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Polar duals of convex and star bodies

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Abstract

In this article, some new inequalities about polar duals of convex and star bodies are established. The new inequalities in special case yield some of the recent results.

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Keywords: polar dual, L_p-mixed volume, dual L_p-mixed volume, the Bourgain and Milman's inequality

1 Notations and preliminaries

The setting for this article is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denotes the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n.

We use $V(K)$ for the n-dimensional volume of convex body K. $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denotes the support function of $K \in \mathcal{K}^n$; i.e., for $u \in S^{n-1}$

$$h(K, u) = \text{Max}\{u \cdot x : x \in K\}, \tag{1.1}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denotes the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K, u) = \text{Max}\{\lambda \geq 0 : \lambda u \in K\}. \tag{1.2}$$

If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let S^n denotes the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denotes the radial Hausdorff metric, as follows, if $K, L \in S^n$, then $\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty$ (See [1,2]).

1.1 L_p-mixed volume and dual L_p-mixed volume

If $K, L \in \mathcal{K}^n$, the L_p-mixed volume $V_p(K, L)$ was defined by Lutwak (see [3]):

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \mathrm{d}S_p(K, u), \tag{1.3}$$

where $S_p(K, \cdot)$ denotes a positive Borel measure on S^{n-1}.

The L_p analog of the classical Minkowski inequality (see [3]) states that: If K and L are convex bodies, then

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$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{1.4}$$

with equality if and only if K and L are homothetic.

If K, $L \in S^n$, $p \geq 1$, the L_p-dual mixed volume $\tilde{V}_{-p}(K,L)$ was defined by Lutwak (see [4]):

$$\tilde{V}_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n+p} \rho(L,u)^{-p} \mathrm{d}S(u), \tag{1.5}$$

where $dS(u)$ signifies the surface area element on S^{n-1} at u.

The following dual L_p-Minkowski inequality was obtained in [2]: If K and L are star bodies, then

$$\tilde{V}_{-p}(K,L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{1.6}$$

with equality if and only if K and L are dilates.

1.2 Mixed bodies of convex bodies

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, the notation of mixed body $[K_1,\ldots, K_{n-1}]$ states that (see [5]): corresponding to the convex bodies $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ in $\mathbb{R}^n$, there exists a convex body, unique up to translation, which we denote *by*$[K_1,\ldots, K_{n-1}]$.

The following is a list of the properties of mixed body: It is symmetric, linear with respect to Minkowski linear combinations, positively homogeneous, and for $K_i \in \mathcal{K}^n, i = 1, \ldots, n, L_1 \in \mathcal{K}^n$ and $\lambda_i > 0$,

(1) $V_1([K_1, \ldots, K_{n-1}], K_n) = V(K_1, \ldots, K_{n-1}, K_n)$;
(2) $[K_1 + L_1, K_2, \ldots, K_{n-1}] = [K_1, K_2, \ldots, K_{n-1}] + [L_1, K_2, \ldots, K_{n-1}]$;
(3) $[\lambda_1 K_1, \ldots, \lambda_{n-1} K_{n-1}] = \lambda_1 \ldots \lambda_{n-1} \cdot [K_1, \ldots, K_{n-1}]$;
(4) $\underbrace{[K, \ldots, K]}_{n-1} = K$.

The properties of mixed body play an important role in proving our main results.

1.3 Polar of convex body

For $K \in \mathcal{K}^n$, the polar body of K, K^* is defined:

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

It is easy to get that

$$\rho(K,u)^{-1} = h(K^*,u). \tag{1.7}$$

Bourgain and Milman's inequality is stated as follows (see [6]).

If K is a convex symmetric body in $\mathbb{R}^n$, then there exists a universal constant $c>0$ such that

$$V(K)V(K^*) \geq c^n \omega_n^2. \tag{1.8}$$

Different proofs were given by Pisier [7].

2 Main results

In this article, we establish some new inequalities on polar duals of convex and star bodies.

Theorem 2.1 *If K, K_1, ..., K_{n-1} are convex bodies in $\mathbb{R}^n$and let $L = [K_1, ..., K_{n-1}]$, then the L_p-mixed volumes $V_p(K, L)$, $V_p(K^*, L)$, $V_p(B, L)$ satisfy*

$$V_p(K,L)V_p(K^*,L) \geq V_p(B,L)^2. \tag{2.1}$$

Proof From (1.1) and (1.2), it is easy

$$h(K,u) \geq \rho(K,u), \quad K \in \mathcal{K}^n. \tag{2.2}$$

By definition of L_p-mixed volume, we have

$$V_p(K,L) = \frac{1}{n} \int\limits_{S^{n-1}} h(K,u)^p \mathrm{d}S_p(L;u), \tag{2.3}$$

and

$$V_p(K^*,L) = \frac{1}{n} \int\limits_{S^{n-1}} h(K^*,u)^p \mathrm{d}S_p(L,u). \tag{2.4}$$

Multiply both sides of (2.3) and (2.4), in view of (1.7) and (2.2) and using the Cauchy-Schwarz inequality (see [8]), we obtain

$$\begin{aligned}
&n^2 V_p(K,L)V_p(K^*,L)\\
&= \left(\int\limits_{S^{n-1}} h(K,u)^p \mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)\left(\int\limits_{S^{n-1}} \frac{1}{\rho(K,u)^p}\mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)\\
&\geq \left(\int\limits_{S^{n-1}} h(K,u)^{\frac{p}{2}} \cdot \frac{1}{\rho(K,u)^{\frac{p}{2}}}\mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)^2\\
&\geq \left(\int\limits_{S^{n-1}} \mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)^2\\
&= n^2 V_p^2(B,L).
\end{aligned}$$

Taking $p = n - 1$ in (2.1) and in view of the property (1) of mixed body, we obtain the following result: *If* $K, K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then

$$V(K,K_1,\ldots,K_{n-1})V(K^*,K_1,\ldots,K_n) \geq V(B,K_1,\ldots,K_{n-1})^2. \tag{2.5}$$

This is just an inequality given by Ghandehari [9].

