n$;
1_n	$n \times 1$ vector with all the entries equal to 1;
I	identity matrix (of size defined by the context);
$\mathrm{img}(A)$	image of matrix A;
$\ker(A)$	null space of matrix A;
$A \otimes B$	Kronecker product of matrices A and B;
$\mathrm{spc}(A)$	set of eigenvalues of A $\in \mathbb{R}^{n\times n}$, i.e., $\mathrm{spc}(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$.

Let $G = (A, E, G)$ be a weighted digraph of order n with the set of nodes $A = \{A_1, \ldots, A_n\}$, set of directed edges E belonging to $A \times A$, and a weighted adjacency matrix $G =$

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$(G_{ij})_{n\times n}$. If an information can be transmitted from the jth node to the ith node, a directed edge $e_{ij}\in E$ is denoted, i.e., a directed edge $e_{ij}\in E$ if and only if $G_{ij}\neq 0$. In particular, G is called positive if $G_{ij}>0$ for all i,j, otherwise G is called nonpositive.

For distinct nodes A_{ik}, $k=1,\ldots,l$, let a sequence of edges $(A_i,A_{i1}),(A_{i1},A_{i2}),\ldots,(A_{il},A_j)$ be a directed path from A_i to A_j. If there is a directed path between any pair of distinct notes A_i and A_j for graph G, then it is denoted as a strongly connected graph. Provided that, for some node i, there is a directed path from i to any other node, the node i is called a root of the graph. A directed tree is a direct graph G with the property that there is exactly one root and except the root, every node in G has exactly one parent. For a directed graph of order n, a spanning tree of a directed graph is a directed tree with $n-1$ edges which connect all of the n nodes of the graph. If any subset of edges contains or forms a spanning tree, we say that the graph has a spanning tree.

In this paper, we investigate robustness of consensus to uncertain parameters. In particular, it is supposed that the weighted adjacency matrix of the closed-loop system is affected by uncertain parameters, reflecting for instance missing information on the control gains. We denote such a matrix as $G(\theta)$, where $\theta\in\mathbb{R}^r$ is an uncertain vector constrained as

$$\theta\in\Omega \tag{1}$$

where

$$\Omega=\{\theta\in\mathbb{R}^r:\ s_i(\theta)\geq 0\ \forall i=1,\ldots,h\} \tag{2}$$

for some functions $s_1,\ldots,s_h:\mathbb{R}^r\to\mathbb{R}$. In the sequel, we will assume that the entries of $G(\theta)$ and $s_1(\theta),\ldots,s_h(\theta)$ are polynomials. Moreover, we say that $G(\theta)$ is positive if $G_{ij}(\theta)>0$ for all i,j and for all $\theta\in\Omega$, otherwise $G(\theta)$ is called nonpositive.

For robust first-order consensus, we consider the continuous-time uncertain multi-agent dynamical system described by

$$\dot{x}_i(t)=\sum_{j=1,j\neq i}^{n}G_{ij}(\theta)(x_j(t)-x_i(t)),\qquad i=1,\ldots,n \tag{3}$$

where x_i is the state of the ith node, and $G(\theta)$ is both positive and nonpositive. The robust first-order consensus problem is as follows.

Problem 1: To establish if, for any initial state, the uncertain multi-agent dynamical system (3) achieves robust first-order consensus, i.e.,

$$\lim_{t\to\infty}x_i(t)-x_j(t)=0\quad\forall i,j\forall\theta\in\Omega. \tag{4}$$

In order to address this problem, we rewrite the uncertain multi-agent dynamical system (3) as

$$\dot{x}(t)=-L(\theta)x(t) \tag{5}$$

where $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ is the state vector and $L(\theta)=(L_{ij}(\theta))_{n\times n}$ is the uncertain Laplacian matrix given by

$$\begin{aligned}L_{ij}(\theta)&=-G_{ij}(\theta)\quad\forall i\neq j\\ L_{ii}(\theta)&=-\sum_{j=1,j\neq i}^{n}L_{ij}(\theta).\end{aligned} \tag{6}$$

