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Title	Worst-Case Mahler Measure in Polytopic Uncertain Systems
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Citation	IEEE Transactions on Automatic Control, 2012, v. 57 n. 12, p. 3208-3213
Issued Date	2012
URL	http://hdl.handle.net/10722/184507
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convexifying the Pareto rate region over other existing techniques in terms of resultant user rates. Then, the MCO problem has been transformed into a single-objective optimization problem by using NB. A variety of characteristics for NB in MIMO interference systems such as the uniqueness and the optimality of different NB solutions have been considered. A sufficient condition ensuring the uniqueness of the pure-strategy NB solution in MIMO interference systems has been derived. A method to determine the optimality among FP- and TDM-based NB solutions has been presented as well. Finally, the convexity of the rate region and the existence of the FP-based NB solution have also been demonstrated via numerical studies.

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Worst-Case Mahler Measure in Polytopic Uncertain Systems

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Abstract—The Mahler measure provides a way to quantify the unstable and plays a key role in stabilization problems. This technical brief addresses the computation of the worst-case Mahler measure in systems depending polynomially on uncertain parameters constrained in a polytope. A sufficient condition for establishing an upper bound of the worst-case Mahler measure is provided in terms of linear matrix inequality (LMI) feasibility tests, where a homogeneous parameter-dependent quadratic Lyapunov function (HPD-QLF) is searched for. Moreover, it is shown that the best upper bound guaranteed by this condition can be obtained by solving generalized eigenvalue problems. Then, the conservatism of this methodology is investigated, showing that the upper bound is monotonically nonincreasing with the degree of the HPD-QLF, and that there exists a degree for which the upper bound is guaranteed to be tight. Some numerical examples illustrate the proposed results.

Index Terms—Linear matrix inequality (LMI), Mahler measure, networked control system, robustness, uncertainty.

I. INTRODUCTION

The Mahler measure [1], i.e., the absolute product of the unstable eigenvalues of a matrix, provides a way to quantify the unstable in discrete-time linear systems, see in particular the recent work [2]. This measure plays a key role in control systems. For instance, in networked control systems, an important issue is stabilization with information constraint in the input channel, see e.g., [3]–[6]. This information constraint can be modeled in several ways including data-rate constraint [7], [8], quantization [9], and signal-to-noise ratio [10]. As it has been shown in the literature, solutions for this issue can be obtained in terms of the Mahler measure of the system, see e.g., [11], [12].

Unfortunately, the model of a control system is not exactly known in general. In fact, its coefficients can be affected by uncertain parameters, for instance representing physical quantities that cannot be measured exactly or that are subject to changes. This means that analysis and control issues should consider not just one model but instead a family of admissible ones. In terms of the Mahler measure, hence, it appears important to determine the worst-case value among all the admissible models.

Systems with uncertainty can be modeled in various ways. One of the most used in the literature is known as polytopic description of the uncertainty and consists of expressing the coefficients of the system as functions of uncertain parameters constrained into a bounded convex polytope. This description includes the standard case of uncertain systems affected by scalar parameters constrained into intervals, and has been adopted for addressing numerous issues in systems with uncertainty, such as robust stability, robust performance, and robust control, see e.g., [13]–[17] and references therein among many contributions. Before proceeding it is worth mentioning that the uncertainty can be modeled also in other ways, e.g., through quadratic forms as done in [18].

This technical brief investigates the Mahler measure in uncertain systems affected by polytopic uncertainty. Specifically, a discrete-time linear system is considered, whose coefficients are generic polynomial functions of an uncertain vector constrained in a bounded convex polytope. The problem consists of determining the worst-case Mahler measure of the system for all the admissible uncertainties. A sufficient condition for establishing an upper bound of the worst-case Mahler measure is provided in terms of linear matrix inequality (LMI) feasibility tests, where a homogeneous parameter-dependent quadratic Lyapunov function (HPD-QLF) is searched for. Moreover, it is shown that the best upper bound guaranteed by this condition can be obtained by solving generalized eigenvalue problems. Then, the conservatism of this methodology is investigated, showing that the upper bound is monotonically nonincreasing with the degree of the HPD-QLF, and that there exists a degree for which the upper bound is guaranteed to be tight. Some numerical examples illustrate the proposed results.

