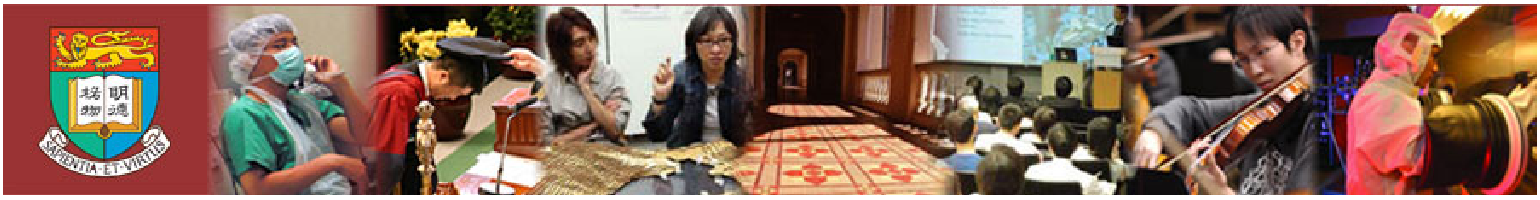

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The TOC-Based Algorithm for Solving Multiple Constraint Resources: A Re-examination

Jun-Qiang Wang, Zhong-Tian Zhang, Jian Chen, Yin-Zhou Guo, Shuo Wang, Shu-Dong Sun, Ting Qu, and George Q. Huang

***Abstract*—In a previous paper in this journal named *The TOC-Based Algorithm for Solving Multiple Constraint Resources*, Ray *et al.* considered an integrated heuristic approach named AHP/TOC comprising the analytic hierarchy process (AHP) and theory of constraints (TOC) in a multiple constraints resource product mix problem. This paper gives three typical examples and points out that the proposed approach would not generate the optimal solution. Furthermore, the reasons are analyzed in terms of the ranking approach of product priority using AHP and the adjustment approach of product mix after a new bottleneck has been identified. We clarify the cases under which the AHP/TOC method can and cannot output the optimal solution. Finally, some possible improvements are given.**

***Index Terms*—Heuristic algorithm, product mix optimization, theory of constraints (TOC).**

Nomenclature

- i Product index, $i = 1, 2, \ldots, n$.
- j Resource index, $j = 1, 2, \ldots m$.
- t_{ij} Processing time of product i on resource j. Generally, $t_{ij} \geq 0$.
- D_i Demand of product i. Generally, $D_i > 0$.
- CM_i Contribution margin of product i. Generally, $\mathrm{CM}_i > 0$.
- CP_j Available capacity of resource j. Generally, $\mathrm{CP}_j > 0$.
- d_i Difference between capacity and demand on resource $j, d_j = \mathrm{CP}_j - \sum_{i=1}^{n} (D_i \cdot t_{ij})$.
- b_j Actual time and maximum available time of resource $j, b_j = \min\{\mathrm{CP}_j, \sum_{i=1}^{n} (D_i \cdot t_{ij})\}$. Generally, $b_j > 0$.
- r_{i1} Normalized values of contribution margin, $r_{i1} = \frac{\mathrm{CM}_i}{\sum_{i=1}^{n} \mathrm{CM}_i}$. Generally, $r_{i1} \in (0, 1)$.
- p_{ij} Normalized values of product processing time, $p_{ij} = \frac{t_{ij}}{\sum_{i=1}^{n} t_{ij}}$, $p_{ij} \in [0, 1)$.
- q_{1j} Normalized values of available time and required time, $q_{1j} = \frac{b_j}{\sum_{j=1}^{m} b_j}$, $q_{1j} \in (0, 1)$.
- φ_i Priority of product i. Generally, $\varphi_i > 0$.
- ρ_1 Weight of the profit. Generally, $\rho_1 > 0$, $\rho_1 + \rho_2 = 1$.
- ρ_2 Weight of the resource. Generally, $\rho_2 > 0$, $\rho_1 + \rho_2 = 1$.

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I. Introduction

In a previous paper in this journal named *The TOC-Based Algorithm for Solving Multiple Constraint Resources*, Ray *et al.* [1] considered an integrated heuristic approach named AHP/TOC comprising the analytic hierarchy process (AHP) and the theory of constraints (TOC) in a multiple constraints resource product mix problem.

Compared with classical TOC approach and integer linear programming (ILP) analysis through an example with five products and four manufacturing resources, the authors pointed out the advantages of the proposed integrated heuristic approach as follows: 1) it is well suited for dealing with the throughput analysis that involves quantitative factors; 2) it eliminates rigorous mathematical expressions; 3) it is simple and straightforward; 4) it generates the optimum solution in all the cases; and 5) it minimizes the time for calculation.

On the basis of the above contributions from the paper, our further study has shown that the proposed AHP/TOC method would not generate the optimum solution in all the cases. The paper discusses the core components of AHP and TOC heuristic (TOCh) of the proposed approach, analyzes the two limitations in terms of ranking approach of product priority using the AHP component, and the adjustment approach of product mix after a new bottleneck has been identified, clarifies the cases under which the AHP/TOC method can and cannot output the optimal solution, and gives two possible improvements.