It is worth pointing out that the uncertain Laplacian matrix has the diffusion property that

$$\sum_{j=1}^{n}L_{ij}(\theta)=0\quad\forall i=1,\ldots,n. \tag{7}$$

For robust second-order consensus problem, we consider the continuous-time uncertain multi-agent dynamical system described by

$$\begin{aligned}\dot{x}_i(t)&=\rho_i(t)\\ \dot{\rho}_i(t)&=\sum_{j=1,j\neq i}^{n}\alpha G_{ij}(\theta)(x_j(t)-x_i(t))\\ &\quad+\sum_{j=1,j\neq i}^{n}\beta G_{ij}(\theta)(\rho_j(t)-\rho_i(t))\end{aligned} \tag{8}$$

where $x_i\in\mathbb{R}$ is the position state of the ith node, $\rho_i\in\mathbb{R}$ is the velocity state of the ith node, and $\alpha,\beta\in\mathbb{R}$ are constants. Different from first-order consensus, second-order consensus requires that not merely do the position states of agents tend to be the same, but also the velocity states of agent converge to a consistent value. Based on this we propose the problem of robust second-order consensus as follows.

Problem 2: We now address the problem of establishing if, for any initial state, the uncertain multi-agent dynamical system (8) achieves robust second-order consensus, i.e.,

$$\begin{matrix}\lim_{t\to\infty}x_i(t)-x_j(t)=0\\ \lim_{t\to\infty}\rho_i(t)-\rho_j(t)=0\end{matrix}\quad\forall i,j\ \forall\theta\in\Omega. \tag{9}$$

In order to address this problem, we rewrite the uncertain multi-agent dynamical system (8) as

$$\begin{aligned}\dot{x}_i(t)&=\rho_i(t)\\ \dot{\rho}_i(t)&=-\sum_{j=1}^{n}\alpha L_{ij}(\theta)x_j(t)-\sum_{j=1}^{n}\beta L_{ij}(\theta)\rho_j(t)\end{aligned} \tag{10}$$

where $x\in\mathbb{R}^n$ is the position state vector and $\rho\in\mathbb{R}^n$ is the velocity state vector. We define the global state vector as $y=(x',\rho')'\in\mathbb{R}^{2n}$. Then, system (10) can be rewritten in compact form as

$$\dot{y}(t)=\tilde{L}(\theta)y(t) \tag{11}$$

where $\tilde{L}(\theta)$ is the uncertain extended Laplacian matrix given by

$$\tilde{L}(\theta)=\begin{bmatrix}0 & I\\ -\alpha L(\theta) & -\beta L(\theta)\end{bmatrix}. \tag{12}$$

B. SOS Polynomials

Let $f(\theta)$ be a polynomial of degree $2m$ in $\theta \in \mathbb{R}^r$. Then, $f(\theta)$ can be always written as

$$f(\theta) = \theta^{\{m\}'}(F + C(\delta))\theta^{\{m\}} \quad (13)$$

where $\theta^{\{m\}}$ is a vector containing all monomials of degree less than or equal to m in θ, F is a symmetric matrix, and $C(\delta)$ is a linear parametrization of the subspace

$$\mathcal{C} = \left\{C = C' : \theta^{\{m\}'}C\theta^{\{m\}} = 0\right\}.$$

The representation (13) is known as Gram matrix method and square matrix representation (SMR). This representation allows one to establish whether a polynomial is SOS via LMIs. Indeed, $f(\theta)$ is SOS if there exist polynomials $f_1(\theta), f_2(\theta), \ldots$ such that

$$f(\theta) = \sum_i f_i(\theta)^2$$

and this condition holds if and only if there exists δ such that the following LMI feasibility test holds:

$$F + C(\delta) \geq 0.$$

This technique can also be used in the case of matrix polynomials. Specifically, let $M(\theta)$ be a symmetric matrix polynomial of size $s \times s$ of degree $2m$ in $\theta \in \mathbb{R}^r$ (this means that all of the entries of $M(\theta)$ are polynomials whose highest degree in θ is $2m$). Then, $M(\theta)$ can be written as

$$M(\theta) = \Delta(\bar{M} + D(\delta), m, s) \quad (14)$$

where

$$\Delta(\bar{M}+D(\delta), m, s) = \left(\theta^{\{m\}} \otimes I\right)' (\bar{M}+D(\delta)) \left(\theta^{\{m\}} \otimes I\right)$$

and I is the $s \times s$ identity matrix, $\bar{M}$ is a symmetric matrix, and $D(\delta)$ is a linear parametrization of the subspace

$$\mathcal{D} = \{D = D' : \Delta(D, m, s) = 0\}.$$

Similarly to the scalar case, $M(\theta)$ is SOS if there exist matrix polynomials $M_1(\theta), M_2(\theta), \ldots$ such that

$$M(\theta) = \sum_i M_i(\theta)' M_i(\theta)$$

and this condition holds if and only if there exists δ such that the following LMI feasibility test holds:

$$\bar{M} + D(\delta) \geq 0.$$

See, for instance, [16]–[18] and references therein for details and algorithms about SOS polynomials.

III. Conditions for Robust Consensus

Here, the robust first-order and second-order consensus conditions are derived, respectively.

A. Robust First-Order Consensus

Lyapunov stability theory is widely used to study the property of dynamical system. For the first time, we associate the robust consensus with Lyapunov stability theory, and we provide a new condition for investigating robust first-order consensus based on matrix inequalities. Specifically, define a matrix $V_1 \in \mathbb{R}^{n \times n-1}$ such that

$$\text{img}(V_1) = \ker(1_n'). \quad (15)$$

Then we get the transformed uncertain Laplacian matrix:

$$\hat{L}(\theta) = V_1' L(\theta) V_1. \quad (16)$$

Theorem 1: Robust first-order consensus for uncertain multi-agent system (with both positive and non-positive weighted digraph) can be achieved if and only if there exists a symmetric function $P_1 : \mathbb{R}^r \to \mathbb{R}^{n-1 \times n-1}$ such that

$$\begin{cases} P_1(\theta) > 0 \\ P_1(\theta)\hat{L}(\theta) + \hat{L}(\theta)' P_1(\theta) > 0 \end{cases} \quad \forall \theta \in \Omega. \quad (17)$$

In order to investigate the condition of Theorem 1, we can exploit SOS matrix polynomials introduced in Section II-B. Indeed, it is easy to verify that (17) holds if there exist matrix polynomials $P_1(\theta)$, $G_{1i}(\theta)$ and a scalar $c > 0$ such that

$$\begin{cases} G_{1i}(\theta) \text{ is SOS} \\ P_1(\theta) - I \text{ is SOS} \\ R_1(\theta) - cI \text{ is SOS} \end{cases} \quad (18)$$

where

$$R_1(\theta) = P_1(\theta)\hat{L}(\theta) + \hat{L}(\theta)' P_1(\theta) - \sum_{i=1}^{h} G_{1i}(\theta) s_{1i}(\theta). \quad (19)$$

In fact, whenever the constraints in (18) hold with $c > 0$, for any $\theta \in \Omega$ it follows that $G_{1i}(\theta) \geq 0$, $P_1(\theta) > 0$, and

$$\begin{aligned} 0 &\leq P_1(\theta)\hat{L}(\theta) + \hat{L}(\theta)' P_1(\theta) - \sum_{i=1}^{h} G_{1i}(\theta) s_{1i}(\theta) - cI \\ &\leq P_1(\theta)\hat{L}(\theta) + \hat{L}(\theta)' P_1(\theta) \end{aligned}$$

i.e., (17) holds.