The technical brief is organized as follows. Section II introduces the problem formulation and some preliminaries on the representation of polynomials. Section III describes the proposed results. Section IV presents some illustrative examples. Lastly, Section V concludes the technical brief with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

The notation used throughout the technical brief is as follows: $\mathbb{R}$: space of real numbers; $\mathbb{C}$: space of complex numbers; 0_n: $n \times 1$ null vector; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of matrix A; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A; $(A)_{i,j}$: entry of matrix A in position (i, j); $\text{conv}\{A, B, \ldots\}$: convex hull of $A, B, \ldots$; $\text{diag}\{A, B, \ldots\}$: block diagonal matrix with blocks $A, B, \ldots$; $\text{Re}(a)$ ($\text{Im}(a)$): real (imaginary) part of $a \in \mathbb{C}$; $|a|$: magnitude of $a \in \mathbb{C}$, i.e., $|a| = \sqrt{\text{Re}(a)^2 + \text{Im}(a)^2}$; $\text{sq}(a)$: $(a_1^2, \ldots, a_n^2)'$, $a \in \mathbb{R}^n$.

We consider polytopic uncertain discrete-time linear systems of the form

$$x(t+1) = A(p)x(t) \tag{1}$$

where t is a nonnegative integer, $x(t) \in \mathbb{R}^n$ is the state vector, $p \in \mathbb{R}^q$ is the uncertain vector, and $A : \mathbb{R}^q \to \mathbb{R}^{n \times n}$ is a matrix polynomial of degree δ. The uncertain vector p is constrained according to

$$p \in \mathcal{P} \tag{2}$$

where $\mathcal{P}$ is the polytope

$$\mathcal{P} = \text{conv}\left\{p^{(1)}, \ldots, p^{(r)}\right\} \tag{3}$$

and $p^{(1)}, \ldots, p^{(r)} \in \mathbb{R}^q$ are given vectors.

Let us introduce the Mahler measure. This measure provides a way to quantify how unstable a matrix is (for discrete-time systems). Specifically, let $X \in \mathbb{R}^{n \times n}$. The Mahler measure of X is defined as

$$M(X) = \prod_{i=1}^{n} \max\{1, |\lambda_i(X)|\} \tag{4}$$

where $\lambda_i(X) \in \mathbb{C}$ is the i-th eigenvalue of X.

1) Problem: The problem that we consider in this technical brief consists of determining the worst-case Mahler measure of the system (1)–(3), i.e., the quantity

$$\mu = \sup_{p \in \mathcal{P}} M(A(p)). \tag{5}$$

B. Representation of Polynomials

Before proceeding, we briefly introduce a key tool that will be exploited in the next sections to derive the proposed conditions. For $s \in \mathbb{R}^r$, let $V(s) = V(s)' \in \mathbb{R}^{u \times u}$ be a symmetric matrix homogeneous polynomial of degree $2m$. Let $s^{\{m\}} \in \mathbb{R}^{\sigma(m)}$ be a vector containing all monomials of degree equal to m in s, where $\sigma(m)$ is the number of such monomials given by

$$\sigma(m) = \frac{(r + m - 1)!}{(r - 1)!m!}. \tag{6}$$

Then, $V(s)$ can be written as

$$V(s) = \left(s^{\{m\}} \otimes I_u\right)(W + L(\alpha))\left(s^{\{m\}} \otimes I_u\right) \tag{7}$$

where $W = W' \in \mathbb{R}^{u\sigma(m) \times u\sigma(m)}$, $L(\alpha) = L(\alpha)' \in \mathbb{R}^{u\sigma(m) \times u\sigma(m)}$ is a linear parametrization of

$$\mathcal{L}(m, u) = \left\{L = L' : \left(s^{\{m\}} \otimes I_u\right)L\left(s^{\{m\}} \otimes I_u\right) = 0\right\} \tag{8}$$

and $\alpha \in \mathbb{R}^{\omega(m,u)}$ is a vector of free parameters, where

$$\omega(m, u) = \frac{1}{2}u\left(\sigma(m)(u\sigma(m) + 1) - (u + 1)\sigma(2m)\right). \tag{9}$$

The representation (7) is known as square matricial representation (SMR) for matrix polynomials and extends the Gram matrix method to the representation of matrix polynomials. In particular, it turns out that $V(s)$ is a sum of squares (SOS) of matrix polynomials if and only if there exists α satisfying the LMI

$$W + L(\alpha) \geq 0. \tag{10}$$

See, e.g., [16] and [19] for details.

