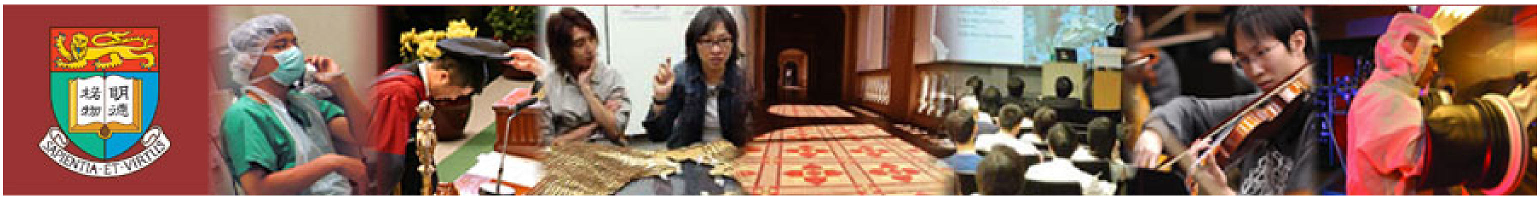

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Weyl-Cartan-Weitzenböck gravity as a generalization of teleparallel gravity

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We consider a gravitational model in a Weyl-Cartan space-time in which the Weitzenböck condition of the vanishing of the sum of the curvature and torsion scalar is imposed. In contrast to the standard teleparallel theories, our model is formulated in a four-dimensional curved spacetime. The properties of the gravitational field are then described by the torsion tensor and Weyl vector fields. A kinetic term for the torsion is also included in the gravitational action. The field equations of the model are obtained from a Hilbert-Einstein type variational principle, and they lead to a complete description of the gravitational field in terms of two fields, the Weyl vector and the torsion, respectively, defined in a curved background. The cosmological applications of the model are investigated for a particular choice of the free parameters in which the torsion vector is proportional to the Weyl vector. The Newtonian limit of the model is also considered, and it is shown that the Poisson equation can be recovered in the weak field approximation. Depending on the numerical values of the parameters of the cosmological model, a large variety of dynamic evolutions can be obtained, ranging from inflationary/accelerated expansions to non-inflationary behaviors. In particular we show that a de Sitter type late time evolution can be naturally obtained from the field equations of the model. Therefore the present model leads to the possibility of a purely geometrical description of the dark energy, in which the late time acceleration of the Universe is determined by the intrinsic geometry of the space-time.

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I. INTRODUCTION

Despite its remarkable achievements at the scale of the Solar System and its intrinsic theoretical beauty, general relativity, the best gravitational theory proposed up to now, is facing presently some severe observational challenges. The observations of high redshift supernovae [1] and the Boomerang/Maxima/WMAP data [2], showing that the location of the first acoustic peak in the power spectrum of the microwave background radiation is consistent with the inflationary prediction $\Omega = 1$, have provided compelling evidence that about 95% of the content of the Universe resides in two unknown forms of matter and energy that we call cold dark matter (CDM) and dark energy (DE), with negative pressure, respectively, the first residing in bound objects as non-luminous matter [3], the latter in the form of a zero-point energy that pervades the whole Universe [4]. Dark matter is thought to be composed of cold neutral weakly interacting massive particles, beyond those existing in the Standard Model of Particle Physics, and not yet detected in accelerators or in dedicated direct and indirect searches [5]. CDM contributes $\Omega_{DM} \approx 0.25$ to the total density of the Universe, and is mainly motivated by the theoretical interpretation of the galactic rotation curves and large scale structure formation. DE provides $\Omega_{DE} \approx 0.73$ to the energy content of the Universe, and is responsible for the acceleration of the distant type Ia supernovae [4].

There are two main theoretical lines of thought along which one could propose explanations of the recent acceleration of the Universe. The first idea is that the acceleration is produced by the presence of a "real" physical field in the Universe, which may be also related to Einstein's cosmological constant. One way of implementing this model are cosmologies based on a mixture of cold dark matter and quintessence, a slowly-varying, spatially inhomogeneous component [6]. An example of the idea of quintessence is the suggestion that it is the energy associated with a scalar field Q with self-interaction potential $V(Q)$. If the potential energy density is greater than the kinetic one, then the pressure $p = \dot{Q}^2/2 - V(Q)$, associated to the Q-field is negative. Quintessential cosmological models have

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been intensively investigated in the physical literature [6]. A different possibility has been followed in [7], where the conditions under which the dynamics of a self-interacting Brans-Dicke (BD) field can account for the accelerated expansion of the Universe have been analyzed.

Neither CDM nor DE have direct laboratory observational or experimental evidence for their existence. Therefore it would be important if a unified dark matter-dark energy scenario could be found, in which these two components are different manifestations of a single fluid. A candidate for such an unification is the so-called generalized Chaplygin gas, which is an exotic fluid with the pressure p given as a function of the density ρ by the equation of state $p = -B/\rho^n$, where B and n are two parameters to be determined [8]. This dark energy-dark matter model was initially suggested with $n = 1$, and then generalized for the case $n \neq 1$. The cosmological implications of the Chaplygin gas model have also been considered [8].

The second line of thought that can be considered for the explanation of the recent observational data of the acceleration of the distant galaxies is based on the assumption that the geometrical framework necessary to describe the Universe must go beyond the Riemannian geometrical approach and the Einstein-Hilbert Lagrangian on which standard general relativity is based. One of the simplest possible theoretical extensions of general relativity are the $f(R)$ gravity models, in which the Lagrangian of the gravitational field is given by an arbitrary function of the Riemann curvature R. $f(R)$ models can indeed explain the late acceleration of the universe, and their cosmological implications have been intensively investigated [9]. The non-trivial inclusion of the matter Lagrangian in the gravitational field Lagrangian has led to the so-called $f(R, L_m)$ theories, which have been extensively investigated recently [10].

An alternative geometrical approach, in which the basic physical variable is not the metric $g_{\mu\nu}$ of the space-time, but a set of tetrad vectors $e^i{}_\mu$ was recently considered in the so called $f(T)$ theories of gravity, where T is the torsion scalar. In this approach the torsion, generated by the tetrad fields, can be used to describe general relativity entirely using only torsion instead of curvature. This is the so-called teleparallel equivalent of General Relativity (TEGR), which was introduced in [11]. Teleparallel gravity is based on the Weitzenböck space model [12], in which torsion exactly compensates curvature and the space-time becomes flat. In the $f(T)$ gravity models the field equations are of second order, unlike in $f(R)$ gravity, which in the metric approach is a fourth order theory. $f(T)$ models have been extensively applied to cosmology, and in particular to explain the late-time accelerating expansion without the need of dark energy [13].