II. Proposed Heuristic Method

According to whether the adjustment conditions are met or not, the AHP/TOC method is summarized into two stages: the product mix determination stage and the product mix adjustment stage. The flow chart of the AHP/TOC method is illustrated in Fig. 1.

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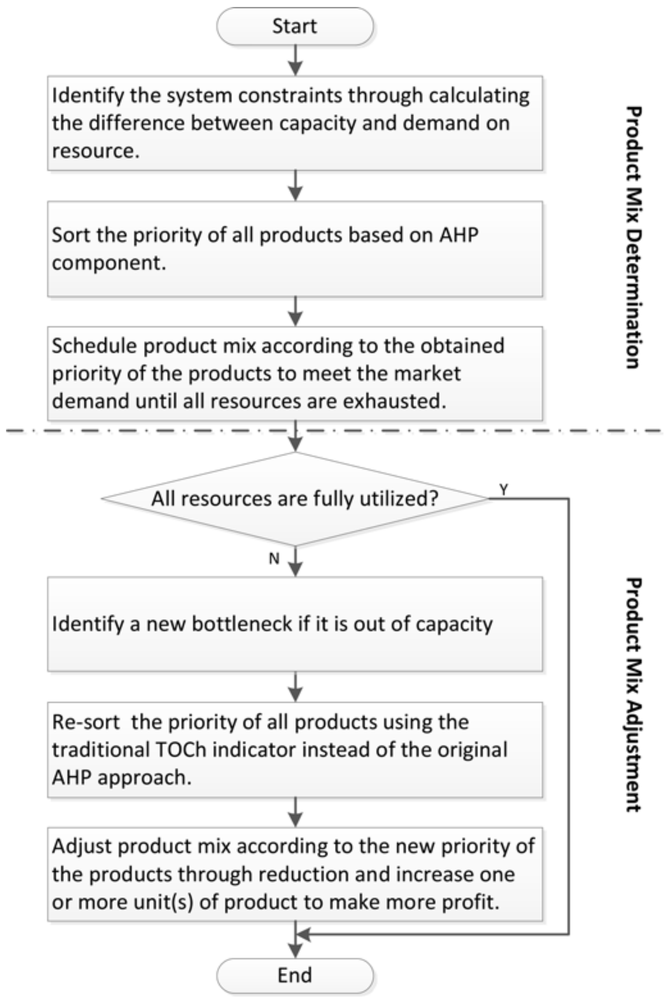

Fig. 1. Flow chart of the proposed method.

Two important components work in the AHP/TOC method: AHP and TOCh component. The AHP component is used to determine the priority of product in the first stage of product mix determination, which is calculated by first multiplying profit decision matrix and the resource decision matrix with their corresponding weight and then adding the two results together. The TOCh component works not only in the first stage of product mix determination to identify the bottleneck resource and to make the initial product mix solution according to the priority of product obtained by AHP component, but also in the second stage of product mix adjustment to identify the new bottleneck resource and to adjust product mix solution according the corresponding priority obtained by the classical TOCh indicator.

Here the adjustment conditions are listed. Assume that P and Q are candidate products for the reduction and increase. Reducing P and increasing Q can be a feasible alternative if the following conditions are met.

1) The priority of product P is prior to Q in the initial product mix solution.
2) Product Q should meet three conditions:
   a) The market demand of Product Q is not fully met.
   b) It is prior to P in at least one of the priority sequences.
   c) In the same priority sequence that it is prior to P, it is prior to all the products whose demands have not been fully met.

The advantages of the AHP/TOC method are highlights as follows in our view.

1) The AHP/TOC method considered the influence of all bottlenecks and treated them equally, when facing the multiple bottleneck scenarios, while the traditional method focused on dominant bottleneck [7] and ignored the influence of other bottlenecks. Then multiple decision makers are involved in the decision-making process and the AHP/TOC method addressed every bottleneck to the decision-making process.
2) The AHP/TOC method incorporates tangible and intangible criteria into the decision-making process using AHP component in order to comprehensively determine the priority of the product. When determining the priority of the product, the AHP/TOC method considered multiple dimension factors, including the contribution margin of each product, the processing time of each product on all resources, required capacity and available capacity of each resource. While the traditional TOCh method only considered the contribution margin and the processing time on bottleneck.
3) The AHP/TOC method integrated the advantages of TOC and AHP. It inherited the advantage of TOCh to explicitly exploit the bottlenecks, used the advantage of multiple criteria decisions of AHP component to determine the priority ranking of each product.
4) The sensitivity analysis was conducted to provide the decision maker with additional insight regarding the robustness of the AHP/TOC method so that the decision maker can make a better decision.

For an example from a fabrication industry with five products and four manufacturing resources, the AHP/TOC method found an optimal solution with a simple and straightforward way.

III. Computational Examples

In order to comprehensively analyze the AHP/TOC method, the two components of product mix determination and product mix adjustment are checked separately. Here we give three typical examples of multiple bottleneck scenarios to verify the effectiveness according to the adjustment conditions of the proposed adjustment approach in [1].