III. PROPOSED RESULTS

This section provides the proposed results. Let us start with the following theorem, which provides an equivalent reformulation of the Mahler measure.

Theorem 1: Let $X \in \mathbb{R}^{n \times n}$. For any integer k satisfying $1 \leq k \leq n$ define

$$f_k(X) = \max_{\lambda \in \text{spc}(\Pi_k(X))} |\lambda| \tag{11}$$

where $\Pi_k(X) \in \mathbb{R}^{c_k \times c_k}$ is a matrix function with size given by

$$c_k = \frac{n!}{(n - k)!k!} \tag{12}$$

and whose (i, j)-th entry is defined as

$$(\Pi_k(X))_{i,j} = \det(Y_k(X, i, j)) \tag{13}$$

where $Y_k(X, i, j) \in \mathbb{R}^{k \times k}$ is the submatrix of X built with the rows indexed by $y(i)$ and the columns indexed by $y(j)$, where $y(l)$ is the l-th k-tuple built with increasing integers in $[1, n]$. Then

$$M(X) = \max_{k=1,\ldots,n} \max\{1, f_k(X)\}. \tag{14}$$

Proof: Let k be an integer satisfying $1 \leq k \leq n$. From the construction of $\Pi_k(X)$ it follows that [20]:

$$\text{spc}(\Pi_k(X)) = \left\{\prod_{j=1}^{k} \lambda_{i_j} : 1 \leq i_j \leq n, \ i_j \neq i_l \ \forall j \neq l\right\}$$

where $\lambda_i \in \mathbb{C}$ is the i-th eigenvalue of X. Moreover, let us observe that

$$\max\{1, f_k(X)\} \leq M(X)$$

if the number of eigenvalues of X with magnitude larger than or equal to 1 is different from k, while

$$\max\{1, f_k(X)\} = M(X)$$

if this number is equal to k. Therefore, $M(X)$ satisfies (14). □

Theorem 1 provides a certain equivalence of the Mahler measure of a matrix X with the spectrum of some matrices obtained by X, specifically showing that the Mahler measure is the maximum between 1 and the largest absolute eigenvalue of these matrices. We can exploit Theorem 1 to determine the worst-case Mahler measure of the system (1)–(3) defined in (5) as follows. First, let us observe that the system (1)–(3) can be equivalently rewritten as

$$x(t+1) = \bar{A}(s)x(t) \tag{15}$$

where $s \in \mathbb{R}^r$ is a vector constrained according to

$$s \in \mathcal{S} \tag{16}$$

where $\mathcal{S}$ is the simplex

$$\mathcal{S} = \left\{ s \in \mathbb{R}^r : \sum_{i=1}^{r} s_i = 1,\ s_i \geq 0 \right\} \tag{17}$$

and $\bar{A} : \mathbb{R}^r \longrightarrow \mathbb{R}^{n \times n}$ is the matrix homogeneous polynomial of degree δ satisfying

$$\bar{A}(s) = A\left(\sum_{i=1}^{r} s_i p^{(i)} \right) \quad \forall s \in \mathcal{S}. \tag{18}$$

Second, let k be any integer satisfying $1 \leq k \leq n$ and let us define the matrix homogeneous polynomial of degree δk

$$\bar{B}_k(s) = \Pi_k(\bar{A}(s)). \tag{19}$$

If there exist $w \in \mathbb{R}$ and $F_k : \mathbb{R}^r \longrightarrow \mathbb{R}^{c_k \times c_k}$ such that

$$\left. \begin{array}{rl} 0 & < \quad F_k(s) \\ 0 & < \quad wF_k(s) - \bar{B}_k(s)'F_k(s)\bar{B}_k(s) \end{array} \right\} \quad \forall s \in \mathcal{S} \tag{20}$$

then one can conclude that

$$f_k\left(\bar{A}(s)\right) < \sqrt{w} \quad \forall s \in \mathcal{S}. \tag{21}$$