Historically, the first extension of general relativity was proposed by Weyl [14], who unified gravitation and electromagnetism by extending the notion of parallel transport in general relativity to include the possibility that lengths, and not only directions, change when vectors are transported along a path. This non-integrability of length provides a geometrical interpretation for the electromagnetic field and an elegant to unify the two known long-range forces of nature. Despite its intrinsic beauty and rich physical structure, Weyl's theory did not become a mainstream topic of research mainly because of Einstein's comment [15] that "...in Weyl's theory the frequency of spectral lines would depend on the history of the atom, in complete contradiction to known experimental facts." A generalization of Weyl's theory was introduced by Dirac [16] who proposed the existence of an unmeasurable metric ds_E, affected by transformations in the standards of length, and of a measurable one, the conformally invariant atomic metric ds_A. In fact, any function $f(x)$ that transforms as $f(x)/\sigma(x)$ under the transformation $g_{\mu\nu} \to \sigma^2 g_{\mu\nu}$ would provide the appropriate relationship between the two metrics as $f(x)ds_E = ds_A$.

The Weyl-Dirac theory and its applications were studied in [18], and some of the early results are summarized in the book [19]. The Weyl-Dirac theory was considered within the framework of the weak field approximation in [20], and it was shown that the resulting gravitational potential differs from that of Newtonian one by a repulsive correction term increasing with distance. If the time variation rate of gravitational coupling is adopted from observational bounds, the theory can explain the rotation curves of typical spiral galaxies without resorting to dark matter. The intergalactic effects and the gravitational lensing of clusters of galaxies have been estimated, and they are consistent with observational data.

The Weyl geometry can be immediately generalized to include the torsion of the space-time [17]. This geometric model is called the Weyl-Cartan geometry, and it was extensively studied from both mathematical and physical point of view [21]. In [22] torsion was included in the geometric framework of the Weyl-Dirac theory to build up an action integral, from which one can obtain a gauge covariant (in the Weyl sense) general relativistic massive electrodynamics. For a recent review of the geometric properties and of the physical applications of the Riemann-Cartan and Weyl-Cartan space-times see [23].

It is the purpose of the present paper to consider an extension of the Weyl-Cartan gravitational model by explicitly including the Weitzenböck condition that cancels torsion and curvature, in the geometric structure of the Weyl-Cartan space. In this way we obtain a curvature-full geometric gravitational model, in which the properties of the gravitational field are determined by two fields, the Weyl vector and the torsion, as well as by some scalars formed from the combination of these basic fields. The basic difference with respect to the $f(T)$ type teleparallel theories is that the model is formulated in the usual four-dimensional curved spacetime, whose properties are described by the metric tensor $g_{\mu\nu}$. However, the properties of the gravitational field are described directly by two vector fields in a curved space-time, and the metric is not the primary object determining the gravitational properties. The field equations of this gravity model are obtained from a variational principle, and the cosmological applications of the field equations are studied in detail. In particular we show that depending on the numerical values of the parameters of the cosmological model, a large variety of dynamic evolutions can be obtained, ranging from inflationary/accelerated expansions to non-inflationary behavior. The weak field limit of the model is also considered, and it is shown that the Poisson equation can be recovered from the Weitzenböck condition and the field equations. This shows that the model proposed in the present paper is phenomenologically viable.

The present paper is organized as follows. The geometrical properties of the Weyl-Cartan space-time are briefly described in Section II. The gravitational field equations corresponding to the Weyl-Cartan-Weitzenböck (WCW) geometry are obtained in Section III. The Newtonian limit of the model is also considered. The cosmological implications of the model are investigated in Section IV. Specific cosmological solutions are presented in Section V. We discuss and conclude our results in Section VI. Finally, two appendices present the relation of this work to phantom models and possible generalizations of the action, respectively.

II. GEOMETRICAL PRELIMINARIES

Weyl generalized the Riemannian geometry of general relativity by supposing that during the parallel transport around a closed path a vector would not only undergo a change of direction, but would also experience a change in length [14]. In order to describe these two simultaneous changes mathematically, Weyl introduced a vector field w^{μ}, which, together with the metric $g_{\mu\nu}$, represent the fundamental fields of the Weyl geometry. It is a remarkable feature of the Weyl model that the properties of w^{μ} coincide precisely with those of the electromagnetic potentials, suggesting that the long-range forces of electromagnetism and gravity have a common geometric origin [16].

If a vector of length l is carried by parallel transport along an infinitesimal displacement δx^{μ} in Weyl space, the change in its length δl is given by $\delta l = l w_{\mu}\delta x^{\mu}$ [16]. For parallel transport around a small closed loop of area $\delta s^{\mu\nu}$ the change of the vector is $\delta l = l W_{\mu\nu}\delta s^{\mu\nu}$, where

$$W_{\mu\nu} = \nabla_{\nu} w_{\mu} - \nabla_{\mu} w_{\nu}, \tag{1}$$

where the covariant derivative is with respect to the metric. Under the local scaling of lengths $\tilde{l} = \sigma(x) l$, the field w_{μ} transforms as $\tilde{w}_{\mu} = w_{\mu} + (\ln\sigma)_{,\mu}$, while the metric tensor coefficients transform according to the conformal transformations $\tilde{g}_{\mu\nu} = \sigma^2 g_{\mu\nu}$ and $\tilde{g}^{\mu\nu} = \sigma^{-2} g^{\mu\nu}$, respectively [19, 23]. A distinctive featur of the Weyl geometry is the presence of the semi-metric connection

$$\bar{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + g_{\mu\nu} w^{\lambda} - \delta^{\lambda}_{\mu} w_{\nu} - \delta^{\lambda}_{\nu} w_{\mu}, \tag{2}$$

where $\Gamma^{\lambda}{}_{\mu\nu}$ is the Christoffel symbol with respect to the metric $g_{\mu\nu}$. In the Weyl geometry $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$ is assumed to be symmetric and with its help one can construct a gauge covariant derivative [23]. The Weyl curvature tensor can be written as

$$\bar{R}_{\mu\nu\alpha\beta} = \bar{R}_{(\mu\nu)\alpha\beta} + \bar{R}_{[\mu\nu]\alpha\beta}, \tag{3}$$

where we define

$$\bar{R}_{[\mu\nu]\alpha\beta} = R_{\mu\nu\alpha\beta} + 2\nabla_{\alpha} w_{[\mu} g_{\nu]\beta} + 2\nabla_{\beta} w_{[\nu} g_{\mu]\alpha} + 2 w_{\alpha} w_{[\mu} g_{\nu]\beta} + 2 w_{\beta} w_{[\nu} g_{\mu]\alpha} - 2 w^2 g_{\alpha[\mu} g_{\nu]\beta}, \tag{4}$$

and

$$\bar{R}_{(\mu\nu)\alpha\beta} = \frac{1}{2}\left(\bar{R}_{\mu\nu\alpha\beta} + \bar{R}_{\nu\mu\alpha\beta}\right) = g_{\mu\nu} W_{\alpha\beta}, \tag{5}$$

and square brackets denote anti-symmetrization. The first contraction of the Weyl curvature tensor is given by

$$\bar{R}^{\mu}{}_{\nu} = \bar{R}^{\alpha\mu}{}_{\alpha\nu} = R^{\mu}{}_{\nu} + 2 w^{\mu} w_{\nu} + 3\nabla_{\nu} w^{\mu} - \nabla_{\mu} w^{\nu} + g^{\mu}{}_{\nu}\left(\nabla_{\alpha} w^{\alpha} - 2 w_{\alpha} w^{\alpha}\right), \tag{6}$$

where $R^{\mu}{}_{\nu}$ is the Ricci tensor constructed by the metric. For the Weyl scalar we obtain

$$\bar{R} = \bar{R}^{\alpha}{}_{\alpha} = R + 6\left(\nabla_{\mu} w^{\mu} - w_{\mu} w^{\mu}\right). \tag{7}$$

