Title	Strategy-proofness and "median voters"
Author(s)	Ching, S
Citation	International Journal Of Game Theory, 1997, v. 26 n. 4, p. 473-490
Issued Date	1997
URL	http://hdl.handle.net/10722/177668
Rights	Creative Commons: Attribution 3.0 Hong Kong License

Strategy-Proofness and "Median Voters"

Stephen Ching*

December 1993 This Version: December 1995

*Department of Economics and Finance, City University of Hong Kong, Kowloon Tong, Hong Kong. I wish to thank William Thomson for his enormous effort in supervision. I am grateful to Jeff Banks, Marcus Berliant, Tarık Kara, Hideo Konishi, Shige Serizawa, John Weymark, two anonymous referees of this journal, and the participants of the Theory Workshop at the University of Rochester, the Econometric Society Meeting, Winter 1994, and the International Game Theory Conference at SUNY, Stony Brook, July 1993 for their useful comments. All remaining errors are my own responsibility.

Abstract

We consider the problem of choosing a level of the public good for an economy in which agents have continuous and single-peaked preferences (Black, 1948). We show that a solution satisfying *strategy-proofness* and *continuity* if and only if it is an augmented median-voter solution. An *augmented median-voter solution* is described in terms of 2^n parameters (which satisfy an anti-monotonicity condition) as follows: $n+1$ of them are selected according to an increasing order of the peaks; the outcome is the median of these $n+1$ parameters and the n peaks. This result establishes a formal connection between strategy-proofness and a generalized notion of median voter. (Similar median formulas were used by Moulin (1980) to describe smaller classes of solutions.) We provide an interpretation of these 2^n parameters in terms of the following properties: *anonymity*, *voter sovereignty*, *unanimity*, and *Pareto efficiency*.

Keywords: strategy-proofness, single-peaked preferences, augmented median-voter solutions

1 Introduction

We consider the problem of choosing a public good level for an economy in which agents have continuous and "single-peaked" preferences (Black, 1948). A preference is *single-peaked* if more is strictly preferred to less up to a point, and less to more beyond that point. The preferred point is commonly referred as the *peak of the preference*. A *solution* is a systematic way to associate a public good level with each economy. We are interested in solutions that provide incentive to the agents to reveal their true preferences. The strongest such incentive compatibility requirement is *strategy-proofness*: no agent can ever benefit from misrepresenting his preference, regardless of whether the other agents misrepresent or not.

Moulin (1980) first characterized the class of strategy-proof solutions satisfying *peak only*; the requirement that solutions depend only on the peaks of the preferences. He described each of them by means of a "minmax formula" with 2^n parameters. The parameters are known as "phantom voters". Let us call these solutions *minmax solutions*. It is well-known that two different sets of 2^n parameters can be used to describe the same minmax solution.

Instead of peak only, we consider the property of "continuity" (with respect to preferences). We show that a solution satisfying strategy-proofness and continuity can be described in terms of 2^n parameters (which satisfy an anti-monotonicity condition) as follows: $n+1$ of them are selected according to an increasing order of the peaks; the outcome is the median of these $n+1$ parameters and the n peaks. We name these solutions *augmented median-voter solutions*. This result establishes a formal connection between strategy-proofness and a generalized notion of median voter. In contrast to the minmax solutions, an augmented median-voter solution is uniquely identified by its 2^n parameters. Similar median formulas were used by Moulin (1980) to describe classes of strategy-proof solutions satisfying additional properties.

We show that several conditions are equivalent to strategy-proofness and continuity together. A solution satisfies strategy-proofness and any one of the following four conditions: (i) continuity; (ii) peak only; (iii) "peak monotonicity"; or (iv) the range is a closed interval; if and only if it is "uncompromising" (Border and Jordan, 1983). Hence, it follows from

Moulin's (1980) characterization and our characterization that the class of minmax solutions and the class of augmented median-voter solutions are the same. Then we show how a minmax solution is related to the same augmented median-voter solution in terms of their parameters.

We also consider the following properties: *voter sovereignty*: no alternative is *a priori* excluded; *unanimity*: if the preferred alternatives of all agents are the same, then this alternative is chosen; *Pareto efficiency*: an alternative is chosen only if there is no other alternative that is preferred by all agents and strictly preferred by at least one agent; and *anonymity*: the names of the agents do not matter. Given an augmented median-voter solution, its parameters actually coincide with the outcomes of the solution for the economies in which the peaks of the agents are equal to either the lowest or highest feasible level. Consequently, the range of the solution is determined by the smallest and largest parameters. More interestingly, whether the solution satisfies voter sovereignty, unanimity, and Pareto efficiency depends only on the same 2 parameters; and whether it satisfies anonymity depends only on the other $2^n - 2$ parameters. The parameters therefore correspond naturally to these properties and we prefer to interpret the parameters in this way.

In addition, this interpretation gives a simple proof of the characterization of *generalized median-voter solutions:* the outcome is the median of these $n-1$ parameters and the peaks of the n agents. The properties involved are anonymity, voter sovereignty, and strategy-proofness. This characterization strengthens Moulin's (1980) result by weakening Pareto efficiency to voter sovereignty, and by dropping peak only. The stronger result is contained in Barberà, Gül, and Stacchetti (1993) and Barberà and Jackson (1994), which can be traced back to Border and Jordan.

The first systematic study of strategy-proofness was conducted by Gibbard (1973) and Satterthwaite (1975) on the abstract Arrovian domain. Their celebrated result is that a solution with at least three outcomes is strategy-proof if and only if it is "dictatorial" (a solution is *dictatorial* if there is an agent whose preferred point is always chosen). Recently, there has been considerable attention to the study of strategy-proofness in economic and political environments.[1] We will discuss in detail the connection

[1]Strategy-proofness has also been studied in matching models. See Roth (1982) and Alcalde and Barberà (1994) for the one-to-one case; Sönmez (1994) for the many-to-one

with five closely related papers: Moulin (1980); Border and Jordan; Barberà, Gül, and Stacchetti; Barberà and Jackson; and Danilov (1994). The reader is referred to Muller and Satterthwaite (1985) for a survey of the earlier literature; Sprumont (1995) and Thomson (1993a) for surveys of the recent literature.