The condition (18) can be formulated via a convex optimization problem by using the representation of matrix polynomials reported in Section II. Indeed, it directly follows that (17) holds if $c^* > 0$, where c^* is the solution of the convex optimization problem

$$c^* = \sup_{c, \bar{G}_{1i}, \bar{P}_1, \delta} c$$
$$\text{s.t.} \begin{cases} \bar{G}_{i1} \geq 0 \\ \bar{F}_1 + D_1(\delta) - cI - \sum_{i=1}^{h} \bar{U}_{1i}(\bar{G}_{1i}) \geq 0 \\ \text{trace}(\bar{P}_1) = 1. \end{cases} \quad (20)$$

The matrices involved in this problem are defined by

$$\begin{aligned} G_{1i}(\theta) &= \Delta(\bar{G}_{1i}, m_i, n-1) \\ G_{1i}(\theta) s_{1i}(\theta) &= \Delta(\bar{U}_{1i}(\bar{G}_{1i}), m_0, n-1) \\ P_1(\theta) &= \Delta(\bar{P}_1, m, n-1) \\ R_1(\theta) &= \Delta(\bar{F}_1 + D_1(\delta), m_0, n-1). \end{aligned}$$

Here, $2m_i$ is the degree of $G_{1i}(\theta)$, $2m$ is the degree of $P_1(\theta)$, and $2m_0$ is the degree of $R_1(\theta) - cI$.

For an interaction topology with positive weighted interaction topology but without parametric uncertainties, it has been found that the topological structure determines whether the consensus can be achieved. The following theorem extends to the case of uncertain multi-agent dynamical systems three existing conditions found for the case of multi-agent dynamical systems without uncertainty [19] and provides a further condition in terms of zeros of a polynomial.

Theorem 2: For a given uncertain Laplacian matrix $L(\theta)$ in (6) and a network $G = (A, E, G(\theta))$ with a positive weighted digraph, i.e., $\exists e_{ij} \in E$ if and only if $G_{ij}(\theta) > 0$, the following statements are equivalent.

1) Robust first-order consensus can be achieved.
2) $\forall\theta \in \Omega$, $L(\theta)$ has exactly one simple eigenvalue 0 and all the other eigenvalues have positive parts.
3) $\forall\theta \in \Omega$, the directed graph G has a spanning tree.
4) $\forall\theta \in \Omega$, $q(\theta) \neq 0$, where

$$q(\theta) = \left.\frac{d}{d\lambda} l(\lambda, \theta)\right|_{\lambda=0} \quad (21)$$

$$l(\lambda, \theta) = \det(\lambda I - L(\theta)). \quad (22)$$

One way of checking the condition of Theorem 2 consists of using SOS polynomials and amounts to solving an LMI problem. Specifically, statement d) in Theorem 2 holds if there exist polynomials $g_i(\theta)$ and a scalar $c > 0$ such that

$$\begin{cases} g_i(\theta) \text{ is SOS} \\ (-1)^k q(\theta) - c - \sum_{i=1}^{h} g_i(\theta) s_{2i}(\theta) \text{ is SOS} \end{cases} \quad (23)$$

where $k \in \{0, 1\}$ is defined by

$$k = \begin{cases} 0, & \text{if } q(\theta_0) > 0 \\ 1, & \text{otherwise} \end{cases}$$

and θ_0 is any vector θ in Ω which can be freely chosen.

B. Robust Second-Order Consensus

Let us consider the problem of establishing robust second-order consensus. For this problem, we exploit the uncertain expanded Laplacian matrix $\tilde{L}(\theta)$. Extending the results given in [12] for the case of multi-agent dynamical systems without uncertainty, one has that robust second-order consensus for the uncertain multi-agent dynamical system (11) can be obtained if and only if $-\tilde{L}(\theta)$ has only one zero eigenvalue of algebraic multiplicity two and all the other eigenvalues are in the open right half plane.