The Weyl geometry can be extended to the more general case of the Weyl-Cartan spaces with torsion, by considering a space time with a symmetric metric tensor $g_{\mu\nu}$, defining the length of a vector, and an asymmetric connection $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$, which defines the law of the parallel transport as $dv^\mu = -v^\sigma \hat{\Gamma}^\mu{}_{\sigma\nu} dx^\nu$ [17, 23]. In this case the connection may be decomposed into three irreducible parts: the Christoffel symbol, the contortion tensor and the non-metricity, so that one can write [17]

$$\hat{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + C^\lambda{}_{\mu\nu} + \frac{1}{2} g^{\lambda\sigma} \left(Q_{\nu\mu\sigma} + Q_{\mu\nu\sigma} - Q_{\lambda\mu\nu}\right). \tag{8}$$

The contorsion $C^\lambda{}_{\mu\nu}$ in equation (8) can be given in terms of the torsion tensor $\hat{\Gamma}^\lambda{}_{[\mu\nu]}$ defined as

$$\hat{\Gamma}^\lambda{}_{[\mu\nu]} = \frac{1}{2}\left(\hat{\Gamma}^\lambda{}_{\mu\nu} - \hat{\Gamma}^\lambda{}_{\nu\mu}\right), \tag{9}$$

as follows

$$C^\lambda{}_{\mu\nu} = \hat{\Gamma}^\lambda{}_{[\mu\nu]} + g^{\lambda\sigma} g_{\mu\kappa} \hat{\Gamma}^\kappa{}_{[\nu\sigma]} + g^{\lambda\sigma} g_{\nu\kappa} \hat{\Gamma}^\kappa{}_{[\mu\sigma]}. \tag{10}$$

The contorsion tensor is antisymmetric with respect to its first two indices.

The non-metricity tensor $Q_{\lambda\mu\nu}$ can be defined as (minus) the covariant derivative of the metric tensor with respect to $\hat{\Gamma}^\lambda{}_{\mu\nu}$ [17],

$$Q_{\lambda\mu\nu} = -\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + g_{\nu\sigma} \hat{\Gamma}^\sigma{}_{\mu\lambda} + g_{\sigma\mu} \hat{\Gamma}^\sigma{}_{\nu\lambda} \tag{11}$$

By comparing equations (2) and (8) it follows that the Weyl geometry is a particular case of the Weyl-Cartan geometry, in which the torsion is zero, and the non-metricity is given by $Q_{\lambda\mu\nu} = -2g_{\mu\nu} w_\lambda$. Therefore in a Weyl-Cartan space the connection is written as

$$\hat{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + g_{\mu\nu} w^\lambda - \delta^\lambda_\mu w_\nu - \delta^\lambda_\nu w_\mu + C^\lambda{}_{\mu\nu}, \tag{12}$$

where

$$C^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu} - g^{\lambda\beta} g_{\sigma\mu} T^\sigma{}_{\beta\nu} - g^{\lambda\beta} g_{\sigma\nu} T^\sigma{}_{\beta\mu}, \tag{13}$$

is the contortion of the Weyl-Cartan space, in which the Weyl-Cartan torsion $T^\lambda{}_{\mu\nu}$ is defined as

$$T^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\hat{\Gamma}^\lambda{}_{\mu\nu} - \hat{\Gamma}^\lambda{}_{\nu\mu}\right). \tag{14}$$

By the use of the connection, one can obtain the curvature tensor as

$$K^\lambda{}_{\mu\nu\sigma} = \hat{\Gamma}^\lambda{}_{\mu\sigma,\nu} - \hat{\Gamma}^\lambda{}_{\mu\nu,\sigma} + \hat{\Gamma}^\alpha{}_{\mu\sigma} \hat{\Gamma}^\lambda{}_{\alpha\nu} - \hat{\Gamma}^\alpha{}_{\mu\nu} \hat{\Gamma}^\lambda{}_{\alpha\sigma}. \tag{15}$$

Using equation (12), one can obtain the curvature tensor $K^\lambda{}_{\mu\nu\sigma}$ in the terms of Riemann tensor and some new terms containing Weyl vector, torsion and contortion. By contracting the resulting curvature tensor, one can obtain the Ricci scalar of the model as follows

$$\begin{aligned} K = K^{\mu\nu}{}_{\mu\nu} = R &+ 6\nabla_\nu w^\nu - 4\nabla_\nu T^\nu - 6 w_\nu w^\nu + 8 w_\nu T^\nu \\ &+ T^{\mu\alpha\nu} T_{\mu\alpha\nu} + 2T^{\mu\alpha\nu} T_{\nu\alpha\mu} - 4T_\nu T^\nu. \end{aligned} \tag{16}$$

where we define $T_\mu = T^\nu{}_{\mu\nu}$ and all covariant derivatives are of the metric.

III. FIELD EQUATIONS OF THE WEYL-CARTAN-WEITZENBÖCK GRAVITY

We work in a four dimensional Weyl-Cartan space-time with non-metricity and torsion. In such a framework, at each point of space time, there is a symmetric metric tensor $g_{\mu\nu}$, a Weyl vector w_μ and a torsion tensor $T^\lambda{}_{\mu\nu}$. Using the units in which $16\pi G = 1$, one may consider the following action

$$I = \int d^4x \sqrt{-g} \left(K + \frac{1}{4} W^{\mu\nu} W_{\mu\nu} + \beta \nabla_\mu T \nabla^\mu T + L_m\right), \tag{17}$$

where $T=T_\mu T^\mu$ and

$$W_{\mu\nu}=\nabla_\nu w_\mu-\nabla_\mu w_\nu. \tag{18}$$

The first term produces the kinetic term for the metric. The second term is the usual kinetic term for the Weyl vector. We also add a kinetic term for the torsion. We note that this is not the most general kinetic term for the torsion. In Appendix B we will discuss an alternative possibility for the kinetic term. In addition, we assume that the matter Lagrangian only depends on the metric and not on the torsion and the Weyl vector.