The paper is organized as follows: Section 2 describes the model and the main results. Section 3 contains the proofs and other related results. Section 4 concludes.

2 The Model and the Main Results

We consider the problem of choosing a level of the public good in an interval $[0, M]$ for a group of agents $N = \{1, \ldots, n\}$. Each agent $i \in N$ is equipped with a continuous preference relation R_i over $[0, M]$. Let P_i be the strict relation associated with R_i, and I_i the indifference relation. The preference relation R_i is *single-peaked* if there exists a number $p(R_i) \in [0, M]$ such that for all $x, y \in [0, M]$ with $[y < x < p(R_i)]$ or $[p(R_i) < x < y]$, we have xP_iy. Let $\mathcal{R}_s$ be the class of all single-peaked preference relations. The preference relation $R_i \in \mathcal{R}_s$ can be described by the function $r_i : [0, M] \rightarrow [0, M]$ such that for all $x \in [0, p(R_i)]$, $r_i(x) = y$ if there exists $y \in [p(R_i), M]$ such that yI_ix, or $r_i(x) = M$ otherwise; for all $x \in [p(R_i), M]$, $r_i(x) = y$ if there exists $y \in [0, p(R_i)]$ such that yI_ix, or $r_i(x) = 0$ otherwise. A preference profile is a list $R = (R_1, \ldots, R_n) \in \mathcal{R}_s^n$. Since the interval is fixed, an *economy* is simply denoted by a preference profile. A *solution* is a function $\varphi : \mathcal{R}_s^n \rightarrow [0, M]$ which associates a public good level with each economy.

Our main property is the incentive compatible requirement that no agent can ever benefit from misrepresenting his preference in the direct revelation game associated with the solution.

Strategy-proofness: For all $R \in \mathcal{R}_s^n$, all $i \in N$, and all $R_i' \in \mathcal{R}_s$, $\varphi(R) R_i \varphi(R_i', R_{-i})$.

Moulin (1980) first characterized the class of strategy-proof solutions satisfying *peak only*: for all $R, R' \in \mathcal{R}_s^n$, if $p(R_i) = p(R_i')$ for all $i \in N$, then

---

case.

$\varphi(R) = \varphi(R')$. He described such a solution in terms of a list of parameters $b = (b_S)_{S \subseteq N} \in [0, M]^{2^n}$ as follows:

$$\forall R \in \mathcal{R}_s^n,\ X^b(R) = \min_{S \subseteq N} \{\max\{\{p(R_i)\}_{i \in S}, b_S\}\}.$$

Let $\mathcal{X} = \{X^b | b \in [0, M]^{2^n}\}$ be the class of *minmax solutions*. The parameters are known as "phantom voters".

Example 1: Let $N = \{1, 2\}$, $R \in R_s^2$ be such that $p(R_1) = 2$ and $p(R_2) = 5$, and $b_\emptyset = 5$, $b_{\{1\}} = 7$, $b_{\{2\}} = 3$, $b_N = 1$. Then $X^b(R) = \min\{5, \max\{2, 7\}, \max\{5, 3\}, \max\{2, 5, 1\}\} = 5$. If we change $b_{\{1\}}$ to $b'_{\{1\}} = 5$, $X^{b'}(R) = 5 = X^b(R)$. Indeed, X^b and $X^{b'}$ represent the same solution.

Peak only is a strong property. It requires that solutions depend only on the peaks of the preferences.[2] Instead, we consider the requirement that if the preferences of an agent change "a little," then the chosen level does not change much. Formally, given two preference relations $R_i, R'_i \in \mathcal{R}_s$, let $d(R_i, R'_i) = \max\{|r_i(x) - r'_i(x)||x \in [0, M]\}$ be the distance between R_i and R'_i. A sequence of preference relations $\{R_i^v\}$ in $\mathcal{R}_s$ converges to R_i, written as $R_i^v \to R_i$, if as $v \to \infty$, $d(R_i, R_i^v) \to 0$.

Continuity: For all $R \in \mathcal{R}_s^n$, all $i \in N$, and all $\{R_i^v\}$ in $\mathcal{R}_s$, if $R_i^v \to R_i$, then $\varphi(R_i^v, R_{-i}) \to \varphi(R)$.[3]

We show that a solution satisfying strategy-proofness and continuity can be described in term of a list of parameters $a = (a_S)_{S \subseteq N} \in [0, M]^{2^n}$ as follows:

$$\forall S, Q \subseteq N \text{ s.t. } S \supseteq Q,\ a_S \leq a_Q \text{ and} \tag{1}$$

$$\forall R \in \mathcal{R}_s^n,\ V^a(R) = \text{med}\{p(R_{i_1}), p(R_{i_2}), \ldots, p(R_{i_n}), \underbrace{a_\emptyset, a_{\{i_1\}}, a_{\{i_1, i_2\}}, \ldots, a_N}_{n+1}\}, \tag{2}$$

where $p(R_{i_1}) \leq p(R_{i_2}) \leq \ldots \leq p(R_{i_n})$.

[2] However, Barberà and Jackson pointed out that peak only is not so demanding in the presence of strategy-proofness. Intuitively, a strategy-proof solution is unlikely to depend on detailed information of the preferences. See Remark 3 for a formal statement of their result.

[3] This intuitive definition of continuity was formulated by Sprumont (1991). It coincides with the continuity condition in the Hausdorff sense.