Starting from this result, we provide a new condition for investigating robust second-order consensus based on matrix inequalities. Specifically, define vectors as

$$u_1 = \begin{pmatrix} 1_n \\ 0_n \end{pmatrix} \quad u_2 = \begin{pmatrix} 0_{n-1} \\ 1_n \end{pmatrix}. \quad (24)$$

Let $V_2 \in \mathbb{R}^{2n \times 2n-1}$ and $V_3 \in \mathbb{R}^{2n-1 \times 2n-2}$ be matrices such that

$$\begin{aligned} \text{img}(V_2) &= \ker(u_1') \\ \text{img}(V_3) &= \ker(u_2'). \end{aligned} \quad (25)$$

Let us define the transformed uncertain expanded Laplacian matrix:

$$\breve{L}(\theta) = -V_3' V_2' \tilde{L}(\theta) V_2 V_3. \quad (26)$$

Theorem 3: Robust second-order consensus for uncertain multi-agent system with both positive and non-positive weighted digraph can be achieved if and only if there exists a symmetric function $P_2 : \mathbb{R}^r \to \mathbb{R}^{2n-2 \times 2n-2}$ such that

$$\begin{cases} P_2(\theta) > 0 \\ P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)' P_2(\theta) > 0 \end{cases} \quad \forall\theta \in \Omega. \quad (27)$$

In order to investigate the existence of a function $P_2(\theta)$ satisfying condition (27), we can exploit SOS matrix polynomials. It is easy to verify that (27) holds if there exist matrix polynomials $P_2(\theta)$, $G_{3i}(\theta)$ and a scalar $c > 0$ such that

$$\begin{cases} G_{3i}(\theta) \text{ is SOS} \\ P_2(\theta) - I \text{ is SOS} \\ R_2(\theta) - cI \text{ is SOS} \end{cases} \quad (28)$$

where

$$R_2(\theta) = P_2(\theta)\breve{L}(\theta) + \breve{L}(\theta)' P_2(\theta) - \sum_{i=1}^{h} G_{3i}(\theta) s_{3i}(\theta).$$

Before concluding this section, let us remark that the proposed results for establishing robust consensus in uncertain multi-agent systems require the solution of optimization problems, in contrast to existing conditions for establishing consensus in uncertainty-free multi-agent systems where one just needs to check the eigenvalues of the Laplacian matrices. Unfortunately, this is unavoidable, as it happens also for the simpler problem of establishing robust stability of uncertain linear systems, see for instance [17].

IV. Numerical Examples

Here, we present some illustrative examples where robust first-order and second-order consensus are investigated for uncertain multi-agent dynamical systems. The optimization problems are solved with the standard MATLAB toolbox SeDuMi. The SMR matrices are built using the algorithms reported in [17] and references therein.

A. Example 1

In this example, we consider the uncertain four-agent system shown in Fig. 1. It is assumed that the network is affected by an uncertain parameter, specifically

$$G(\theta) = \begin{bmatrix} 1 & 2-2\theta & 5+\theta & 2+\theta \\ 3\theta & 1 & 0 & 0 \\ 0 & 4-3\theta & 1 & 0 \\ 2+3\theta & 0 & 0 & 1 \end{bmatrix}$$

where θ is constrained in the set Ω chosen as $\Omega = [0, 1]$. Hence, we have $n = 4$ and $r = 1$. Moreover, Ω can be described as in (2) with

$$s_1(\theta) = \theta(1 - \theta).$$

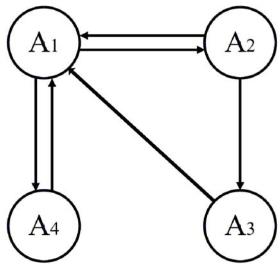

Fig. 1. Digraph of a four-agent system.