In terms of the dynamical variables $\left(g_{\mu\nu},w_\mu,T^\lambda_{\ \mu\nu}\right)$ one can express the action explicitly as

$$\begin{aligned}I=\int d^4x\sqrt{-g}\bigg(&R+T^{\mu\alpha\nu}T_{\mu\alpha\nu}+2T^{\mu\alpha\nu}T_{\nu\alpha\mu}-4T_\mu T^\mu\\&+\frac{1}{4}W^{\mu\nu}W_{\mu\nu}+\beta\nabla_\mu T\nabla^\mu T-6w_\mu w^\mu+8w_\mu T^\mu+L_m\bigg).\end{aligned} \tag{19}$$

A. The Weitzenböck condition and the field equations of the Weyl-Cartan-Weitzenböck gravity

Now, we assume the Weitzenböck condition

$$R+T^{\mu\alpha\nu}T_{\mu\alpha\nu}+2T^{\mu\alpha\nu}T_{\nu\alpha\mu}-4T_\mu T^\mu=0. \tag{20}$$

Imposing this condition we obtain the following form for the action,

$$I=\int d^4x\sqrt{-g}\bigg(\frac{1}{4}W^{\mu\nu}W_{\mu\nu}+\beta\nabla_\mu T\nabla^\mu T-6w_\mu w^\mu+8w_\mu T^\mu+L_m\bigg). \tag{21}$$

It is worth mentioning that, without the kinetic term for the torsion, the field equations obtained by varying the action with respect to the torsion will be satisfied only when the Weyl vector vanishes which is the teleparallel gravity. The field equations, obtained by variation of the action with respect to the Weyl vector field and torsion are given by

$$\frac{1}{2}\nabla^\nu W_{\nu\mu}-6w_\mu+4T_\mu=0, \tag{22}$$

$$2\left(w^\rho\delta^\sigma_\mu-w^\sigma\delta^\rho_\mu\right)-\beta\left(T^\rho\delta^\sigma_\mu-T^\sigma\delta^\rho_\mu\right)\Box T=0. \tag{23}$$

The variation of the action with respect to the metric gives us

$$\begin{aligned}&\frac{1}{2}\left(W_{\mu\rho}W_\nu^{\ \rho}-\frac{1}{4}g_{\mu\nu}W_{\rho\sigma}W^{\rho\sigma}\right)-6\left(w_\mu w_\nu-\frac{1}{2}g_{\mu\nu}w_\rho w^\rho\right)\\&+\beta\Big(\nabla_\mu T\nabla_\nu T-\frac{1}{2}g_{\mu\nu}\nabla_\rho T\nabla^\rho T-2T_\mu T_\nu\Box T\Big)+4\left(T_\mu w_\nu+T_\nu w_\mu-g_{\mu\nu}T_\rho w^\rho\right)-\frac{1}{2}T^m_{\mu\nu}=0,\end{aligned} \tag{24}$$

where $T^m_{\mu\nu}$ is the energy-momentum tensor.

The torsion tensor can be decomposed into its irreducible components under the local $O(1,3)$ group as

$$T_{\mu\nu\rho}=\frac{2}{3}(t_{\mu\nu\rho}-t_{\mu\rho\nu})+\frac{1}{3}\left(Q_\nu g_{\mu\rho}-Q_\rho g_{\mu\nu}\right)+\epsilon_{\mu\nu\rho\sigma}S^\sigma, \tag{25}$$

where Q_μ and S^μ are two unknown vectors and $t_{\mu\nu\rho}$ is symmetric with respect to the interchange of μ and ν and satisfies the following identities

$$t_{\mu\nu\rho}+t_{\nu\rho\mu}+t_{\rho\mu\nu}=0,\quad g^{\mu\nu}t_{\mu\nu\rho}=0=g^{\mu\rho}t_{\mu\nu\rho}. \tag{26}$$

As one can see from equations (22), (23) and (24), the torsion tensor appears in the contracted form, and so only the Q_μ vector is relevant in the model. We can then consistently set the tensor $t_{\mu\nu\rho}$ to zero and obtain the vector S^μ from the Weitzenböck condition (20). Contraction of equation (25) with $g^{\mu\rho}$ results in $T_\mu=Q_\mu$. Now by contracting equation (23) with δ^μ_σ one obtains

$$2w^\rho-\beta T^\rho\Box T=0. \tag{27}$$

Multiplication by T_ρ and the assumption $T \neq 0$ leads to

$$\Box T = \frac{2}{\beta}\frac{1}{T} w^\rho T_\rho. \tag{28}$$

Now, putting this back to equation (23) one obtains

$$T^\alpha T_\alpha \left(w^\rho \delta^\sigma_\mu - w^\sigma \delta^\rho_\mu\right) = w^\alpha T_\alpha \left(T^\rho \delta^\sigma_\mu - T^\sigma \delta^\rho_\mu\right), \tag{29}$$

with the solution

$$T_\mu = Q_\mu = A w_\mu, \tag{30}$$

where A is an arbitrary constant. The value of this constant can be obtained by taking the covariant derivative of equation (22) which leads to

$$\nabla^\mu \left(T_\mu - \frac{3}{2} w_\mu\right) = 0, \tag{31}$$

So, we must have $A = 3/2$.

Substituting this to equation(22), we obtain the following differential equation for the Weyl vector

$$\nabla^\nu \nabla_\mu w_\nu - \Box w_\mu = 0. \tag{32}$$

This equation gives the Weyl vector as a function of the metric in general. With a given matter energy-momentum tensor, one can then obtain the Weyl vector and the metric components from equations (24) and (32). The vector S^μ however can be obtained from the teleparallel condition (20) as a function of the metric and Weyl vector using solution (30).

It is interesting to note that because our theory is generally covariant, we have $\nabla^\mu T^m_{\mu\nu} = 0$ [24]. To see this, we take the divergence of equation (24) which leads to

$$\begin{aligned}\nabla^\mu \Bigg[&\frac{1}{2}\left(W_{\mu\rho} W_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} W_{\rho\sigma} W^{\rho\sigma}\right) - 6\left(w_\mu w_\nu - \frac{1}{2} g_{\mu\nu} w_\rho w^\rho\right) \\ &+ \beta\Big(\nabla_\mu T \nabla_\nu T - \frac{1}{2} g_{\mu\nu} \nabla_\rho T \nabla^\rho T - 2 T_\mu T_\nu \Box T\Big) + 4\left(T_\mu w_\nu + T_\nu w_\mu - g_{\mu\nu} T_\rho w^\rho\right)\Bigg] = 0.\end{aligned} \tag{33}$$

This equation leads to nothing new since it is always zero by using the field equations. Note that the first parenthesis is the well-known electromagnetic energy-momentum tensor and it is divergence free by using the first bianchi identity and equation (32). The remaining terms reduce to

$$-3\nabla_\nu w^2 + \frac{81\beta}{16}\left(\Box w^2 \nabla_\nu w^2 + \nabla^\mu w^2 \nabla_\mu \nabla_\nu w^2 - \nabla^\mu w^2 \nabla_\nu \nabla_\mu w^2\right) = 0, \tag{34}$$

where $w^2 = w^\mu w_\mu$. However, using equations (28) and (30) one can easily see that the LHS of the above equation gives zero.

B. The Newtonian limit

Now we examine our theory in the Newtonian limit. Using equations (25) and (30), one can write the Weitzenböck condition equation (20) as

$$R = 6(w_\mu w^\mu - S_\mu S^\mu). \tag{35}$$

Taking the trace of equation (24) and using equations (28) and (30) results in

$$-24 w^2 - \frac{81\beta}{8} \nabla_\mu w^2 \nabla^\mu w^2 = T^m, \tag{36}$$

where $w^2 = w_\mu w^\mu$ and T^m is the trace of the matter energy-momentum tensor. Assuming that matter can be described by a perfect fluid with energy density ρ and thermodynamic pressure p, after substituting equation (35) into equation (36) one obtains

$$R = \frac{1}{4}(\rho - 3p) - 6S^2 - \frac{81\beta}{32} \nabla_\mu w^2 \nabla^\mu w^2, \tag{37}$$

where $S^2 = S_\mu S^\mu$.