A. *Example 1*

Example 1 is slightly modified based on [2]. Specifically, the processing time of P2 on R2 is set as 20 instead of original 30, as shown in Table I.

Using the AHP/TOC method, the solving process of the priority of product is intuitively demonstrated step by step in Table II. From Table II, the priority of P2 (0.541) is prior to P1 (0.459). Therefore, P2 is prior to be made than P1. Then the limited

TABLE I
KNOWN CONDITIONS AND CAPACITY ANALYSIS OF EXAMPLE 1.

Product	D_i	CM_i	Resource			
			R1	R2	R3	R4
P1	100	45	15	15	15	10
P2	50	60	10	20	5	5
Required capacity $\sum_{i=1}^{n}(D_i \cdot t_{ij})$			2000	2500	1750	1250
Available capacity CP_j			1700			
Difference $d_j = CP_j - \sum_{i=1}^{n}(D_i \cdot t_{ij})$			-200	-800	-50	450
Actual time and maximum available time $b_j = \min\{CP_j, \sum_{i=1}^{n}(D_i \cdot t_{ij})\}$			1700	1700	1700	1250

TABLE II
CALCULATION PROCESS USING THE AHP/TOC METHOD FOR EXAMPLE 1

	Profit (weight ρ_1: 0.833)	Resource (weight ρ_2: 0.167)						φ_i
	r_{i1}		R1	R2	R3	R4	$\sum_{i=1}^{4}(p_{ij} \cdot q_{1j})$	$\rho_1 r_{i1} + \rho_2 \sum_{j=1}^{4}(p_{ij} \cdot q_{1j})$
			p_{i1}	p_{i2}	p_{i3}	p_{i4}		
P1	0.429		0.6	0.429	0.75	0.667	0.608	0.459
P2	0.571		0.4	0.571	0.25	0.333	0.393	0.541
		q_{1j}	0.268	0.268	0.268	0.197		

resources are preferentially allocated to the product with high priority. Consequently, the initial product mix is 50P2 and 46P1.

In the process of product mix, R2 is the first resource out of its available capacity. Therefore, resource R2 becomes the new bottleneck. The priority of each product on R2 is recalculated as P1 = P2 = 3. Compared with the initial priority obtained by the AHP component, P1 < P2, so it does NOT meet the adjustment conditions and it is not necessary to adjust the obtained product mix solution. Thus, the product mix solution resulting from the proposed method is 46P1 and 50P2, and the corresponding throughput is US$ 5070.

Also, we use the ILP approach and output the optimal product mix solution of 48P1 and 49P2 with the total throughput of US$ 5100.

Therefore, regarding the example 1 modified from [2], the AHP/TOC method outputs a feasible solution when facing the situation that the priority of each product on new bottleneck is equal, but different from the one obtained using the AHP component.

B. Example 2

Example 2 is newly designed as shown in Table III. For example 2, the calculation process of the priority of product using the AHP/TOC method is intuitively displayed as shown in Table IV. From Table IV, the priority of P2 (0.668) is prior to P1 (0.332). Certainly P2 is prior to be made. Therefore, the initial product mix is 7P2 and 0P1. In the process of product mix, R2 is the first resource out of its available capacity, therefore R2 is the new bottleneck. The priority of each product on R2 is recalculated as P1(1.08) < P2(1.2). The relationship between P1 and P2 is the same as the initial priority, so it does NOT meet the adjustment conditions and it is not necessary to adjust. Therefore, the product mix is 0P1 and 7P2, and the corresponding throughput is US$ 168.

TABLE III
KNOWN CONDITIONS AND CAPACITY ANALYSIS OF EXAMPLE 2

Product	D_i	CM_i	Resource	
			R1	R2
P1	10	13	2	12
P2	10	24	18	20
Required capacity $\sum_{i=1}^{n}(D_i \cdot t_{ij})$			200	320
Available capacity CP_j			150	
Difference $d_j = CP_j - \sum_{i=1}^{n}(D_i \cdot t_{ij})$			-50	-170
Actual time and maximum available time $b_j = \min\{CP_j, \sum_{i=1}^{n}(D_i \cdot t_{ij})\}$			150	150

TABLE IV
CALCULATION PROCESS USING THE AHP/TOC METHOD FOR EXAMPLE 2

	Profit (weight ρ_1: 0.833)	Resource (weight ρ_2: 0.167)				φ_i
	r_{i1}		R1	R2	$\sum_{i=1}^{2}(p_{ij} \cdot q_{1j})$	
P1	0.351		0.1	0.375	0.238	0.332
P2	0.649		0.9	0.625	0.763	0.668
		q_{1j}	0.5	0.5		

TABLE V
KNOWN CONDITIONS AND CAPACITY ANALYSIS OF EXAMPLE 3

Product	D_i	CM_i	Resource						
			R1	R2	R3	R4	R5	R6	R7
P1	70	80	20	5	10	0	5	5	20
P2	60	60	10	10	5	30	5	5	5
P3	50	50	10	5	10	15	20	5	10
P4	150	30	5	15	10	5	5	15	0
Required capacity $\sum_{i=1}^{n}(D_i \cdot t_{ij})$			3250	3450	3000	3300	2400	3150	2200
Available capacity CP_j			2400						
Difference $d_j = CP_j - \sum_{i=1}^{n}(D_i \cdot t_{ij})$			-850	-1050	-600	-900	0	-750	200
Actual time and maximum available time $b_j = \min\{CP_j, \sum_{i=1}^{n}(D_i \cdot t_{ij})\}$			2400	2400	2400	2400	2400	2400	2200

While using the ILP approach, the optimal solutions are 4P1 and 5P2, and the total throughput is US$ 172.