This suggests that we can start by looking for a matrix function $F_k(s)$ satisfying (20). To this end, we focus our attention on matrix polynomials of a generic degree. Let us observe that, since $s \in \mathcal{S}$, we can assume without loss of generality that $F_k(s)$ is homogeneous. Such a $F_k(s)$ defines a Lyapunov function candidate of the form

$$\tilde{v}(x(t)) = \tilde{x}(t)'F_k(s)\tilde{x}(t) \tag{22}$$

for the system

$$\tilde{x}(t+1) = \frac{\bar{B}_k(s)}{\sqrt{w}}\tilde{x}(t) \tag{23}$$

where $\tilde{x} \in \mathbb{R}^{c_k}$. In particular, this class of Lyapunov functions is known as HPD-QLFs, see e.g., [16], [21]. Hence, the problems are how to check the existence of a HPD-QLF satisfying (20) for a given w, and how to compute the smallest w for which (20) can be satisfied by a HPD-QLF.

To this end, let $F_k(s) = F_k(s)' \in \mathbb{R}^{c_k \times c_k}$ be a symmetric matrix homogeneous polynomial of degree m, where m is a nonnegative integer. We can parametrize $F_k(s)$ as

$$F_k(s) = Q_k(z)\left(s^{\{m\}} \otimes I_{c_k} \right) \tag{24}$$

where $Q_k(z)$ is a linear parametrization of the subspace

$$\mathcal{Q}_k = \{ Q_k = (Q_{k,1}, \ldots, Q_{k,\sigma(m)}) : Q_{k,i} \in \mathbb{R}^{c_k \times c_k}, \\ Q_{k,i} = Q'_{k,i},\ i = 1, \ldots, \sigma(m) \} \tag{25}$$

and z is a vector. Then, let us define

$$G_k(s) = \bar{B}_k(s)'F_k(s)\bar{B}_k(s) \tag{26}$$

and let $R_k(z)$ be the matrix defined by

$$G_k(s) = R_k(z)\left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right). \tag{27}$$

Let us express $Q_k(z)$ and $R_k(z)$ as

$$\begin{aligned} Q_k(z) &= (Q_{k,1}(z), \ldots, Q_{k,\sigma(m)}(z)) \\ R_k(z) &= (R_{k,1}(z), \ldots, R_{k,\sigma(m+2\delta k)}(z)) \end{aligned} \tag{28}$$

where $Q_{k,1}(z), \ldots, Q_{k,\sigma(m)}(z)) \in \mathbb{R}^{c_k \times c_k}$ and $R_{k,1}(z), \ldots, R_{k,\sigma(m+2\delta k)}(z) \in \mathbb{R}^{c_k \times c_k}$, and let us define

$$\begin{aligned} C_k(z) &= \mathrm{diag}\left(Q_{k,1}(z), \ldots, Q_{k,\sigma(m)}(z) \right) \\ D_k(z) &= \mathrm{diag}\left(R_{k,1}(z), \ldots, R_{k,\sigma_{m+2\delta k}}(z) \right).. \end{aligned} \tag{29}$$

Let $L_k(\alpha)$ and $M_k(\beta)$ be linear parametrizations of $\mathcal{L}(m, c_k)$ and $\mathcal{L}(m+2\delta k, c_k)$, respectively, and let us define

$$E_k(z, \alpha) = (N_k \otimes I_{c_k})' \left(\mathrm{diag}(v_k) \otimes (C_k(z) + L_k(\alpha)) \right) \\ \times (N_k \otimes I_{c_k}) \tag{30}$$

where v_k is the vector defined by

$$v'_k s^{\{2\delta k\}} = \left(\sum_{i=1}^{r} s_i \right)^{2\delta k} \tag{31}$$

and N_k is the matrix defined by

$$s^{\{2\delta k\}} \otimes s^{\{m\}} = N_k s^{\{m+2\delta k\}}. \tag{32}$$

Theorem 2: Let us consider the system (1)–(3). Let m be a nonnegative integer and $w \in \mathbb{R}$ a given scalar. Suppose that, for all integers k satisfying $1 \leq k \leq n$, there exist z, α and β satisfying the following LMIs:

$$\begin{cases} 0 < C_k(z) + L_k(\alpha) \\ 0 < wE_k(z, \alpha) - D_k(z) + M_k(\beta). \end{cases} \tag{33}$$