We name this solution an *augmented median-voter solution.* Let $\mathcal{V} = \{V^a | a \in [0, M]^{2^n}$ satisfying (1)$\}$ be the class of augmented median-voter solutions. (Similar median formulas were used by Moulin (1980) to describe classes of strategy-proof solutions satisfying additional properties.)

Theorem: *The augmented median-voter solutions are the only solutions satisfying strategy-proofness and continuity.*

Example 2: Let N, R be the same as in Example 1, and $a = b'$ (in Example 1). Then $V^a(R) = \text{med}\{2, 5, 5, 5, 1\} = 5$. The reader can verify that $V^a = X^{b'}$. (This equivalence is not a coincidence and it follows from Proposition 1.)

In (2), the $n + 1$ parameters are selected from 2^n parameters according to an increasing order of the peaks. If there are at least two agents whose peaks are the same, then the parameters can be selected in more than one way. We want to emphasize that the outcomes of V^a are invariant to all these selections. (See Remark 1 for a proof.)

It is easy to see that an augmented median-voter solution V^a is uniquely identified by the list of parameters a. Let $V^a, V^{a'} \in \mathcal{V}$ be such that $a_Q \neq a'_Q$ for some $Q \subseteq N$. For notational simplicity, let $Q = \{1, \ldots, q\}$. Let $R \in \mathcal{R}_s^n$ be such that $p(R_1) = \ldots = p(R_q) = 0$ and $p(R_{q+1}) = \ldots = p(R_n) = M$. Then $V^a(R) = a_Q \neq a'_Q = V^{a'}(R)$.

Fact: The parameters of an augmented median-voter solution V^a are actually equal to its outcomes for the economies in which the peaks of the agents are equal to either the lowest or highest feasible level. For all $i \in N$, let $\underline{R}_i \in \mathcal{R}_s$ be such that $p(\underline{R}_i) = 0$ and $\overline{R}_i \in \mathcal{R}_s$ be such that $p(\overline{R}_i) = M$. It follows that $V^a(\underline{R}_S, \overline{R}_{-S}) = a_S$ for all $S \subseteq N$, where $(\underline{R}_N, \overline{R}_{-N}) = \underline{R}$ and $(\underline{R}_\emptyset, \overline{R}_{-\emptyset}) = \overline{R}$. This fact is crucial to understand the subsequent interpretation of the parameters a.

Note that each augmented median-voter solution is peak only. In fact, we show that several conditions are equivalent to strategy-proofness and continuity together. A solution satisfies strategy-proofness and any one of the following four conditions: (i) continuity; (ii) peak only; (iii) "peak monotonicity"; or (iv) the range is a closed interval; if and only if it is

"uncompromising".[4] Hence, the class of augmented median-voter solutions and the class of minmax solutions are the same. The following result relates a minmax solution and the same augmented median-voter solution in terms of their parameters.

Proposition 1: *Let $X^b \in \mathcal{X}$ be a minmax solution and $V^a \in \mathcal{V}$ be an augmented median-voter solution. The solutions X^b and V^a are the same if and only if $a_S = \min_{Q \subseteq S}\{b_Q\}$ for all $S \subseteq N$.*

Indeed, Moulin (1980) pointed out that there is no loss of generality to describe the class of minmax solutions with 2^n parameters satisfying (1) (see Remark 2 also). We can deduce from Proposition 1 that two minmax solutions $X^b, X^{b'}$ are the same if and only if the parameters b, b' are such that $\min_{Q \subseteq S}\{b_Q\} = \min_{Q \subseteq S}\{b'_Q\}$ for all $S \subseteq N$. We will provide an interpretation of parameters satisfying (1) in terms of the properties discussed below (see Remark 4).

The class of augmented median-voter solutions is very rich. For instance, it includes two classes of degenerate strategy-proof solutions. One class is the class of *constant solutions* denoted by $\mathcal{C} = \{C^\alpha | \alpha \in [0, M]\}$: for all $C^\alpha \in \mathcal{C}$ and all $R \in \mathcal{R}^n_s$, $C^\alpha(R) = \alpha$. An augmented median-voter solution is a constant solution if $a_S = \alpha$ for all $S \subseteq N$. The other one is the class of *dictatorial solutions* denoted by $\mathcal{D} = \{D^i | i \in N\}$: for all $D^i \in \mathcal{D}$ and all $R \in \mathcal{R}^n_s$, $D^i(R) = p(R_i)$. An augmented median voter solution is a dictatorial solution if $a_S = 0$ for all $S \subseteq N$ such that $S \ni i$, and $a_S = M$ for all $S \subseteq N$ such that $S \not\ni i$, where agent i is the dictator.

A constant solution says that all levels, but one, are excluded. A natural way to rule out the constant solutions is to require no level be *a priori* excluded.

Voter sovereignty: For all $x \in [0, M]$, there exists $R \in \mathcal{R}^n_s$ such that $\varphi(R) = x$.

For strategy-proof solutions, we show that voter sovereignty makes con-

[4] Let us mention some closely related results in the literature. Border and Jordan showed that an uncompromising solution is strategy-proof, continuous, and peak only. Barberà, Gül, and Stacchetti showed that a strategy-proof solution with a closed range is peak only. The reader is recommended to see these two papers for details and also Barberà and Jackson, Kim and Roush (1984), and Peters, van der Stel, and Storcken (1991).

tinuity redundant. It is very similar to the result due to Border and Jordan; Barberà, Gül, and Stacchetti; and Barberà and Jackson of which voter sovereignty and strategy-proofness together imply peak only.

Proposition 2: *If a strategy-proof solution satisfies voter sovereignty, then it is continuous.*

Then we characterize the class of solutions satisfying voter sovereignty and strategy-proofness as a subclass of the augmented median-voter solutions. Each of them satisfies the additional requirements that $a_N = 0$ and $a_\emptyset = M$. Therefore, it can be described in terms of $2^n - 2$ parameters.