Therefore, regarding the example 2, the AHP/TOC method outputs a feasible solution when facing the situation that the priority of each product on a new bottleneck is the same as the one obtained using the AHP component.

C. Example 3

The example comes from the literature of [3], and is shown in Table V. We revised nothing for the example.

For Example 3, the calculation process of the priority of product using the AHP/TOC method is intuitively displayed as shown in Table VI. The priority of product implicates the

TABLE VI
CALCULATION PROCESS USING THE AHP/TOC METHOD FOR EXAMPLE 3

	Profit (weight: 0.833)	Resource (weight: 0.167)									φ_i
	r_{i1}		R1	R2	R3	R4	R5	R6	R7	$\sum_{i=1}^{7}(p_{ij}q_{1j})$	
P1	0.36		0.44	0.14	0.29	0	0.14	0.17	0.57	0.25	0.35
P2	0.27		0.22	0.29	0.14	0.6	0.14	0.17	0.14	0.24	0.27
P3	0.23		0.22	0.14	0.29	0.3	0.57	0.17	0.29	0.28	0.24
P4	0.14		0.11	0.43	0.29	0.1	0.14	0.5	0	0.22	0.15
		q_{1j}	0.14	0.14	0.14	0.14	0.14	0.14	0.13		

TABLE VII
ADJUSTMENT PROCESS THROUGH REDUCING P3 AND INCREASING P4 ON RESOURCE R1

	Left Time on R1	Left Time on R2	Left Time on R3	Left Time on R4	Left Time on R6	Throughput (dollars)
P1=70,P2=60,P3=40,P4=0	0	1250	1000	0	1550	11200
P1=70,P2=60,P3=39,P4=2	0	1225	990	5	1525	11210
P1=70,P2=60,P3=38,P4=4	0	1200	980	10	1500	11220
…	0	…	…	…	…	…
P1=70,P2=60,P3=0,P4=80	0	250	600	200	550	11600*

Note: The solution marked * is the optimum solution obtained during this adjustment process.

following relationship: P1 > P2 > P3 > P4, as shown in the last column of Table VI. According to the obtained priority of product, the initial product mix can be determined as 70P1, 60P2, 40P3, and 0P4. Also it can be found that R1 and R4 are exhausted at the same time. Therefore, resource R1 and R4 become the new bottlenecks. Now there are two bottlenecks, R1 and R4. Actually the authors did not consider this scenario. According to their approach, without loss of generality, we select a bottleneck one by one as the new bottleneck to adjust the product mix solution.

1) Resource R1 is Appointed as the New Bottleneck: The order of product priority on R1 is P2(6) = P4(6) > P3(5) > P1(4), which is not completely consistent with the order determined by the AHP component, P1 > P2 > P3 > P4. According to the adjustment conditions, it is reasonable to increase some units of P4 on the condition of reducing some units of P1 or P3 to make more profit. Note that the processing time P1 on R4 is 0 that means P1 never consumes time out of R4. If P1 is reduced by one unit, we cannot gain left time anymore, therefore there is no chance to increase R4.

Therefore, only reducing P3 by some units makes it possible to increase some units of P4. The adjustment process will not stop until P4 cannot be increased anymore or P3 cannot be reduced anymore. The detailed adjustment process is shown in Table VII. The optimum product mix solution is 70P1, 60P2, 0P3, and 80P4, and the throughput is US$ 11 600.

2) Resource R4 is Appointed as the New Bottleneck: The contribution margins per unit constraint minute of the four products on R4 are ∞, 2, 3.5, and 6, respectively. Therefore the order of product priority is P1 > P4 > P3 > P2, which is not completely consistent with the order determined by the AHP component, P1 > P2 > P3 > P4.