Then, (20)–(21) hold. Consequently

$$\mu \leq \max\{1, \sqrt{w}\}. \tag{34}$$

Proof: Suppose that the inequalities in (33) hold. For any $s \in \mathbb{R}_0^r$, let us post- and pre-multiply the first inequality by $s^{\{m\}} \otimes I_{c_k}$ and its transpose, respectively. We get

$$0 < \left(s^{\{m\}} \otimes I_{c_k} \right)' (C_k(z) + L_k(\alpha)) \left(s^{\{m\}} \otimes I_{c_k} \right)$$
$$= F_k(\mathrm{sq}(s))$$

i.e., $F_k(\mathrm{sq}(s))$ is positive definite. Then, let us post- and pre-multiply the second inequality by $s^{\{m+2\delta k\}} \otimes I_{c_k}$ and its transpose, respectively. By defining

$$H_k(s) = \left(\sum_{i=1}^r s_i \right)^{2\delta k} F_k(s)$$

and observing that

$$H_k(\mathrm{sq}(s)) = \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)' E_k(z, \alpha) \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)$$

one gets

$$0 < \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)' (w E_k(z, \alpha) - D_k(z) + M_k(\beta))$$
$$\times \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)$$
$$= J_k(\mathrm{sq}(s))$$

i.e., $J_k(\mathrm{sq}(s))$ is positive definite in s, where

$$J_k(s) = w H_k(s) - G_k(s).$$

Since $F_k(s)$ and $J_k(s)$ are homogeneous in s, it follows (see e.g., [16]) that:

$$\left. \begin{matrix} 0 & < & F_k(s) \\ 0 & < & J_k(s) \end{matrix} \right\} \ \forall s \in \mathcal{S}.$$

Then, let us observe that

$$J_k(s) = w F_k(s) - \bar{B}_k(s)' F_k(s) \bar{B}_k(s) \ \forall s \in \mathcal{S}.$$

This means that (20) holds, which implies that also (21) holds. Therefore, from Theorem 1, it follows that an upper bound of μ can be obtained from w according to (34). $\square$

Theorem 2 provides a sufficient condition for establishing whether a given scalar is an upper bound of the worst-case Mahler measure. This condition requires to check whether, for all integers k satisfying $1 \le k \le n$, there exist variables z, α and β satisfying the LMIs (33). This condition is built for given m and w, which define the degree of $F_k(s)$ and the candidate upper bound of μ, respectively.

Let us define the best upper bound of μ provided by Theorem 2 for a chosen m as

$$\phi(m) = \max\{1, \sqrt{w^*}\} \tag{35}$$

where

$$w^* = \max_{k=1,\dots,n} w_k^* \tag{36}$$

and

$$w_k^* = \inf_w \ w$$
$$\text{s.t. } \exists z, \alpha, \beta : (33) \text{ holds.} \tag{37}$$

It turns out that computing w_k^* involves the solution of a bilinear matrix inequality (BMI) because w multiplies z in (33). One way to handle this problem is to perform a line-search on w where the LMI condition (33) is checked for any fixed w, for instance via a bisection algorithm. Another way to compute w_k^* is to observe that (37) is a generalized eigenvalue problem: indeed, the first LMI in (33) ensures that $E_k(z, \alpha) > 0$, and consequently (37) is a generalized eigenvalue problem which belongs to the class of quasi-convex optimization problems [22].

The following result provides a monotonicity property for the upper bound $\phi(m)$ with respect to m.