Another way to rule out constant solutions is to impose some optimality requirement on a solution φ, such as *unanimity*: for all $R \in \mathcal{R}_s^n$, if $p(R_i) = p(R_j)$ for all $i, j \in N$, then $\varphi(R) = p(R_i)$; or *Pareto efficiency*: for all $R \in \mathcal{R}_s^n$, there is no $x \in [0, M]$ such that $xR_i\varphi(R)$ for all $i \in N$, and $xP_i\varphi(R)$ for some $i \in N$. In this model, Pareto efficiency can be conveniently written as follows: for all $R \in \mathcal{R}_s^n$, $\varphi(R) \in [\min_{i\in N}\{p(R_i)\}, \max_{i\in N}\{p(R_i)\}]$. Note that voter sovereignty is weaker than unanimity, which in turn is weaker than Pareto efficiency.

It is interesting that even though the subclass of augmented median-voter solutions satisfying voter sovereignty is characterized without any optimality requirement, it nonetheless satisfies Pareto efficiency (and *a fortiori* unanimity). Let V^a be an augmented median-voter solution in this subclass. Since $a_N = 0$ and $a_\emptyset = M$, then the outcomes of V^a are always the median of n peaks and $n - 1$ parameters, so $V^a(R) \in [\min_{i\in N}\{p(R_i)\}, \max_{i\in N}\{p(R_i)\}]$ for all $R \in \mathcal{R}_s^n$. Therefore, for strategy-proof solutions, voter sovereignty, unanimity, and Pareto efficiency are all equivalent (Border and Jordan).

A dictatorial solution says that there is an agent whose preferred point is always chosen. To rule out dictatorial solutions, we impose the nondiscriminatory requirement that solutions do not depend on the names of the agents. (Other similar requirements are "symmetry" and "no-envy" (Foley, 1967). Note that both of them are vacuous in models with only public commodities.)

A permutation of order n is a bijection $\pi : N \to N$. Let Π be the collection of all such permutations. Given $\pi \in \Pi$, let $R_\pi = (R_{\pi(1)}, \ldots, R_{\pi(n)})$.

Anonymity: For all $R \in \mathcal{R}_s^n$ and all $\pi \in \Pi$, $\varphi(R) = \varphi(R_\pi)$.

The class of solutions satisfying anonymity, strategy-proofness, and continuity can be characterized as a subclass of the class the augmented median-voter solutions. Each of them satisfies the additional requirements that $a_S = a_{S'}$ for all $S, S' \subset N$ such that $|S| = |S'| > 0$. Therefore, it can be described in terms of $n+1$ parameters. (The same class of solutions is characterized by Moulin (1980) with continuity replaced by peak only.)

Altogether, we conclude that any solution satisfying anonymity, voter sovereignty, and strategy-proofness can be described in terms of a list of parameters $a = (a_i)_{i=1}^{n-1} \in [0, M]^{n-1}$ as follows:

$$\forall R \in \mathcal{R}_s^n,\ W^a(R) = \text{med}\{p(R_1), \ldots, p(R_n), a_1, \ldots, a_{n-1}\}.$$

Moulin (1980) named this solution a *generalized Condorcet-winner solution.*

Corollary: *The generalized Condorcet-winner solutions are the only solutions satisfying anonymity, voter sovereignty, and strategy-proofness.*

Moulin (1980) first characterized the generalized Condorcet-winner solutions by means of anonymity, Pareto efficiency, strategy-proofness, and peak only.[5] The corollary strengthens this characterization. The stronger result is contained in Barberà, Gül, and Stacchetti and Barberà and Jackson, which can be traced to Border and Jordan.

3 The Proofs and Other Results

We first show that several conditions are equivalent to strategy-proofness and continuity together. The following lemma says that a solution satisfies strategy-proofness and continuity if and only if its outcome is the median of the peak of an agent and two outcomes of the solution obtained by choosing

[5]Moulin (1984) also characterized the generalized Condorcet-winner solutions mainly by two standard independence axioms: "Arrow independence of irrelevant alternatives" and "Nash independence of irrelevant alternatives". Thomson (1993b) characterized a subclass of the generalized Condorcet-winner solutions using "replacement-domination" as the main axiom. Thomson and Ching (1992) consider the problem in a variable population environment and characterize the same solutions on the basis of "population-monotonicity".

the peak of that agent equal to either 0 or M, the preferences of the other agents being held fixed.

Lemma 1: *A solution φ satisfies strategy-proofness and continuity if and only if for all $R \in \mathcal{R}_s^n$ and all $i \in N$,*

$$\varphi(R) = med\{p(R_i), \varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i})\}. \tag{3}$$

Proof: Let φ be a solution satisfying strategy-proofness and continuity. Let $R \in \mathcal{R}_s^n$ and $i \in N$. We first show that $\varphi(\underline{R}_i, R_{-i}) \leq \varphi(\overline{R}_i, R_{-i})$. Suppose, by contradiction, that $\varphi(\overline{R}_i, R_{-i}) < \varphi(\underline{R}_i, R_{-i})$. Then $\varphi(\overline{R}_i, R_{-i}) < \varphi(\underline{R}_i, R_{-i}) \leq p(\overline{R}_i)$, so $\varphi(\underline{R}_i, R_{-i})\overline{P}_i\varphi(\overline{R}_i, R_{-i})$, contradicting strategy-proofness. We proceed by distinguishing two cases.

Case 1: $p(R_i) \in (\varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i}))$.