According to the adjustment conditions, it is reasonable to increase some units of P4 by reducing some units of P2 or P3 to make more profit. Thus, there exist two alternative solutions of product mix adjustment. 1) The first one is that P2 first and P3 second are reduced in order to increase P4, and the adjustment process is shown in Table VIII. The optimum product mix solution is 70P1, 0P2, 38P3, and 124P4, and the throughput is US$ 11 220. 2) The second one is that P3 first and P2 second are reduced in order to increase P4, and the adjustment process is shown in Table IX. The optimum product mix

TABLE VIII
ADJUSTMENT PROCESS OF REDUCING P2, P3 AND INCREASING P4 ON RESOURCE R4

	Left Time on R1	Left Time on R2	Left Time on R3	Left Time on R4	Left Time on R6	Throughput (dollars)
P1=70,P2=60,P3=40,P4=0	0	1250	1000	0	1550	11200
Part 1: Reduce P2, Increase P4.						
P1=70,P2=59,P3=40,P4=2	0	1230	985	20	1525	11200
P1=70,P2=58,P3=40,P4=4	0	1210	970	40	1500	11200
…	0	…	…	…	…	…
P1=70,P2=0,P3=40,P4=120	0	50	100	1200	50	11200
Part 2: Reduce P3, Increase P4.						
P1=70,P2=0,P3=39,P4=122	0	25	90	1205	25	11210
P1=70,P2=0,P3=38,P4=124	0	0	80	1210	0	11220*
P1=70,P2=0,P3=37,P4=124	10	5	90	1225	5	11170
P1=70,P2=0,P3=36,P4=124	20	10	100	1240	10	11120
P1=70,P2=0,P3=35,P4=125	25	0	100	1250	0	11100

Note: The solution marked * is the optimum solution for this adjustment process.

TABLE IX
ADJUSTMENT PROCESS OF REDUCING P3, P2 AND INCREASING P4 ON RESOURCE R4

	Left Time on R1	Left Time on R2	Left Time on R3	Left Time on R4	Left Time on R6	Throughput (dollars)
P1=70,P2=60,P3=40,P4=0	0	1250	1000	0	1550	11200
Part 1: Reduce P3, Increase P4.						
P1=70,P2=60,P3=39,P4=2	0	1225	990	5	1525	11210
P1=70,P2=60,P3=38,P4=4	0	1200	980	10	1500	11220
…	0	…	…	…	…	…
P1=70,P2=60,P3=0,P4=80	0	250	600	200	550	11600*
Part 2: Reduce P2, Increase P4.						
P1=70,P2=59,P3=0,P4=82	0	230	585	220	525	11600*
P1=70,P2=58,P3=0,P4=84	0	210	570	240	500	11600*
…	0	…	…	…	…	…
P1=70,P2=48,P3=0,P4=104	0	10	420	440	250	11600*
P1=70,P2=47,P3=0,P4=105	5	5	415	465	240	11570
P1=70,P2=46,P3=0,P4=106	10	0	410	490	230	11540
P1=70,P2=45,P3=0,P4=106	20	10	415	520	235	11480
P1=70,P2=44,P3=0,P4=107	25	5	410	545	225	11450

Note: The solution marked * is the optimum solution for this adjustment process.

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TABLE X
OPTIMAL INTEGER SOLUTION TO THE EXAMPLE FROM [3]

	①	②	③	④
P1	51	54	53	52
P2	38	44	42	40
P3	50	38	42	46
P4	100	100	100	100

TABLE XI
RESULT COMPARISONS BETWEEN AHP/TOC AND OPTIMAL SOLUTION

Research	Number of Product	Number of Resource	Number of Bottleneck	Throughput Obtained by AHP/ TOC	Optimal Solution
Luebbe & Finch (1992) [11]	4	4	3	6660	6660
	4	4	2	9110	9110
Plenert (1993) [2]	2	4	3	4875	4875
	4	5	5	14370	14370
Fredendall & Lea (1997) [7]	5	6	2	2230	2230
Hsu & Chung (1998) [3]	4	7	5	11600	11860
Example 1 in this paper	2	4	3	5070	5100
Example 2 in this paper	2	2	2	168	172

Note: The priority of the decision matrix in all examples is the same with the original paper, which is 0.167 for resources and 0.833 for profit.

solution is 70P1, 60P2, 0P3, and 80P4, and the throughput is US$ 11 600.

Compared with the optimum solution of these adjustments, the optimum throughput is adjusted to US$ 11 600 through either reducing some units of P3 to increase some units of P4 when Resource R1 is appointed as the new bottleneck, or reducing some units of P3 first and P2 second to increase some units of P4 when Resource R4 is appointed as the new bottleneck. There are 13 product mix solutions corresponding to the same optimum throughput, US$ 11 600, such as 70P1-60P2-0P3-80P4, 70P1-59P2-0P3-82P4, and 70P1-58P2-0P3-84P4.

Actually the optimal throughput is US$ 11 860 [3], [4]. The ILP approach and revised TOCh approach [3] can obtain only one solution, the one shown in the first column of Table X. The IA_TOC approach [4] can find four different solutions with the same optimal throughput, US$ 11 860, as shown in Table X. Therefore, regarding the example 3 from [3], the AHP/TOC method just gains a feasible solution, although the adjustment process in AHP/TOC method is activated.

All related examples are checked and the corresponding results are shown in Table XI. Regarding the examples from [2], [7], and [11], the AHP/TOC method all gains an optimal solution.

IV. ANALYSIS OF LIMITATIONS

By analyzing the two core components of AHP and TOCh, we highlight how the two key factors play important roles in the AHP/TOC method: the priority of product and the adjustment process. Here we analyzed the reasons in terms of two aspects, the priority of product and the adjustment process.