According to (6), the Laplacian matrix $L(\theta)$ is given by

$$L(\theta) = \begin{bmatrix} 9 & -2+2\theta & -5-\theta & -2-\theta \\ -3\theta & 3\theta & 0 & 0 \\ 0 & 4-3\theta & 4-3\theta & 0 \\ -2-3\theta & 0 & 0 & 2+3\theta \end{bmatrix}.$$

We observe that $G(\theta)$ is positive since all its entries are nonnegative for all $\theta \in \Omega$. This implies that we can use either condition (17) or statement 4) of Theorem 2 to investigate robust first-order consensus.

First, we use condition (17) by looking for a constant matrix function $P_1(\theta)$. By solving (20), we can find $c^* = 0.9792$. Therefore, robust first-order consensus is achieved.

Then, let us use statement 4) of Theorem 2. In particular, the polynomial $q(\theta)$ is given by

$$q(\theta) = 18\theta^3 + 6\theta^2 - 112\theta - 56.$$

According to statement 4) of Theorem 2, robust first-order consensus is achieved if and only if $q(\theta) \neq 0$ for all $\theta \in [0,1]$. In this case, it is easy to see that $q(\theta)$ satisfies this property since $q(\theta)$ is an univariate polynomial with roots 2.79, -1.3316 and -1.7917 which are all lying outside $[0,1]$. Nevertheless, let us use condition (23). In this case, $k = 1$ and by simply choosing a multiplier $g_1(\theta)$ of degree 2 we find that this condition holds with $c = 56$, which proves that statement 4) of Theorem 2 is satisfied. Fig. 2 shows the process of robust first-order consensus with the initial states and θ randomly chosen in $[0,1]$ for five times.

Next, let us consider the problem of establishing whether this uncertain network is able to achieve robust second-order consensus with $\alpha = \beta = 1$ in the system (8), and we look for a constant matrix function $P_2(\theta)$ satisfying (27). Nevertheless, let us use the condition (28), and we can find $c^* = 0.0913$. Therefore, robust second-order consensus is achieved with chosen α and β. In this case, the uncertain extended Laplacian matrix is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 9 & l_1 & l_2 & l_3 & -9 & l_1 & l_2 & l_3 \\ l_4 & -l_4 & 0 & 0 & l_4 & -l_4 & 0 & 0 \\ 0 & l_5 & -l_5 & 0 & 0 & l_5 & -l_5 & 0 \\ l_6 & 0 & 0 & -l_6 & l_6 & 0 & 0 & -l_6 \end{bmatrix}$$

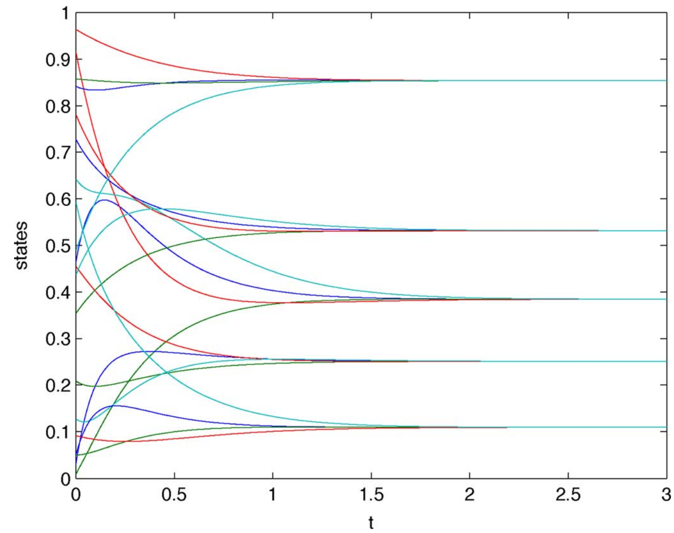

Fig. 2. Trajectories of robust first-order consensus.

where $l_1 = 2-2\theta$, $l_2 = 5+\theta$, $l_3 = 2+\theta$, $l_4 = 3\theta$, $l_5 = 4-3\theta$, and $l_6 = 2+3\theta$.