Then $\text{med}\{p(R_i), \varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i})\} = p(R_i)$. Suppose, by contradiction, that $\varphi(R) \neq p(R_i)$ and, without loss of generality, that $\varphi(R) < p(R_i)$. Let $R_i' \in \mathcal{R}_s$ be such that $p(R_i') = p(R_i)$ and $\varphi(\overline{R}_i, R_{-i})P_i'\varphi(R)$. Let $\{R_i^v\}$ in $\mathcal{R}_s$ be a sequence of preference relations such that for all $v \in [0,1] \cap \mathbb{Q}$ and all $x_i \in [0, p(R_i)]$, $r_i^v(x_i) = (1-v)r_i(x_i) + vr_i'(x_i)$. Note that $R_i^0 = R_i$ and $R_i^1 = R_i'$. Since $\varphi(R) < p(R_i)$, then continuity, strategy-proofness, and $\{R_i^v\}$ in $\mathcal{R}_s$ together imply that for all $v \in [0,1] \cap \mathbb{Q}$, $\varphi(R_i^v, R_{-i}) = \varphi(R)$, so $\varphi(\overline{R}_i, R_{-i})P_i'\varphi(R_i', R_{-i})$, contradicting strategy-proofness.

Case 2: $p(R_i) \notin (\varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i}))$.

Suppose, without loss of generality, that $p(R_i) \leq \varphi(\underline{R}_i, R_{-i})$. Then $\text{med}\{p(R_i), \varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i})\} = \varphi(\underline{R}_i, R_{-i})$. Suppose, by contradiction, that $\varphi(R) \neq \varphi(\underline{R}_i, R_{-i})$ and, without loss of generality, that $\varphi(R) < \varphi(\underline{R}_i, R_{-i})$. Then $p(\underline{R}_i) \leq \varphi(R) < \varphi(\underline{R}_i, R_{-i})$, so $\varphi_i(R)\underline{P}_i\varphi_i(\underline{R}_i, R_{-i})$, contradicting strategy-proofness.

Conversely, let φ be a solution satisfying (3). Obviously, φ is continuous. It remains to show that φ is strategy-proof. Let $R \in \mathcal{R}_s^n$ and $i \in N$. The case that $p(R_i) \in (\varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i}))$ is trivial. Suppose then that $p(R_i) \notin (\varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i}))$ and, without loss in generality, that $p(R_i) \leq \varphi(\underline{R}_i, R_{-i})$. By (3), $\varphi(R) = \varphi(\underline{R}_i, R_{-i})$ and for all

$R_i' \in \mathcal{R}_s$, $\varphi(\underline{R}_i, R_{-i}) \leq \varphi(R_i', R_{-i})$. Altogether, $p(R_i) \leq \varphi(R) \leq \varphi(R_i', R_{-i})$, so $\varphi(R) R_i \varphi(R_i', R_{-i})$. □

This proof can easily be modified to replace continuity by peak only. First, a solution satisfying (3) is peak only. Then, since continuity is only used in Case 1, we only need to modify the proof as follows: after R_i' (in Case 1) is selected, peak only implies that $\varphi_i(R_i', R_{-i}) = \varphi_i(R)$, so $\varphi_i(\overline{R}_i, R_{-i}) P_i' \varphi_i(R_i', R_{-i})$, contradicting strategy-proofness. Therefore, continuity in Lemma 1 can be replaced by peak only.

Moreover, a solution satisfying (3) is *peak monotonic*: for all $R \in \mathcal{R}_s^n$, all $i \in N$, and all $R_i' \in \mathcal{R}_s$, if $p(R_i) \leq p(R_i')$, then $\varphi(R) \leq \varphi(R_i', R_{-i})$. Since peak monotonicity implies peak only, then continuity in Lemma 1 can also be replaced by peak monotonicity. Furthermore, a similar proof can be used to show that a solution satisfies (3) if and only if it is *uncompromising*: for all $R \in \mathcal{R}_s^n$, all $i \in N$, and all $R_i' \in \mathcal{R}_s$, if $[\varphi(R) < p(R_i)$ and $\varphi(R) \leq p(R_i')]$ or $[\varphi(R) > p(R_i)$ and $\varphi(R) \geq p(R_i')]$, then $\varphi(R) = \varphi(R_i', R_{-i})$. Hence, we have established the following result:

Lemma 2: *A solution satisfies strategy-proofness and any one of the following conditions: (i) continuity; (ii) peak only; or (iii) peak monotonicity; if and only if it is uncompromising.*

Remark 1: Let $V^a \in \mathcal{V}$ be an augmented median-voter solution. Recall that the $n+1$ parameters in (2) are selected according to an increasing order of the peaks. Here, we show that the outcomes are invariant to all these selections. For notational simplicity, let $R \in \mathcal{R}_s^n$ be such that $p(R_1) \leq \ldots \leq p(R_i) = p(R_{i+1}) \leq \ldots \leq p(R_n)$. We distinguish three cases:

Case 1: $\text{med}\{\ldots, p(R_i), p(R_{i+1}), \ldots, a_{\{1,\ldots,i\}}, a_{\{1,\ldots,i+1\}}, \ldots\} = p(R_i)$.

It is without loss of generality to assume that

$$\underbrace{p(R_1), \ldots, p(R_{i-1})}_{i-1} \leq p(R_i) = \underbrace{p(R_{i+1}), \ldots, p(R_n)}_{n-i} \text{ and}$$

$$\underbrace{a_N, \ldots, a_S, \ldots, a_{\{1,\ldots,i\}}}_{n+1-i} \leq p(R_i) \leq \underbrace{a_{\{1,\ldots,i-1\}}, \ldots, a_\emptyset}_{i},$$

where $S \supseteq \{1, \ldots, i+1\}$.