In the Newtonian limit, $R = -2R_{00} = -2\nabla^2\phi$, where ϕ is the Newtonian potential corresponding to the (00) component of the metric as $g_{00} = -(1+2\phi)$. Also, as explicitly shown in the next Section, equation (44), one can expect that the norm of the vector S_μ must be proportional to the norm of the Weyl vector. So we assume that in the Newtonian limit the condition $S^2 = bw^2$, $b =$ constant, also holds. Consequently, if we reintroduce the factor $16\pi G$, we obtain the generalized Poisson equation

$$\nabla^2\phi = -4\pi\frac{1-b}{2}G(\rho-3p) + \frac{81(1-b)}{64}\beta\nabla_\mu w^2\nabla^\mu w^2, \tag{38}$$

which for $b = 3$ takes the form of the generalized Poisson equation for weak gravitational fields of the Weyl-Cartan-Weitzenböck gravity,

$$\nabla^2\phi = 4\pi G(\rho-3p) - \frac{81}{32}\beta\nabla_\mu w^2\nabla^\mu w^2. \tag{39}$$

If $\rho \gg p$ and for $(81/32)\beta\nabla_\mu w^2\nabla^\mu w^2 \ll \rho$, we obtain the standard Poisson equation of Newtonian gravity, $\nabla^2\phi = 4\pi G\rho$.

IV. COSMOLOGICAL EVOLUTION EQUATIONS IN WCW GRAVITY

To consider the cosmological solutions of the model we take the flat FRW metric as

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right), \tag{40}$$

and assume a perfect fluid form for the matter content of the universe. ¿From the symmetry reasons shown by the metric above one may show that only the time component of the Weyl vector is non-zero. We therefore write

$$w^\mu = \left[\psi(t), 0, 0, 0\right], \tag{41}$$

where $\psi(t)$ is an arbitrary function of time. Equation (30) implies that

$$Q_\mu = -\frac{3}{2}\left[\psi(t), 0, 0, 0\right]. \tag{42}$$

The argument above shows that with a given matter energy-momentum tensor, one may obtain all the variables in terms of the arbitrary temporal component of the Weyl vector. As we have seen in section III B, one expects that for a time-like ansatz for the Weyl vector, the vector S^μ must also be time-like. So, let's assume the following ansatz for S^μ

$$S^\mu = \left[\Sigma(t), 0, 0, 0\right], \tag{43}$$

where $\Sigma(t)$ is an arbitrary time dependent function. The constraint equation (20) gives the function $\Sigma(t)$ as

$$\Sigma^2 = \psi^2 + 2H^2 + \dot{H}, \tag{44}$$

where $H = \dot{a}/a$ is the Hubble parameter. Now, equation (24) can be used to write

$$H = \frac{2}{\dot{\Psi}}\left[\frac{1}{k} - \frac{1}{6}\ddot{\Psi}\right], \tag{45}$$

$$\rho = -6\Psi + \frac{k}{2}\dot{\Psi}^2, \tag{46}$$

$$p = 6\Psi + \frac{k}{2}\dot{\Psi}^2, \tag{47}$$

where we have denoted

$$\Psi = \psi^2, \tag{48}$$

and

$$k = \frac{81\beta}{8}, \tag{49}$$

respectively. Equation (45) can be written as a second order linear differential equation

$$\ddot{\Psi} + 3H\dot{\Psi} - \frac{6}{k} = 0, \tag{50}$$

or, equivalently

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\frac{d}{dt}\Psi\right) - \frac{6}{k} = 0, \tag{51}$$

giving the dependence of Ψ on the Hubble function H and of the scale factor a. By taking the time derivative of equation (46) we obtain

$$\frac{d\rho}{dt} = -3kH\dot{\Psi}^2, \tag{52}$$

which proves the statement at the end of section III A on the conservation of the energy momentum tensor in this setup

$$\frac{d\rho}{dt} + 3H\left(\rho + p\right) = 0. \tag{53}$$

To go any further it would be useful to count the number of equations and unknowns that one would have to deal with. A simple inspection shows that we have three equations (45), (46) and (47) and four unknowns, namely H, Ψ, p and ρ. In order to close the system of field equations we have to specify a supplementary relation between ρ and p. Once the equation of state is given, the system of field equations (45)-(47) is closed, and the time evolution of the scale factor of the Universe and of the Weyl vector can be obtained by assuming some appropriate initial conditions.

¿From equations (45) - (47) it follows that for large varying Ψ the pressure acts like matter with a stiff equation of state, and as a cosmological constant for small temporal variations of ψ

$$p \approx \begin{cases} \rho = \frac{k}{2}\dot{\Psi}^2, & \dot{\Psi}^2 \gg 12\Psi/k, \\ -\rho = 6\Psi, & \dot{\Psi}^2 \ll 12\Psi/k. \end{cases} \tag{54}$$

Therefore, this result gives a geometrical interpretation of the dark energy in terms of the Weyl vector. The equation of states are given by

$$w = \frac{p}{\rho} \approx \begin{cases} 1, & \dot{\Psi}^2 \gg 12\Psi/k, \\ -1, & \dot{\Psi}^2 \ll 12\Psi/k. \end{cases} \tag{55}$$

V. COSMOLOGICAL MODELS IN WCW GRAVITY

In this section we present a number of explicit, analytical and numerical cosmological solutions of the field equations (45)-(47) of the WCW gravity model. As an indicator of the accelerating expansion we use the deceleration parameter, defined as

$$q = \frac{d}{dt}\frac{1}{H} - 1. \tag{56}$$

Positive values of q indicate a decelerating behavior, while an accelerating expansion of the Universe requires a negative q. In order to illuminate the energy-momentum part of the geometric content of the Universe, we write it as

$$\rho(t) = \rho_m + \rho_T(t), \tag{57}$$

$$p(t) = p_m + p_T(t), \tag{58}$$

where ρ_m and p_m represent the contribution of the energy density and pressure of ordinary matter (dust and radiation) to the energy-momentum tensor, and $\rho_T(t)$ and $p_T(t)$ are the effective energy-density and pressure corresponding to the Weyl vector and the torsion field.