Theorem 3: Let us consider the system (1)–(3), and let m be a nonnegative integer. Then

$$\phi(m+1) \le \phi(m). \tag{38}$$

Proof: From the definition of $\phi(m)$ in (35), the property (38) can be proved by showing that, if the inequalities in (33) are feasible for some $m = m_0$ and for any w and k satisfying $1 \le k \le n$, then they are feasible also for $m = m_0 + 1$ and for such w and k. To this end, let us denote in the sequel of this proof the quantities corresponding to the case $m = m_0 + 1$ with the "hat" symbol, i.e., $\hat{m} = m_0 + 1$. Let us observe that there exists $\hat{z}$ such that

$$\hat{F}_k(s) = \left(\sum_{i=1}^r s_i \right) F_k(s)$$

since $\hat{F}_k(s)$ is a generic symmetric matrix homogeneous polynomial of degree $\hat{m}$ parametrized by $\hat{z}$. Moreover, one can write

$$\hat{F}_k(\mathrm{sq}(s)) = \left(s^{\{\hat{m}\}} \otimes I_{c_k} \right)' \tilde{C}_k \left(s^{\{\hat{m}\}} \otimes I_{c_k} \right)$$

where

$$\tilde{C}_k = \left(\tilde{N}(m) \otimes I_{c_k} \right)' (I_r \otimes (C_k(z) + L_k(\alpha))) \left(\tilde{N}(m) \otimes I_{c_k} \right)$$

and $\tilde{N}(m)$ is the matrix defined by

$$s \otimes s^{\{m\}} = \tilde{N}(m) s^{\{\hat{m}\}}.$$

Since $\tilde{N}(m)$ is a full column rank matrix and since $C_k(z) + L_k(\alpha)$ is positive definite, it follows that $\tilde{C}_k$ is positive definite, and hence there exists $\hat{\alpha}$ such that

$$\hat{C}_k(\hat{z}) + \hat{L}_k(\hat{\alpha}) = \tilde{C}_k > 0.$$

Next, as $F_k(s)$ is replaced by $\hat{F}_k(s)$, one has that the matrices $G_k(s)$, $H_k(s)$ and $J_k(s)$ are replaced by

$$\hat{G}_k(s) = \bar{B}_k(s)' \hat{F}_k(s) \bar{B}_k(s)$$
$$\hat{H}_k(s) = \left(\sum_{i=1}^r s_i \right)^{2\delta k} \hat{F}_k(s)$$
$$\hat{J}_k(s) = w \hat{H}_k(s) - \hat{G}_k(s).$$

This implies that

$$\hat{J}_k(\mathrm{sq}(s)) = w \hat{H}_k(\mathrm{sq}(s)) - \hat{G}_k(\mathrm{sq}(s))$$
$$= \left(\sum_{i=1}^r s_i^2 \right) J_k(\mathrm{sq}(s))$$
$$= \left(s^{\{\hat{m}+2\delta k\}} \otimes I_{c_k} \right)' \tilde{D}_k \left(s^{\{\hat{m}+2\delta k\}} \otimes I_{c_k} \right)$$

where

$$\tilde{D}_k = \left(\tilde{N}(m + 2\delta k) \otimes I_{c_k} \right)' \left(I_r \otimes (w E_k(z, \alpha) - D_k(z) \right.$$
$$\left. + M_k(\beta)) \right) \left(\tilde{N}(m + 2\delta k) \otimes I_{c_k} \right).$$

Since $w E_k(z, \alpha) - D_k(z) + M_k(\beta)$ is positive definite, it follows that $\tilde{D}_k$ is positive definite, and since

$$\hat{H}_k(\mathrm{sq}(s)) = \left(s^{\{\hat{m}+2\delta k\}} \otimes I_{c_k} \right)' \hat{E}_k(\hat{z}, \hat{\alpha}) \left(s^{\{\hat{m}+2\delta k\}} \otimes I_{c_k} \right)$$

one can conclude that there exists $\hat{\beta}$ such that

$$w \hat{E}_k(\hat{z}, \hat{\alpha}) - \hat{D}_k(\hat{z}) + \hat{M}_k(\hat{\beta}) = \tilde{D}_k > 0.$$

$\square$

Theorem 3 states an interesting property of the upper bound $\phi(m)$ of μ, specifically that $\phi(m)$ is monotonically nonincreasing with m.