B. Example 2

In this example, we consider the uncertain matrix $G(\theta)$ given by

$$G(\theta) = \begin{bmatrix} 1 & 5+2\theta_1\theta_2 & 2-3\theta_1^2 & -3\theta_1\theta_2 \\ 2\theta_2^2-3\theta_1\theta_2 & 1 & 0 & 0 \\ 0 & 4+2\theta_1^2 & 1 & 0 \\ 3\theta_1\theta_2+6 & 0 & 0 & 1 \end{bmatrix}$$

where $\theta \in \mathbb{R}^2$ is constrained in the set Ω chosen as $\Omega = [-1,1]^2$. Hence, we have $n = 4$ and $r = 2$. Moreover, Ω can be described as in (2) with

$$s_i(\theta) = 1 - \theta_i^2 \quad \forall i = 1,2.$$

In this case, $G(\theta)$ is not positive, hence let us use condition (17) to investigate robust first-order consensus. We look for a constant matrix function $P_1(\theta)$ satisfying (17), and, by solving (20), we find $c^* = 0.769$. Therefore, robust first-order consensus is achieved.

Next, let us consider the problem of establishing whether this uncertain network is able to achieve robust second-order consensus with $\alpha = 1, \beta = 0.25$ in the system (8). We look for a constant matrix function $P_2(\theta)$ satisfying (27). Let us use the condition (28), and we find $c^* = -0.0024$, which does not prove (28). We repeat the procedure by looking for a matrix function $P_2(\theta)$ of degree 2, and we find a positive c^*. Therefore, robust second-order consensus is achieved.

C. Example 3

With a topology shown in Fig. 3, an uncertain six-agent system is considered in this example. It is assumed that the

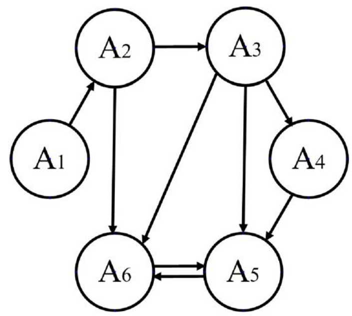

Fig. 3. Digraph of a six-agent system.

network is affected by two uncertain parameters, i.e., θ_1 and θ_2. Specifically the uncertain matrix $G(\theta)$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3+2\theta_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3-\theta_2 & 1 & 0 & 2\theta_1+\theta_2 & 0 \\ 0 & 0 & 5+2\theta_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 3-4\theta_2 \\ 0 & 5 & 2-3\theta_1 & 0 & 2-\theta_2 & 1 \end{bmatrix}$$

where $\theta \in \mathbb{R}^2$ is constrained in the set Ω chosen as $\Omega = \{\theta : \|\theta\| \leq 1\}$. Hence, we have $n = 4$ and $r = 2$. Moreover, Ω can be described as in (2) with

$$s_1(\theta) = 1 - \theta_1^2 - \theta_2^2.$$

Also, in this case, $G(\theta)$ is not positive, hence let us use condition (17) to investigate robust first-order consensus. We look for a constant matrix function $P_1(\theta)$ satisfying (17), and, by solving (20), we find $c^* = 0.1135$, i.e., robust first-order consensus is achieved.

Next, let us consider the problem of establishing whether this uncertain network is able to achieve robust second-order consensus with $\alpha = 1, \beta = 0.6$ in the system (8). We look for a constant matrix function $P_2(\theta)$ satisfying (27). Let us use the condition (28), and we find $c^* < 0$, which does not prove (28). We repeat the procedure by looking for a matrix function $P_2(\theta)$ of degree 1, and we find $c^* = 0.0034$, i.e., robust second-order consensus is achieved.