Let $p(R_1) \leq \ldots \leq p(R_{i+1}) = p(R_i) \leq \ldots \leq p(R_n)$ be the new ordering. There are two subcases: (i) If $a_{\{1,\ldots,i-1,i+1\}} \leq p(R_{i+1})$, then

$$\begin{array}{lll} \underbrace{p(R_1),\ldots,p(R_{i-1})}_{i-1} & \leq p(R_{i+1}) & = \underbrace{p(R_i),p(R_{i+2}),\ldots,p(R_n)}_{n-i} \text{ and} \\ \underbrace{a_N,\ldots,a_S,\ldots,a_{\{1,\ldots,i-1,i+1\}}}_{n+1-i} & \leq p(R_{i+1}) & \leq \underbrace{a_{\{1,\ldots,i-1\}},\ldots,a_\emptyset}_{i}. \end{array}$$

(ii) If $p(R_{i+1}) < a_{\{1,\ldots,i-1,i+1\}}$, then

$$\begin{array}{lll} \underbrace{p(R_1),\ldots,p(R_i)}_{i} & = p(R_{i+1}) & \leq \underbrace{p(R_{i+2}),\ldots,p(R_n)}_{n-1-i} \text{ and} \\ \underbrace{a_N,\ldots,a_S,\ldots,a_{\{1,\ldots,i+1\}}}_{n-i} & \leq p(R_{i+1}) & < \underbrace{a_{\{1,\ldots,i-1,i+1\}},a_{\{1,\ldots,i-1\}},\ldots,a_\emptyset}_{i+1}. \end{array}$$

In both subcases, $\text{med}\{\ldots,p(R_{i+1}),p(R_i),\ldots,a_{\{1,\ldots,i-1,i+1\}},a_{\{1,\ldots,i+1\}},\ldots\} = p(R_{i+1}) = p(R_i)$.

Case 2: $\text{med}\{\ldots,p(R_i),p(R_{i+1}),\ldots,a_{\{1,\ldots,i\}},a_{\{1,\ldots,i+1\}},\ldots\} = p(R_j) \neq p(R_i)$.

Suppose, without loss of generality, that $p(R_i) < p(R_j)$. We can choose such an agent j satisfying the following inequalities:

$$\begin{array}{lll} \underbrace{p(R_1),\ldots,p(R_i),p(R_{i+1}),\ldots p(R_{j-1})}_{j-1} & \leq p(R_j) & \leq \underbrace{p(R_{j+1}),\ldots,p(R_n)}_{n-j} \text{ and} \\ \underbrace{a_N,\ldots,a_{\{1,\ldots,j\}}}_{n+1-j} & \leq p(R_j) & \leq \underbrace{a_{\{1,\ldots,j-1\}},\ldots,a_S,\ldots,a_\emptyset}_{j}, \end{array}$$

where $S \subseteq \{1,\ldots,j-1\}$.

Let $p(R_1) \leq \ldots \leq p(R_{i+1}) = p(R_i) \leq \ldots \leq p(R_n)$ be the new ordering. Then

$$\begin{array}{lll} \underbrace{p(R_1),\ldots,p(R_{i+1}),p(R_i),\ldots,p(R_{j-1})}_{j-1} & \leq p(R_j) & \leq \underbrace{p(R_{j+1}),\ldots,p(R_n)}_{n-j} \text{ and} \\ \underbrace{a_N,\ldots,a_{\{1,\ldots,j\}}}_{n+1-j} & \leq p(R_j) & \leq \underbrace{a_{\{1,\ldots,j-1\}},\ldots,a_{S'},\ldots,a_\emptyset}_{j}, \end{array}$$

where $a_{S'}$ is selected according to the new ordering, $S' \subseteq \{1, \ldots, j-1\}$.

So, $\text{med}\{\ldots, p(R_{i+1}), p(R_i), \ldots, a_{\{1,\ldots,i-1,i+1\}}, a_{\{1,\ldots,i+1\}}, \ldots\} = p(R_j)$.

Case 3: $\text{med}\{\ldots, p(R_i), p(R_{i+1}), \ldots, a_{\{1,\ldots,i\}}, a_{\{1,\ldots,i+1\}}, \ldots\} = a_{\{1,\ldots,j\}} \neq p(R_i)$.

Suppose, without loss of generality, that $p(R_i) < a_{\{1,\ldots,j\}}$. We can choose such an agent j satisfying the following inequalities:

$$\underbrace{p(R_1), \ldots, p(R_i), p(R_{i+1}), \ldots p(R_j)}_{j} \leq a_{\{1,\ldots,j\}} \leq \underbrace{p(R_{j+1}), \ldots, p(R_n)}_{n-j} \text{ and}$$
$$\underbrace{a_N, \ldots, a_{\{1,\ldots,j+1\}}}_{n-j} \leq a_{\{1,\ldots,j\}} \leq \underbrace{a_{\{1,\ldots,j-1\}}, \ldots, a_S, \ldots, a_\emptyset}_{j},$$

where $S \subseteq \{1, \ldots, j-1\}$.

Let $p(R_1) \leq \ldots \leq p(R_{i+1}) = p(R_i) \leq \ldots \leq p(R_n)$ be the new ordering. Then

$$\underbrace{p(R_1), \ldots, p(R_{i+1}), p(R_i), \ldots p(R_j)}_{j} \leq a_{\{1,\ldots,j\}} \leq \underbrace{p(R_{j+1}), \ldots, p(R_n)}_{n-j} \text{ and}$$
$$\underbrace{a_N, \ldots, a_{\{1,\ldots,j+1\}}}_{n-j} \leq a_{\{1,\ldots,j\}} \leq \underbrace{a_{\{1,\ldots,j-1\}}, \ldots, a_{S'}, \ldots, a_\emptyset}_{j},$$

where $a_{S'}$ is selected according to the new ordering, $S' \subseteq \{1, \ldots, j-1\}$.

So, $\text{med}\{\ldots, p(R_{i+1}), p(R_i), \ldots, a_{\{1,\ldots,i-1,i+1\}}, a_{\{1,\ldots,i+1\}}, \ldots\} = a_{\{1,\ldots,j\}}$.

In all three cases, the outcome is the same for the two orderings. □

Moulin (1980) first characterized the class of solutions satisfying strategy-proofness and peak only as the class of minmax solutions. We show how to relate a minmax solution and an augmented median-voter solution in terms of their parameters.