At this point the question is whether and how ϕ_m approximates the sought worst-case Mahler measure depending on m. The following result provides an important answer to this question.

Theorem 4: Let us consider the system (1)–(3). Then, there exists a nonnegative integer m_0 such that

$$\mu = \phi(m) \ \forall m \geq m_0. \tag{39}$$

Proof: Let k be any integer satisfying $1 \leq k \leq n$. Let $w \in \mathbb{R}$ be any scalar satisfying (21), i.e.,

$$\max_{\lambda \in \mathrm{spc}(\bar{B}_k(s))} |\lambda| < \sqrt{w} \ \forall s \in \mathcal{S}.$$

This means that there exists a matrix function $P(s) = P(s)'$ such that

$$\left. \begin{array}{ll} 0 & < \quad P(s) \\ 0 & < \quad w \left(\sum_{i=1}^{r} s_i \right)^{2\delta k} P(s) - \bar{B}_k(s)' P(s) \bar{B}_k(s) \end{array} \right\} \ \forall s \in \mathcal{S}.$$

Such a matrix function $P(s)$ can be obtained from the equation

$$w \left(\sum_{i=1}^{r} s_i \right)^{2\delta k} P(s) - \bar{B}_k(s)' P(s) \bar{B}_k(s) = I_{c_k}$$

which also says that $P(s)$ is a matrix rational function. Let us express $P(s)$ as

$$P(s) = \frac{P_1(s)}{p_2(s)}$$

where $P_1(s)$ and $p_2(s)$ are homogeneous, with $p_2(s) > 0$ for all $s \in \mathcal{S}$. For a nonnegative integer a let us define

$$P_3(s) = \left(\sum_{i=1}^{r} s_i \right)^{a} p_2(s) P(s).$$

It follows that $P_3(s)$ is a matrix polynomial and that

$$\left. \begin{array}{l} 0 < P_3(s) \\ 0 < w \left(\sum_{i=1}^{r} s_i \right)^{2\delta k} P_3(s) - \bar{B}_k(s)' P_3(s) \bar{B}_k(s) \end{array} \right\} \ \forall s \in \mathcal{S}.$$

Consequently, there exists a such that the coefficient matrices of $P_3(s)$ (say $P_{31}, P_{32}, \ldots$) and $w \left(\sum_{i=1}^{r} s_i \right)^{2\delta k} P_3(s) - \bar{B}_k(s)' P_3(s) \bar{B}_k(s)$ (say $P_{41}, P_{42}, \ldots$) are positive definite (see e.g., [19]). Hence, let m be the degree of $P_3(s)$, and let z such that $F_k(s) = P_3(s)$. Let us observe that

$$F_k(\mathrm{sq}(s)) = \left(s^{\{m\}} \otimes I_{c_k} \right)' \bar{C}_k \left(s^{\{m\}} \otimes I_{c_k} \right)$$
$$\text{where}$$
$$\bar{C}_k = \mathrm{diag}(P_{31}, P_{32}, \ldots)$$

which is positive definite, and hence there exists α such that $C_k(z) + L_k(\alpha) = \bar{C}_k > 0$. Then, let us observe that

$$J_k(\mathrm{sq}(s)) = \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)' \bar{D}_k \left(s^{\{m+2\delta k\}} \otimes I_{c_k} \right)$$
$$\text{where}$$
$$\bar{D}_k = \mathrm{diag}(P_{41}, P_{42}, \ldots)$$

which is positive definite, and hence there exists β such that $w E_k(z) - D_k(z) + M_k(\beta) = \bar{D}_k > 0$. Therefore, there exists m such that the condition (21) is equivalent to the existence of z, α, β satisfying (33). From Theorem 3 we conclude that (39) holds. $\square$

Theorem 4 states an important result, specifically that the upper bound $\phi(m)$ coincides with the sought worst-case Mahler measure μ of the system (1)–(3) for a sufficiently large integer m.

IV. ILLUSTRATIVE EXAMPLES

In this section we present two illustrative examples of the proposed results. The matrices in the condition (33) have been generated with the algorithms reported in [16]. The computations have been done in Matlab.