A. Cosmological models with a linear geometric equation of state

As a first example of a cosmological model in WCW gravity we consider the case in which the effective geometric pressure and density of the Weyl field satisfy a linear, barotropic type equation of state of the form

$$p_T = (\Gamma - 1)\,\rho_T, \tag{59}$$

where Γ is a constant. With this equation of state the gravitational field equations become

$$\frac{1}{a}\frac{da}{dt} = -\frac{4}{\sqrt{\left[2\,(\Gamma-2)\,k\right]\left[\,(\Gamma-1)\,\rho_m - p_m + 6\Gamma\Psi\right]}}\left\{1 + \frac{(\Gamma-1)\,a\,(d\rho_m/da) - a\,(dp_m/da)}{6\,[(\Gamma-1)\,\rho_m - p_m + 6\Gamma\Psi]}\right\}^{-1}, \tag{60}$$

and

$$\frac{d\Psi}{dt} = \sqrt{\frac{2}{k\,(\Gamma-2)}\bigg(\,(\Gamma-1)\,\rho_m - p_m + 6\Gamma\Psi\bigg)}. \tag{61}$$

In obtaining the above solution we have assumed that $\dot{\Psi} \geq 0$. The system of equations (60) and (61) must be supplemented with the scale factor dependence on ρ_m and p_m, which can be obtained from the ordinary matter equation of state $p_m = p_m\,(\rho_m)$ and from the energy conservation equation, and must be integrated with some appropriate initial conditions $a\,(0) = a_0$ and $\psi\,(0) = \psi_0$, respectively. In the following we consider that the matter content of the Universe consists of pressure-less dust, with $p_m = 0$. Hence the energy density of the matter varies as $\rho_m = \rho_0/a^3$, where ρ_0 is the present day matter density. By introducing a dimensionless time τ, and the dimensionless Weyl vector θ, defined as

$$\tau = 6\Gamma\sqrt{\frac{2}{k\,(\Gamma-1)\,(\Gamma-2)\,\rho_0}}t, \tag{62}$$

and

$$\theta = \frac{6\Gamma}{(\Gamma-1)\,\rho_0}\Psi, \tag{63}$$

respectively, the cosmological field equations of the WCW gravity model with a linear geometric equation of state take the form

$$\frac{da}{d\tau} = -\frac{2}{3\Gamma}\frac{\sqrt{1+a^3\theta}}{1+2a^3\theta}a^{5/2}, \tag{64}$$

$$\frac{d\theta}{d\tau} = \sqrt{\frac{1}{a^3} + \theta}. \tag{65}$$

One can solve the above equations analytically for $\theta(\tau)$ as a function of $a(\tau)$,

$$\theta(\tau) = -\frac{\Gamma}{2(\Gamma-1)}a^{-3} + C_1 a^{-3\Gamma}, \tag{66}$$

where C_1 is an integration constant. The solution for the scale factor can then be obtained implicitly as a function of time

$$\tau = \frac{3\Gamma}{\sqrt{2(\Gamma-1)}}\int \frac{a^{-3\Gamma-\frac{5}{2}}\left[a^{3\Gamma} - 2C_1(\Gamma-1)a^3\right]}{\sqrt{2C_1(\Gamma-1)a^{3-3\Gamma} + \Gamma - 2}}\mathrm{d}\; a. \tag{67}$$

The time variations of the scale factor a and of the dimensionless Weyl vector θ are presented, for different values of Γ, in Figs. 1 and 2, respectively.

The time variation of the deceleration parameter for this class of cosmological models is presented in Fig. 3. As one can see from the figure, depending on the value of Γ, a large variety of cosmological behaviors can be obtained. Here, the universe may start from an accelerating state, with negative q, and depending on the value of Γ, can end either

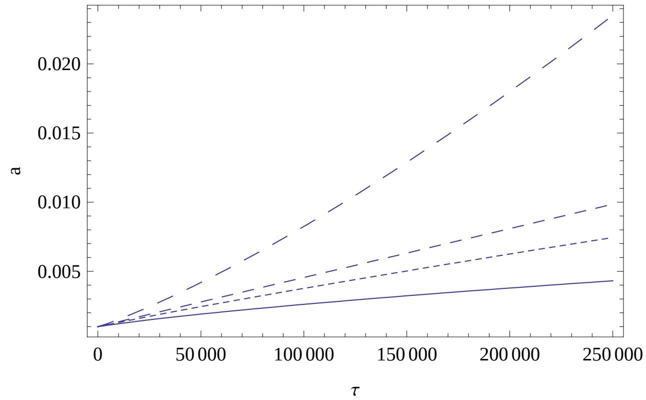

FIG. 1: Time variation of the scale factor of the Universe in the WCW gravity model with linear barotropic equation of state for different values of Γ: $\Gamma=-1$ (solid curve), $\Gamma=-3/4$ (dotted curve), $\Gamma=-2/3$ (dashed curve) and $\Gamma=-1/2$ (long dashed curve), respectively. The initial conditions for the cosmological evolution are $a(0)=0.001$ and $\theta(0)=10^{-6}$.

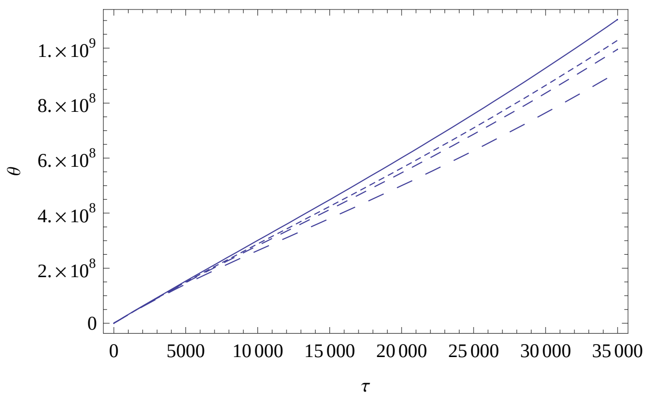

FIG. 2: Time variation of the Weyl vector in the WCW gravity model with linear barotropic equation of state for different values of Γ: $\Gamma=-1$ (solid curve), $\Gamma=-3/4$ (dotted curve), $\Gamma=-2/3$ (dashed curve) and $\Gamma=-1/2$ (long dashed curve), respectively. The initial conditions for the cosmological evolution are $a(0)=0.001$ and $\theta(0)=10^{-6}$.

in a decelerating phase, or with a constant negative value of the deceleration parameter. This class of models may be relevant for the description of the inflationary epoch of the very early evolution of the Universe. For $\Gamma<-1$, the cosmological models are decelerating with $q>0$ for all times. In the absence of matter $\rho_m=p_m=0$ and for $\psi\geq 0$, equations (60) and (61) have the simple solution

$$\psi(t)=\sqrt{\frac{3\Gamma}{(\Gamma-2)\,k}}\;t,\;\; a(t)=a_0t^n, \tag{68}$$

where

$$n=-\frac{2}{3\Gamma}. \tag{69}$$

In order to have an expansionary evolution it is necessary that $\Gamma<0$. The deceleration parameter for this model is

$$q=\frac{1}{n}-1=-\frac{3\Gamma}{2}-1. \tag{70}$$

The Universe will experience an accelerated, power law inflationary expansion if the condition $\Gamma>-2/3$ is satisfied.