V. Conclusion

In this paper, we have addressed robust first-order consensus and robust second-order consensus for a class of uncertain multi-agent dynamical systems. Specifically, we have considered a generic framework where the system is described by a weighted adjacency matrix whose entries are polynomial functions of an uncertain vector constrained in a semialgebraic set. For this uncertain topology, we have provided necessary and sufficient conditions for ensuring robust consensus in both cases of positive and non-positive weighted adjacency matrices. Moreover, we have shown how these conditions can be easily investigated through convex programming by using standard software. Various future directions can be taken starting from the results proposed in this paper, for instance one can consider switching topology adopting the frameworks introduced in [20], [21] and LMI techniques for switching systems as the one introduced in [22]. Also, multi-agent dynamical systems with rational dependence on the uncertainty and/or time-varying uncertainty can be considered adopting the methodology proposed in [23].

Appendix

A. Proof of Theorem 1

We observe that 1_n is an eigenvector of $L(\theta)$ corresponding to the eigenvalue zero. Moreover, observe that $V_1' L(\theta) V_1$ has the same eigenvalues of $L(\theta)$ except that the algebraic multiplicity of the eigenvalue zero has been decreased by one, i.e.,

$$\operatorname{spc}(\hat{L}(\theta)) \cup \{0\} = \operatorname{spc}(L(\theta)). \quad (29)$$

Let us define a dynamical system

$$\dot{\hat{x}}(t) = -\hat{L}(\theta)\hat{x}(t). \quad (30)$$

We observe that $x = \gamma 1_n$ is the equilibrium point of (30), $\forall \gamma \in \mathbb{R}$. Hence, the robust first-order consensus can be achieved is equivalent to the statement that (30) is asymptotically stable. According to (29) and the Lyapunov stability theorem, (30) is asymptotically stable for all $\theta \in \Omega$ if and only if $L(\theta)$ has exactly one simple eigenvalue 0 and all the other eigenvalues have positive parts. From Lyapunov stability theorem for linear systems, this is equivalent to say that there exists $P_1(\theta)$ such that (17) holds for all $\theta \in \Omega$. Therefore, the theorem holds.

B. Proof of Theorem 2

Assume the Laplacian matrix $L(\theta)$ is constructed by (6). Then, the first three statements are equivalent and follow directly from the analogous ones found for the case of multi-agent dynamical systems without uncertainty [19]. From [19, Lemma 3.3], one has that $\Re(\lambda_i(L(\theta))) \geq 0, \forall i = 1, 2 \ldots, n, \forall \theta \in \Omega$. Moreover, statement d) implies that $L(\theta)$ has exactly one zero eigenvalue, $\forall \theta \in \Omega$. Thus, statements 2) and 4) are equivalent. Therefore, the theorem holds.

C. Proof of Theorem 3

Observe that u_1 is an eigenvector of $\tilde{L}(\theta)$ corresponding to the eigenvalue zero. Moreover, observe that $V_2' \tilde{L}(\theta) V_2$ has the same eigenvalues of $\tilde{L}(\theta)$ except that the algebraic multiplicity of the eigenvalue zero has been decreased of one. Similarly, it follows that $V_3' V_2' \tilde{L}(\theta) V_2 V_3$ has the same eigenvalues of $\tilde{L}(\theta)$ except that the algebraic multiplicity of the eigenvalue zero has been decreased of two. Hence, it follows that robust second-order consensus can be achieved if and only if $-\tilde{L}(\theta)$ has all of the eigenvalues in the open right half plane for all $\theta \in \Omega$. From Lyapunov stability theorem for linear systems, this is equivalent to say that there exists $P_2(\theta)$ such that (27) holds for all $\theta \in \Omega$. Therefore, the theorem holds.

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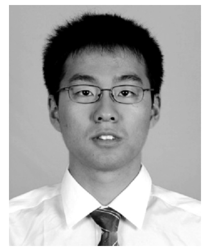

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