A. Example 1

Let us consider the uncertain system

$$\begin{cases} x(t+1) = A(p)x(t) \\ A(p) = A_0 + p A_1 \\ p \in [0, 1] \end{cases}$$
$$A_0 = \begin{pmatrix} 3.4 & 2.9 \\ -1.6 & -1.6 \end{pmatrix}, A_1 = \begin{pmatrix} -2.9 & -4.1 \\ 4.5 & 0.3 \end{pmatrix}$$

and the problem of determining the robust Mahler measure μ in (5). This system can be rewritten as in (15) with

$$\begin{cases} x(t+1) = \bar{A}(s)x(t) \\ \bar{A}(s) = s_1 \bar{A}_1 + s_2 \bar{A}_2 \\ s \in \mathcal{S} \end{cases}$$
$$\bar{A}_1 = \begin{pmatrix} 3.4 & 2.9 \\ -1.6 & -1.6 \end{pmatrix}, \bar{A}_2 = \begin{pmatrix} 0.5 & -1.2 \\ 2.9 & -1.3 \end{pmatrix}.$$

For all k satisfying $1 \leq k \leq 2$, we compute the matrix $\bar{B}_k(s)$. We find that

$$k = 1 \rightarrow \bar{B}_1(s) = \bar{A}(s)$$
$$k = 2 \rightarrow \bar{B}_2(s) = -0.6 s_1^2 - 15.55 s_1 s_2 + 2.83 s_2^2.$$

Hence, we compute the upper bound $\phi(m)$ in (35). With $m = 0$ we find $w_1^* = 16.152$ and $w_2^* = 12.724$, which provide $\phi(0) = 4.019$. Hence, we increase m, and with $m = 1$ we find $w_1^* = 6.037$ and

$w_2^* = 12.724$ which provide the new upper bound $\phi(1) = 3.567$. It is possible to verify that this upper bound is indeed equal to the sought robust Mahler measure, i.e., $\phi(1) = \mu$.

B. Example 2

Let us consider the uncertain system

$$
\begin{cases}
x(t+1) = A(p)x(t) \\
p \in [-1, 1]
\end{cases}
$$

$$
A(p) = \begin{pmatrix} 0.1 & 1.4 & 1.3 - 0.5p \\ -1.3 & 0.8 & 0.4 \\ -0.8 + p & 0.5 & 0.1 \end{pmatrix}
$$

and the problem of determining the robust Mahler measure μ in (5). This system can be rewritten as in (15) with

$$
\begin{cases}
x(t+1) = \bar{A}(s)x(t) \\
s \in \mathcal{S}
\end{cases}
$$

$$
\bar{A}(s) = \begin{pmatrix} 0.1s_1 + 0.1s_2 & 1.4s_1 + 1.4s_2 & 1.8s_1 + 0.8s_2 \\ -1.3s_1 - 1.3s_2 & 0.8s_1 + 0.8s_2 & 0.4s_1 + 0.4s_2 \\ -1.8s_1 + 0.2s_2 & 0.5s_1 + 0.5s_2 & 0.1s_1 + 0.1s_2 \end{pmatrix}.
$$

We compute the upper bound $\phi(m)$ in (35). With $m = 0$ we find $w_1^* = 4.928$, $w_2^* = 24.256$ and $w_3^* = 0.341$, which provide $\phi(0) = 4.925$. It is possible to verify that this upper bound is indeed equal to the sought robust Mahler measure, i.e., $\phi(0) = \mu$.

V. CONCLUSION

This technical brief has investigated the Mahler measure in systems depending polynomially on uncertain parameters constrained in a polytope. It has been shown that a sufficient condition for establishing an upper bound of the worst-case Mahler measure can be obtained in terms of LMI feasibility tests, where a HPD-QLF is searched for, and that the best upper bound guaranteed by this condition can be computed through generalized eigenvalue problems. Moreover, it has been shown that the upper bound is monotonically nonincreasing with the degree of the HPD-QLF, and that there exists a finite degree for which the upper bound is guaranteed to be tight.

VI. ACKNOWLEDGEMENT

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