Let $L = B$, we have the following interesting result:

Let K be a convex body and K^* its polar dual, then

$$V_p(K,B)V_p(K^*,B) \geq \omega_n^2. \tag{2.6}$$

Taking $p = n$-1 in (2.6), we have the following result which was given in [9]:

$$W_{n-1}(K)W_{n-1}(K^*) \geq \omega_n^2,$$

with equality if and only if K is an n-ball.

Corollary 2.2 *The L_p-mixed volume of K and K^*, $V_p(K, K^*)$ satisfies*

$$V_p(K^*, K)^n \geq \omega_n^{2(n-p)} V(K)^{2p-n}. \tag{2.7}$$

Proof In view of the property (4) of the mixed body, we have

$$V_p(K, [K, \ldots, K]) = V_p(K, K) = V(K).$$

Form (1.4) and taking for $K_1 = K_2 = \ldots = K_{n-1} = K$ in (2.1), we have

$$\begin{aligned} V(K)V_p(K^*, K) &\geq V_p^2(B, K) \\ &\geq V(B)^{\frac{2(n-p)}{n}} V(K)^{\frac{2p}{n}} \\ &= \omega_n^{\frac{2(n-p)}{n}} V(K)^{\frac{2p}{n}}. \end{aligned}$$

Taking $p = n$-1 in (2.7), we have the following result:

$$V(K^*, \underbrace{K, \ldots, K}_{n-1})^n \geq \omega_n^2 V(K)^{n-2}.$$

This is just an inequality given by Ghandehari [9]. The cases $p = 1$ and $n = 2$ give Steinhardt's and Firey's result (see [7]).

A reverse inequality about $\tilde{V}(K^*, \underbrace{K, \ldots, K}_{n-1})$ was given by Ghandehari [9].

$$\tilde{V}(K^*, \underbrace{K, \ldots, K}_{n-1})^n \leq \omega_n^2 V(K)^{n-2}.$$

Theorem 2.3 *Let K be a star body in $\mathbb{R}^n$, K^* be the polar dual of K, then there exist a universal constant c>0 such that*

$$V(K)^{n+2p} \tilde{V}_{-p}(K^*, K)^n \geq (c^n \omega_n^2)^{n+p}, \tag{2.8}$$

where c is the constant of Bourgain and Milman's inequality.

Proof From (1.6) and (1.8), we have

$$\begin{aligned} \tilde{V}_{-p}(K^*, K) &\geq V(K^*)^{\frac{n+p}{n}} V(K)^{-\frac{p}{n}} \\ &= (V(K^*)V(K))^{\frac{n+p}{n}} V(K)^{-\frac{n+2p}{n}} \\ &\geq (c^n \omega_n^2)^{\frac{n+p}{n}} V(K)^{-\frac{n+2p}{n}}. \end{aligned}$$

The following theorem concerning L_p-dual mixed volumes will generalize Santaló inequality.

Theorem 2.4 *Let K_1 and K_2 be two star bodies, $K_1^*$and K_2^*be the polar dual of K_1 and K_2, then there exists a constant c, L_p-dual mixed volumes $\tilde{V}_{-p}(K_1, K_2)$and $\tilde{V}_{-p}(K_1, K_2)\tilde{V}_{-p}(K_1^*, K_2^*) \geq c^n \omega_n^2$ satisfy*

$$\tilde{V}_{-p}(K_1, K_2)\tilde{V}_{-p}(K_1^*, K_2^*) \geq c^n \omega_n^2. \tag{2.9}$$

Proof From (1.6), we have

$$\tilde{V}_{-p}(K_1, K_2) \geq (K_1)^{\frac{n+p}{n}} V(K_2)^{-\frac{p}{n}}. \tag{2.10}$$

For K_1^* and K_2^*, we also have

$$\tilde{V}_{-p}(K_1^*, K_2^*) \geq V(K_1^*)^{\frac{n+p}{n}} V(K_2^*)^{-\frac{p}{n}}. \tag{2.11}$$

Multiply both sides of (2.10) and (2.11) and using Bourgain and Milman's inequality, we obtain

$$\begin{aligned}\tilde{V}_{-p}(K_1, K_2)\tilde{V}_{-p}(K_1^*, K_2^*) &\geq (V(K_1)V(K_1^*))^{-\frac{p}{n}}(V(K_2)V(K_2^*))^{-\frac{p}{n}} \\ &\geq (c^n\omega_n^2)^{\frac{n+p}{n}}(c^n\omega_n^2)^{-\frac{p}{n}} \\ &= c^n\omega_n^2.\end{aligned}$$

Taking for $K_1 = K_2 = K$ in (2.9) and in view of $\tilde{V}_{-p}(K_1, K_2) = \tilde{V}_{-p}(K, K) = V(K)$, (2.9) changes to the Bourgain and Milman's inequality (1.8).

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Authors' contributions
C-JZ, L-YC and W-SC jointly contributed to the main results Theorems 2.1, 2.3, and 2.4. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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