Proof of Proposition 1: Let $V^a \in \mathcal{V}$ and $X^b \in \mathcal{X}$ be such that $a_S = \min_{Q \subseteq S}\{b_Q\}$ for all $S \subseteq N$. We first show that $X^b = X^a$. Let $R \in \mathcal{R}^n_s$. Let $S \subseteq N$ and $Q^* \subseteq S$ be such that $b_{Q^*} = a_S$, then

$$\begin{aligned}\max\{\{p(R_i)\}_{i \in Q^*}, b_{Q^*}\} &= \min\{\max\{\{p(R_i)\}_{i \in Q^*}, b_{Q^*}\}, \max\{\{p(R_i)\}_{i \in S}, b_S\}\} \\ &= \min\{\max\{\{p(R_i)\}_{i \in Q^*}, b_{Q^*}\}, \max\{\{p(R_i)\}_{i \in S}, a_S\}\}.\end{aligned}$$

Repeating the argument, we obtain

$$\begin{aligned} X^b(R) &= \min_{S\subseteq N}\{\max\{\{p(R_i)\}_{i\in S}, b_S\}\} \\ &= \min_{S\subseteq N}\{\max\{\{p(R_i)\}_{i\in S}, a_S\}\} \quad = X^a(R). \end{aligned}$$

We next show that $X^a = V^a$. For notational simplicity, let $p(R_1) \leq \ldots \leq p(R_n)$. There are three mutually exclusive cases:

Case 1: $a_\emptyset \leq p(R_1)$ or $p(R_n) \leq a_N$.

Suppose, without loss of generality, that $a_\emptyset \leq p(R_1)$. Then

$$\max\{\{p(R_i)\}_{i\in S}, a_S\} \begin{cases} = a_\emptyset & \text{if } S = \emptyset \\ \geq p(R_1) & \text{otherwise.} \end{cases}$$

Therefore, $X^a(R) = a_\emptyset = V^a(R)$.

Case 2: $a_{\{1,\ldots,i\}} < p(R_i) < a_{\{1,\ldots,i-1\}}$ for some $i = 1, \ldots, n$. Note that $\{1, \ldots, 0\} = \emptyset$. Then

$$\max\{\{p(R_j)\}_{j\in S}, a_S\} \begin{cases} \geq a_{\{1,\ldots,i-1\}} & \text{if } S\subseteq\{1,\ldots,i-1\} \\ = p(R_i) & \text{if } S = \{1,\ldots,i\} \\ \geq p(R_i) & \text{otherwise.} \end{cases}$$

Therefore, $X^a(R) = p(R_i) = V^a(R)$.

Case 3: $p(R_i) \leq a_{\{1,\ldots,i\}} \leq p(R_{i+1})$ for some $i = 1, \ldots, n-1$.

Then

$$\max\{\{p(R_j)\}_{j\in S}, a_S\} \begin{cases} \geq a_{\{1,\ldots,i\}} & \text{if } S\subseteq\{1,\ldots,i-1\}; \\ = a_{\{1,\ldots,i\}} & \text{if } S = \{1,\ldots,i\}; \\ \geq p(R_{i+1}) & \text{otherwise.} \end{cases}$$

Therefore, $X^a(R) = a_{\{1,\ldots,i\}} = V^a(R)$.

Similarly, we can show that if $V^a = X^b$, then $a_S = \min_{Q\subseteq S}\{b_Q\}$ for all $S\subseteq N$. □

Now, we are ready to show that the augmented median-voter solutions are the only solutions satisfying strategy-proofness and continuity.

Proof of the Theorem: It follows from Moulin's (1980) characterization of the minmax solutions, Lemma 2, and Proposition 1. □

Remark 2: Border and Jordan characterized the class of uncompromising solutions in terms of a list of parameters $a' = (a'_S)_{S \subseteq N} \in [0, M]^{2^n}$ satisfying a monotonicity condition as follows:

$$\forall S, Q \subseteq N \text{ s.t. } S \supseteq Q, a'_S \geq a'_Q. \tag{1'}$$

They used the following maxmin formula for the characterization:[6]

$$\forall R \in \mathcal{R}_s^n, I^{a'}(R) = \max_{S \subseteq N} \{\min\{\{p(R_i)\}_{i \in S}, a'_S\}\}.$$

We can slightly modify the augmented median-voter solutions to describe the solution $I^{a'}$ as follows:

$$\forall R \in \mathcal{R}_s^n, U^{a'}(R) = \text{med}\{p(R_{i_1}), p(R_{i_2}), \ldots, p(R_{i_n}), \underbrace{a'_N, a'_{\{i_2,\ldots,i_n\}}, a'_{\{i_3,\ldots,i_n\}}, \ldots, a'_\emptyset}_{n+1}\}, \tag{2'}$$

where $p(R_{i_1}) \leq p(R_{i_2}) \leq \ldots \leq p(R_{i_n})$. Note that in (2'), the $n+1$ parameters are selected in a way different from (2). The two solutions $V^a, U^{a'}$ are the same if and only if $a_S = a'_{-S}$ for all $S \subseteq N$.

Example 3: Let N, R be the same as in Example 1, and $a'_S = a_{-S}$ (in Example 1) for all $S \subseteq N$. Then $I^{a'}(R) = \max\{1, \min\{2,3\}, \min\{5,5\}, \min\{2,5,5\}\} = 5$ and $U^{a'}(R) = \text{med}\{2,5,5,5,1\} = 5$. It can be verified that $I^{a'} = U^{a'} = V^a = X^b$.