A. Ranking Approach of Product Priority Using the AHP Component

The priority of product is obtained by the AHP component and can be mathematically summarized as follows:

$$\varphi_i = \rho_1 \cdot r_{i1} + \rho_2 \sum_{j=1}^{m} (p_{ij} \cdot q_{1j}) = \rho_1 \cdot \frac{\mathrm{CM}_i}{\sum_{i=1}^{n} \mathrm{CM}_i} + \rho_2 \cdot \sum_{j=1}^{m} \left(\frac{t_{ij}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} \right)$$

$\exists\,\forall$ product P and Q, the relationship exists as follows:

$$\varphi_P - \varphi_Q = \rho_1 \cdot \frac{\mathrm{CM}_P - \mathrm{CM}_Q}{\sum_{i=1}^{n} \mathrm{CM}_i} + \rho_2 \cdot \sum_{j=1}^{m} \left(\frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} \right).$$

Apparently it is difficult to directly compare φ_P with φ_Q and to easily point out which one is higher than the other, since it is related with many factors. Therefore, we consider some facts with explicit management meanings to indirectly compare φ_P with φ_{Q}. In general, we assume without loss of generality that the priority of product P is not lower than Q when comparing φ_P with φ_Q.

A1. If $\mathrm{CM}_P = \mathrm{CM}_Q$ and $t_{Pj} = t_{Qj}\ \forall j$, then $\varphi_P = \varphi_Q$.

If the contribution margins of the two products are equal, and the processing times of the two products on any resource are equal, then the priority of the two products are equal.

A2. If $\mathrm{CM}_P = \mathrm{CM}_Q$ and $t_{Pj} < t_{Qj}\ \forall j$, then $\varphi_P > \varphi_Q$.

If the contribution margins of the two products are equal, and the processing times of product *P* on any resource are all less than Q, then the priority of product P is higher than Q.

A3. If $\mathrm{CM}_P > \mathrm{CM}_Q$ and $t_{Pj} = t_{Qj}\ \forall j$, then $\varphi_P > \varphi_Q$.

If the processing times of the two products on any resource are equal, and the contribution margins of product P is higher than Q, then the priority of product P is higher than Q.

A4. If $\mathrm{CM}_P > \mathrm{CM}_Q$ and $t_{Pj} < t_{Qj}\ \forall j$, then $\varphi_P > \varphi_Q$.

If the contribution margins of product *P* is higher than Q, and the processing times of product *P* on any resource are all less than Q, then the priority of product P is higher than Q.

The four facts are easily understood and used as the judgment criteria for comparing the priority of product. Using these judgment criteria, we clarify the cases under which constraints the AHP/TOC method can and cannot work well, as shown in Table XII. Here two conditions are implied: $b_j > 0$ and $\sum_{i=1}^{n} t_{ij} > 0$.

From Table XII, these cases are analyzed as follows.

1) For Case ①, ⑤, ⑦, ⑩ and ⑫, the proposed approach [1] can generate the right product priority.
2) For Case ②, ③, ⑧ and ⑪, it definitely fails to output the wrong product priority.

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TABLE XII
ANALYSIS OF PRODUCT PRIORITY DETERMINED BY THE AHP COMPONENT

Case	$\theta = \rho_1 \cdot \frac{CM_P - CM_Q}{\sum_{i=1}^{n} CM_i}$		$\vartheta = \rho_2 \cdot \sum_{j=1}^{m} \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j}$		$\varphi_P - \varphi_Q = \theta + \vartheta$		Does the comparison result comply with the judgment criteria?	Which criterion?
	Value	**Management meanings**	**Value**	**Management meanings**	**Value**	**Management meanings**		
①	0	the contribution margins of the two products are equal, $CM_P = CM_Q$	$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} \equiv 0$	the processing times of the two products on any resource are equal, $\forall j, t_{Pj} \equiv t_{Qj}$.	0	Equal priority, P=Q.	Yes	A1
②			$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} > 0$	the processing time of P on any resource is more than Q, $\forall j, t_{Pj} > t_{Qj}$.	Positive	the priority of P is larger than Q, P>Q.	No	A2
③			$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} < 0$	the processing time of P on any resource is less than Q, $\forall j, t_{Pj} < t_{Qj}$.	Negative	the priority of P is lower than Q, P<Q.	No	A2
④			Others	Cannot determine the value of $\sum_{j=1}^{m} \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j}$ if only know part value of products rather than all products	Cannot determine	/	Cannot judge whether the conclusion is right or not.	/
⑤	Positive	the contribution margins of P is more than Q, $CM_P > CM_Q$	$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} \equiv 0$	the processing times of the two products on any resource are equal, $\forall j, t_{Pj} \equiv t_{Qj}$.	Positive	the priority of P is larger than Q, P>Q.	Yes	A3
⑥			$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} > 0$	the processing time of P on any resource is more than Q, $\forall j, t_{Pj} > t_{Qj}$.	Positive	the priority of P is larger than Q, P>Q.	Cannot judge whether the conclusion is right or not.	/
⑦			$\forall j, \frac{t_{Pj} - t_{Qj}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} < 0$	the processing time of P on any resource is less than Q, $\forall j, t_{Pj} < t_{Qj}$.	It depends on ρ_1 and ρ_2	if $\theta + \vartheta > 0$, then $\varphi_P - \varphi_Q > 0$, the priority of P is larger than Q, P>Q.	Yes	A4
⑧						If $\theta + \vartheta < 0$, then $\varphi_P - \varphi_Q < 0$, the priority of P is lower than Q, P<Q.	No	A4