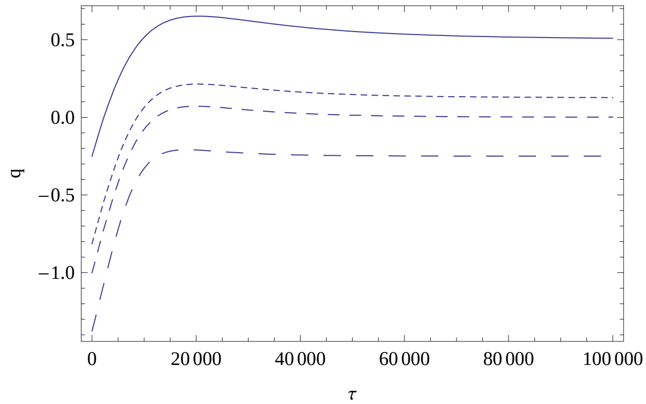

FIG. 3: Time variation of the deceleration parameter in the WCW gravity model with linear barotropic equation of state for different values of Γ: $\Gamma=-1$ (solid curve), $\Gamma=-3/4$ (dotted curve), $\Gamma=-2/3$ (dashed curve) and $\Gamma=-1/2$ (long dashed curve), respectively. The initial conditions for the cosmological evolution are $a(0)=0.001$ and $\theta(0)=10^{-6}$.

B. Exponentially accelerating solutions

In the following we consider first the case of a vacuum Universe with $\rho_m=p_m=0$. From equation (45) it immediately follows that $H=$ constant if $\dot{\Psi}=\psi_0^2=$constant$\neq 0$, giving

$$\psi(t)=\psi_0\sqrt{t}. \tag{71}$$

For the geometric density and pressure we obtain

$$\rho_T=\psi_0^2\left(\frac{k}{2}\psi_0^2-6t\right),\ p_T=\psi_0^2\left(\frac{k}{2}\psi_0^2+6t\right). \tag{72}$$

The expansion of the Universe is accelerating with the scale factor given by $a=a_0\exp\left(H_0t\right)$ where the Hubble constant is $H_0=2/k\psi_0^2$. In order to find the general conditions for a de Sitter type expansion we start from equation (50), written as

$$\frac{d^2}{dt^2}\Psi+3H\frac{d}{dt}\Psi-\frac{6}{k}=0. \tag{73}$$

For an accelerated expansion $H=H_0=$ constant, and therefore the general solution of equation (73) is given by

$$\psi(t)=\sqrt{\frac{2}{H_0k}t-\frac{C_1}{3H_0}e^{-3H_0t}+C_2}, \tag{74}$$

where C_1 and C_2 are arbitrary constants of integration and we assume that $\psi\geq 0$. The energy density and pressure of the Universe are given by

$$\rho_T=-6C_2+\frac{4C_1}{H_0}e^{-3H_0t}+\frac{{C_1}^2k}{2}e^{-6H_0t}+\frac{2\left(1-6H_0t\right)}{{H_0}^2k}, \tag{75}$$

and

$$p_T=6C_2+\frac{C_1^2k}{2}e^{-6H_0t}+\frac{2\left(1+6H_0t\right)}{{H_0}^2k}, \tag{76}$$

respectively. At late times, the energy density and the pressure satisfy the relation $\rho_T+p_T=4/H_0^2k$.

C. Power law expansion

Let us assume that the scale factor behaves as $a(t)=a_0t^{\lambda}$. The temporal component of the Weyl vector can be obtained from equation (73) with the result

$$\psi(t)=\sqrt{\frac{3}{k(3\lambda+1)}t^2+\frac{C_1}{1-3\lambda}t^{1-3\lambda}+C_2}, \tag{77}$$

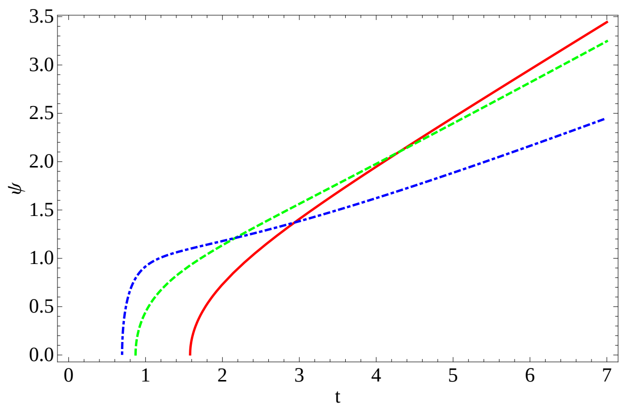

FIG. 4: Time evolution of the temporal component of the Weyl vector for $C_1 = 1$, $C_2 = 1$, $\beta = 1/4$ and $\lambda = 1/2$ (solid curve), $\lambda = 2/3$ (dashed curve), and $\lambda = 8/5$ (dot-dashed curve).

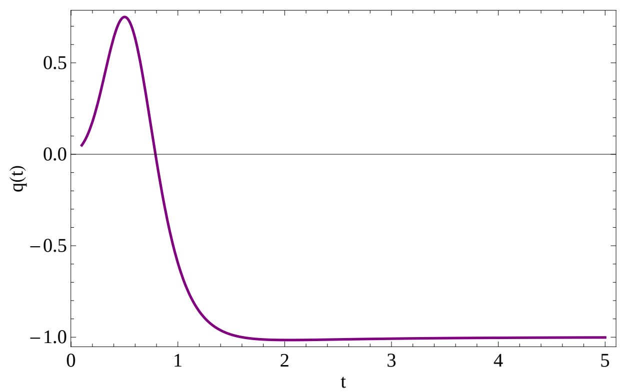

FIG. 5: Deceleration parameter as a function of time for $\psi_1 = 0.1$, $\psi_2 = 10$, $\psi_3 = -1$, $\psi_4 = -1$ and $\beta = 1/4$.

where C_1 and C_2 are integration constants and we assume that $\psi \geq 0$. In this case, the energy-density and pressure are given by

$$\rho_T = -\frac{54\lambda}{k(1+3\lambda)^2}t^2 - \frac{36C_1\lambda}{1-9\lambda^2}t^{1-3\lambda} + \frac{kC_1^2}{2}t^{-6\lambda} - 6C_2, \tag{78}$$

and

$$p_T = -\frac{18(2+3\lambda)}{k(1+3\lambda)^2}t^2 + \frac{12C_1}{1-9\lambda^2}t^{1-3\lambda} + \frac{kC_1^2}{2}t^{-6\lambda} + 6C_2, \tag{79}$$

respectively. Fig. 4 shows the behavior of the Weyl vector for three cases of matter dominated, radiation dominated and the late time acceleration phase of the universe with $\lambda = 2/3, 1/2$ and $8/5$, respectively.

As can be seen form the above solutions one may construct a model starting from a matter dominated universe and ending with a self-accelerating universe by choosing appropriate values for $\psi(t)$. In order to do this one must choose a function which behaves like (77) for small t, and behaves like (74) for large t. Let us assume that

$$\psi(t)^2 = \psi_1 t^{\frac{6}{5}} + \psi_2 t + \frac{\psi_3}{t} + \frac{\psi_4}{t^2}, \tag{80}$$

with ψ_i some constants. In Fig. 5 we have plotted the deceleration parameter as a function of time. As can be seen from the figure, the universe accelerates at late times with the deceleration parameter $q \approx -1$, and decelerates in the early epoch, in agreement with observational data.