TABLE XII
(Continued)

⑨			Others	Cannot determine the value of $\sum_{j=1}^{m}\frac{t_{Pj}-t_{Qj}}{\sum_{i=1}^{n}t_{ij}}\cdot\frac{b_j}{\sum_{j=1}^{m}b_j}$ if only know part value of products rather than all products	Cannot determine	/	Cannot judge whether the conclusion is right or not.	
⑩	Negative	the contribution margins of P is less than Q, $CM_P < CM_Q$	$\forall j, \frac{t_{Pj}-t_{Qj}}{\sum_{i=1}^{n}t_{ij}}\cdot\frac{b_j}{\sum_{j=1}^{m}b_j}\equiv 0$	the processing times of the two products on any resource are equal, $\forall j, t_{Pj}\equiv t_{Qj}$.	Negative	the priority of P is lower than Q, P<Q.	Yes	A3
⑪			$\forall j, \frac{t_{Pj}-t_{Qj}}{\sum_{i=1}^{n}t_{ij}}\cdot\frac{b_j}{\sum_{j=1}^{m}b_j} > 0$	the processing time of P on any resource is more than Q, $\forall j, t_{Pj} > t_{Qj}$.	It depends on ρ_1 and ρ_2	If $\theta+\vartheta>0$, then $\varphi_P-\varphi_Q>0$, the priority of P is larger than Q, P>Q.	No	A4
⑫						If $\theta+\vartheta<0$, then $\varphi_P-\varphi_Q<0$, the priority of P is lower than Q, P<Q.	Yes	A4
⑬			$\forall j, \frac{t_{Pj}-t_{Qj}}{\sum_{i=1}^{n}t_{ij}}\cdot\frac{b_j}{\sum_{j=1}^{m}b_j} < 0$	the processing time of P on any resource is less than Q, $\forall j, t_{Pj} < t_{Qj}$.	Negative	the priority of P is lower than Q, P<Q.	Cannot judge whether the conclusion is right or not	
⑭			Others	Cannot determine the value of $\sum_{j=1}^{m}\frac{t_{Pj}-t_{Qj}}{\sum_{i=1}^{n}t_{ij}}\cdot\frac{b_j}{\sum_{j=1}^{m}b_j}$ if only know part value of products rather than all products	Cannot determine	/	Cannot judge whether the conclusion is right or not.	

3) For other cases, we have no idea to decompose the situation into some specific cases to further discuss their priorities since we cannot judge the complicated comparison of the two products (please see the formula to learn more, $\varphi_P - \varphi_Q = \rho_1 \cdot ((\mathrm{CM}_P - \mathrm{CM}_Q)/\sum_{i=1}^{n}\mathrm{CM}_i) + \rho_2 \cdot \sum((t_{Pj}-t_{Qj})/\sum_{i=1}^{n}t_{ij})\cdot(b_j/\sum_{j=1}^{m}b_j))$ is positive or negative if only know part value.

In brief, we arrive at the following conclusion from the analysis of product priority determined by the AHP component.

1) If the processing times of the two products on any resource are equal, then the ranking approach works well. Specifically, the product priority is only dependent on the contribution margins.
2) If the contribution margins of the two products are equal, and the processing time of P on any resource is more than or less than Q, then the ranking approach does not work.
3) If the contribution margins of P is more (or less) than Q and the processing time of P on any resource is less (or more) than Q, whether the ranking approach works or not is dependent on the weights of profit and resource. Setting the right weights of profit and resource will determine the effectiveness of the ranking approach.

Generally, the priority of product is obtained by the classical TOCh indicator, contribution margin per unit constraint minute of all products on bottleneck. (i.e., [2], [5], [6])

$$\varphi_i = \frac{\mathrm{CM}_i}{t_{i,BN}}$$

where $t_{i,BN}$ is the processing time of product i on the bottleneck resource.

From the expression, we can see there is a positive relationship between φ_i and CM_i, and a negative relationship between φ_i and $t_{i,BN}$. That means the product with higher throughput and shorter processing time on the bottleneck should have the larger priority of product, vice versa.

But in the expression of AHP to determine the priority, $\varphi_i = \rho \sum_{j=1}^{m} \frac{t_{ij}}{\sum_{i=1}^{n} t_{ij}} \cdot \frac{b_j}{\sum_{j=1}^{m} b_j} + (1-\rho) \frac{\mathrm{CM}_i}{\sum_{i=1}^{n} \mathrm{CM}_i}$, there is a positive relationship between φ_i and $t_{i,j}$, that is to say, the longer processing time the product uses, the larger priority the product has. This presents a paradox. According to the common sense, the classical TOCh indicator provides more reasonable management meanings than the proposed AHP component. That is a core point of reason.

Even if the right priority of product is obtained, it would not ensure the output of the optimal product mix solution. At this point, an increasing series of studies are launched. Lee and Plenert [5] and Plenert [6] clearly demonstrated the TOCh cannot find the optimum product mix when production must be done over integer quantities in the case of multiple bottlenecks. Fredendall and Lea [7] and Hsu and Chung [3] insisted that not just the dominate constraint, but all kinds of constraints should be fully utilized, then presented their revised TOCh approaches. For a single bottleneck example, the TOCh approach is usually proved to gain an optimal solution. However, Linhares [8] illustrated that the TOCh approach may fail even in the case of a single bottleneck, and presented that the failure stems from the NP-Complete nature of the product mix problem itself, not from the nature of the method. Wang *et al.* [4] presented the failure is not associated with single or multiple bottlenecks scenarios, also not concerned with the real or integer number of the product quantity, but only related with whether each bottleneck is fully utilized or not. This is the point at the issue.

B. Adjustment Approach of Product Mix

The adjustment process utilizes the product priority determined by the classical TOCh indicator to judge which one product should be reduced or increased to increase the system throughput. Let the left time on new bottleneck be $t_{\mathrm{left},BN}$, and the processing times of P and Q on the new bottleneck be $t_{P,BN}$, $t_{Q,BN}$, respectively.

When reducing one unit of P, this creates $(t_{\mathrm{left},BN} + t_{P,BN})$ of available time on the new bottleneck resource. Schedule this available time to increase Q. Let N be the unit(s) of Q that will be increased when utilized the available time, then $N = (t_{\mathrm{left},BN} + t_{P,BN})/t_{Q,BN}$, and the system throughput will increase $(N \times \mathrm{CM}_Q)$.

Assume reducing P and increasing Q at the same time can ensure no less than the original throughput, then the action can execute

$$N \times \mathrm{CM}_Q - \mathrm{CM}_P \geq 0$$

$$\frac{(t_{\mathrm{left},BN} + t_{P,BN})}{t_{Q,BN}} \times \mathrm{CM}_Q - \mathrm{CM}_P \geq 0$$

and we can obtain the formula

$$\frac{\mathrm{CM}_Q \times (t_{\mathrm{left},BN} + t_{P,BN})}{t_{Q,BN}\mathrm{CM}_P} \geq 1.$$

Thus, we can transform the adjustment conditions in the proposed approach [1] to the formula mentioned above. Actually it is the same with the judgment condition used in the literature [7]. However, Aryanezhad and Komijan [9] pointed out it would fail for an example of five products and four resources, and presented an improved algorithm named by TOCh-AK. Later, Sobreiro and Nagano [10] presented a constructive heuristic based on the TOCh and the Knapsack problem.

V. Conclusion

This paper has discussed the work by Ray *et al.* [1] which proposed an AHP/TOC model and provided the decision maker with an innovative clue and optional way to use TOCh for product mix optimization in the multiple bottleneck scenarios. The work is considered as one of the significant efforts in addressing the product mix optimization problem which is best known to be NP-Complete [10]. We discuss its two key components of product mix determination and product mix adjustment in detail, and analyze two limitations in terms of ranking approach of product priority using the AHP component and the adjustment approach of product mix after a new bottleneck has been identified. Furthermore, we clarify the cases under which the proposed method can and cannot output the optimal solution.

Having discussed the two limitations, we give indications on possible improvements. The first limitation is related to the exploration of the rich information of multidimensional factors related to products and machines to rank product priority. One possible improvement is to introduce the multiattribute decision theory, considering equally the influence of every bottleneck. The second limitation is related to the reallocation problem of limited resources to product if any bottleneck is not fully utilized. It is more complicated than the original product mix optimization problem, since it relates to the adjustment of the determined solution of product mix besides the utilization of left time on bottlenecks. A neighborhood search algorithm based on effective heuristic rule is a possible improvement for exploring the reallocation problem.

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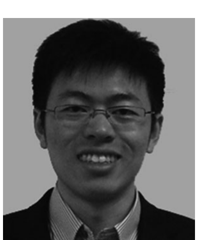

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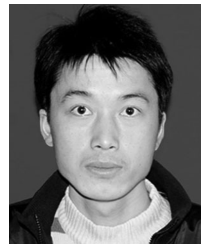

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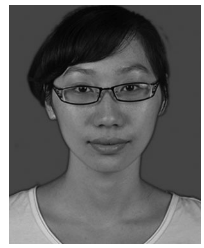

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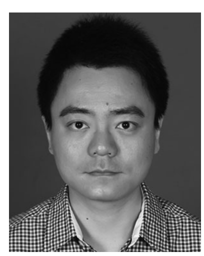

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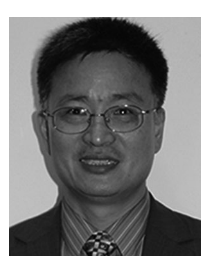

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