VI. DISCUSSIONS AND FINAL REMARKS

In this paper we have investigated a geometrical gravitational theory based on the imposition of the Weitzenböck condition, a relation between the curvature and torsion, in a Weyl-Cartan space-time. As a result, the Einstein-Hilbert action of the gravitational field becomes independent of the curvature, thus leading to the theoretical possibility of the description of the gravitational phenomena in terms of two interacting fields, the Weyl vector and torsion, in a curved background geometry, characterized by a symmetric metric tensor. A dynamical torsion kinetic term has also been added to the gravitational action. By performing the independent variation of the action with respect to the vector fields and metric, the field equations are given by a set of three independent second order partial differential equations, describing the dynamics of the two vector fields in the background geometry, and in the presence of the standard matter fields. The model does not contain an explicit dynamical equation for the metric, which has to be obtained generally by solving the Weitzenböck constraint equation.

The weak field limit of the model has also been investigated, and it was shown that the Poisson equation can be recovered from the Weitzenböck condition and the field equations. The existence of a Newtonian/weak field limit shows that the present model is phenomenologically viable.

The cosmological implications of the model have been analyzed by assuming a flat Friedmann-Robertson-Walker type metric. The resulting field equations can be reduced to three independent equations, giving the Hubble function and the total energy density and pressure as a function of the Weyl vector and its time derivative only.

Present day observations do not rule out the possibility of the presence of some extra gravitational effects, beyond the standard general relativity model, acting at the Solar System, galactic and cosmological levels. The predictions of the WCW gravity model could lead to some new effects, as compared to the predictions of general relativity, or other generalized gravity models, in several problems of current interest such as cosmology, dark matter, gravitational collapse or the generation of gravitational waves. The study of these phenomenon may also provide some specific signatures and effects which could distinguish and discriminate between various gravitational models. In order to explore in more detail the connections between the WCW gravity model and the properties of standard gravity at different length scales, some explicit physical models are necessary to be built. In particular the properties of the static, spherically symmetric gravitational fields in vacuum and inside stars should be considered and predictions of the model should be compared with the existing observational data at the Solar System level. These studies will be pursued in the forthcoming works.

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Appendix A: Relation to Phantom model

An interesting result of the model is the form of the geometric equation of state in the absence of matter, $\rho_m = p_m = 0$, given by equations (46) and (47), respectively, as well as the evolution equation for the Weyl field, given by equation (50). By rescaling the Weyl field as $\Psi \to \phi/\sqrt{k}$, $k > 0$, the density and the pressure of the Weyl field take a form similar to the energy and pressure of the standard scalar field model, with $\rho_T = \dot{\phi}^2/2 - V(\phi)$, and $p_T = \dot{\phi}^2/2 + V(\phi)$, respectively, with $V(\phi) = (6/\sqrt{k})\phi$. However, as compared to the case of the standard scalar field models, the potential has the opposite sign. Moreover, in the Klein - Gordon equation $\ddot{\phi} + 3H\dot{\phi} - dV/d\phi = 0$, describing the evolution of the Weyl field, the term $dV/d\phi$ appears again with the wrong sign as compared to the case of the standard Klein - Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0. \tag{A1}$$

Fields satisfying equation (50) instead of equation (A1) are known as phantom fields [25]. In a cosmological context the hypothetical phantom scalar field would cause super-acceleration of the universe, with the Hubble parameter increasing with time, $\dot{H} > 0$, and leading to a Big Rip singularity at a finite time in the future. Hence in the present model the Weyl field behaves like an equivalent phantom scalar field [25], and super-accelerating cosmological models can indeed be explicitly obtained.

The case of scalar fields with "wrong" sign in the Klein - Gordon equation was discussed recently in [26], in the context of the analysis of the correspondence between scalar fields and effective perfect fluids. In the framework of the scalar field description of fluid systems, the Klein - Gordon equation with the "wrong" sign can be obtained from a Lagrangian of the form $L_{\rho_\phi} = -a^3\rho_\phi = -a^3\left[\dot{\phi}^2/2 + V(\phi)\right]$, while the "correct" sign in the Klein - Gordon equation is obtained for $L_{p_\phi} = a^3 p_\phi = a^3\left[\dot{\phi}^2/2 - V(\phi)\right]$. In the present model a similar effect of the sign change in the Klein-Gordon equation is related to the change in the sign of the potential in the geometric density and pressure associated to the Weyl field.

Appendix B: generalizing the action

One can add another type of the kinetic term for the torsion to the action, namely, $\alpha T_{\mu\nu}T^{\mu\nu}$, $\alpha =$ constant. Another interesting term which can be added to the action is an interaction term between torsion and the Weyl vector, $\gamma T^{\alpha}_{\mu\nu}\nabla_\alpha W^{\mu\nu}$, $\gamma =$ constant, which was originally introduced by Israelit [19, 22]. With this additional terms the equations of motion can be generalized to

$$\frac{1}{2}\nabla^\nu W_{\nu\mu} - 6w_\mu + 4T_\mu + \gamma\nabla^\nu\nabla_\alpha T^{\alpha}_{\mu\nu} = 0, \tag{B1}$$

$$4\left(w^\rho\delta^\sigma_\mu - w^\sigma\delta^\rho_\mu\right) + 2\alpha\left(\delta^\sigma_\mu\nabla_\alpha T^{\rho\alpha} - \delta^\rho_\mu\nabla_\alpha T^{\sigma\alpha}\right) - 2\beta\left(T^\rho\delta^\sigma_\mu - T^\sigma\delta^\rho_\mu\right)\Box T + \gamma\nabla_\mu W^{\rho\sigma} = 0, \tag{B2}$$

and

$$\begin{aligned}
&\frac{1}{4}\left(2W_{\mu\rho}W_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}W_{\rho\sigma}W^{\rho\sigma}\right) - 6\left(w_\mu w_\nu - \frac{1}{2}g_{\mu\nu}w_\rho w^\rho\right) + \alpha(2T_{\mu\rho}T_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}) \\
&- \gamma\left[\frac{1}{2}g_{\mu\nu}\nabla_\gamma W^{\rho\sigma}T^\gamma{}_{\rho\sigma} - \nabla_\rho T^{\rho\sigma}{}_\nu W_{\mu\sigma} - \nabla_\rho T^{\rho\sigma}{}_\mu W_{\nu\sigma} + \frac{1}{2}\nabla_\rho\Big(T_\nu{}^{\rho\sigma}W_{\mu\sigma} - T_{\nu\mu}{}^\sigma W^\rho_\sigma + T_\mu{}^{\rho\sigma}W_{\nu\sigma} - T_{\mu\nu}{}^\sigma W^\rho_\sigma\Big)\right] \\
&+ \beta\Big(\nabla_\mu T\nabla_\nu T - \frac{1}{2}g_{\mu\nu}\nabla_\rho T\nabla^\rho T - 2T_\mu T_\nu\Box T\Big) + 4\left(T_\mu w_\nu + T_\nu w_\mu - g_{\mu\nu}T_\rho w^\rho\right) - \frac{1}{2}T^m_{\mu\nu} = 0,
\end{aligned} \tag{B3}$$

respectively. The effects of such terms as above on the theory would require further investigations which are ongoing.

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