Remark 3: Barberà and Jackson dropped peak only in Moulin's (1980) characterization of the class of solutions satisfying strategy-proofness and peak only. They showed that the range of a strategy-proof solution is closed. Note that the restriction of a single-peaked preference relation $R_i \in \mathcal{R}_s$ to a closed set has at most two "peaks." If there are two such peaks, they identified a class of tie-breaking rules to select one of them such that each of these rules t_i preserves strategy-proofness. Then they showed that a

[6]Border and Jordan also showed that a solution is uncompromising if and only if one of the following two conditions holds: (i) there are at most 2^n parameters; or (ii) the solution has a closed graph. However, these two are not closed form characterization of the uncompromising solutions.

strategy-proof solution can be described by a slightly extended minmax solution as follows:[7]

$$\forall R \in \mathcal{R}_s^n, \bar{X}^b(R) = \min_{S \subseteq N} \{\max\{\{t_i(R)\}_{i \in S}, b_S\}\}.$$

For notational simplicity, let $t_1(R) \leq \ldots \leq t_n(R)$. From Proposition 1, the solution $\bar{X}^b$ can be described in term of the list of parameters a such that $a_S = \min_{Q \subseteq S}\{b_Q\}$ for all $S \subseteq N$ as follows:

$$\bar{X}^b(R) = \bar{V}^a(R) = \text{med}\{t_1(R), \ldots, t_n(R), a_\emptyset, a_{\{1\}}, \ldots, a_N\}.$$

Therefore, we can also use a slight extension of the augmented median-voter solutions to characterize all strategy-proof solutions.

Next, we show that voter sovereignty and strategy-proofness together imply continuity. The following definition is useful. A solution φ is *own peak only* if for all $R \in \mathcal{R}_s^n$, all $i \in N$, and all $R_i' \in \mathcal{R}_s$ such that $p(R_i) = p(R_i')$, we have $\varphi(R) = \varphi(R_i', R_{-i})$.

Proof of Proposition 2: Let φ be a solution satisfying voter sovereignty and strategy-proofness. Because of Lemma 2, it is enough to show that φ is peak only.

Claim 1: The solution φ satisfies unanimity.

Let $R \in \mathcal{R}_s^n$ be such that $p(R_i) = p(R_j)$ for all $i, j \in N$. By voter sovereignty, there exists $R' \in \mathcal{R}_s^n$ such that $\varphi(R') = p(R_1)$. By strategy-proofness, $\varphi(R') = \varphi(R_1, R'_{-1})$. Repeating the argument shows that $\varphi(R_1, R'_{-1}) = \varphi(R_1, R_2, R'_{-12}) = \ldots = \varphi(R)$. Claim 1 is established.

Claim 2: The solution φ is own peak only.

Let $R \in \mathcal{R}_s^n$, $i \in N$, and $R_i' \in \mathcal{R}$ be such that $p(R_i) = p(R_i')$. The case that $\varphi(R) = p(R_i')$ is trivial. Suppose then, without loss of generality, that $\varphi(R) < p(R_i')$. There are two cases:

[7] Zhou (1991) considered a model with many public commodities and preferences are continuous and strictly convex. He showed a variant of the Gibbard-Satterthwaite theorem that a solution with a range of at least dimension two is strategy-proof if and only if it is dictatorial. Barberà and Jackson then used this result to complete this characterization of the class of strategy-proof solutions.

Case 1: $r_i'(\varphi(R)) < r_i(\varphi(R))$.

Suppose, by contradiction, that $\varphi(R) \neq \varphi(R_i', R_{-i})$. There are three cases: (i) If $\varphi(R_i', R_{-i}) < \varphi(R)$, then $\varphi(R_i', R_{-i}) < \varphi(R) < p(R_i')$, so $\varphi(R) P_i' \varphi(R_i', R_{-i})$. (ii) If $\varphi(R) < \varphi(R_i', R_{-i}) \leq r_i'(\varphi(R))$, then $\varphi(R) < \varphi(R_i', R_{-i}) < r_i(\varphi(R))$, so $\varphi(R_i', R_{-i}) P_i \varphi(R)$. (iii) If $r_i'(\varphi(R)) < \varphi(R_i', R_{-i})$, then $p(R_i') < r_i'(\varphi(R)) < \varphi(R_i', R_{-i})$, so $\varphi(R) P_i' \varphi(R_i', R_{-i})$. All three cases contradict strategy-proofness.

Case 2: $r_i(\varphi(R)) \leq r_i'(\varphi(R))$.

Let $S = \{j \in N | p(R_j) > p(R_i')\}$. For notational simplicity, let $S = \{1, \ldots, s\}$ be such that $p(R_1) \leq \ldots \leq p(R_s)$. For all $j \in S$, let $R_j' \in \mathcal{R}_s$ be such that $p(R_j') = p(R_i')$ and $r_j'(\varphi(R)) \in (p(R_j), r_j(\varphi(R)))$. Repeating the argument in Case 1 shows that $\varphi(R) = \varphi(R_S', R_{-S})$. Let $R' = (R'_{S \cup \{i\}}, R_{-S \cup \{i\}})$. An argument similar to Case 1 shows that there are two subcases: (i) $\varphi(R_S', R_{-S}) = \varphi(R')$ and (ii) $\varphi(R') \in [r_i(\varphi(R)), r_i'(\varphi(R))]$.

Claim 2.1: If (i) holds, then $\varphi(R') = \varphi(R_i', R_{-i})$.

We first show that $\varphi(R') = \varphi(R_1, R'_{-1})$. Suppose, by contradiction, that $\varphi(R') \neq \varphi(R_1, R'_{-1})$. An argument similar to Case 1 shows that $\varphi(R_1, R'_{-1}) \in [r_1'(\varphi(R')), r_1(\varphi(R'))]$. Note that $p(R_1) < \varphi(R_1, R'_{-1})$. For all $j \in S \backslash \{1\}$, let $R_j'' \in \mathcal{R}_s$ be such that $p(R_j'') = p(R_1)$ and $r_j'(\varphi(R_1, R'_{-1})) < r_j''(\varphi(R_1, R'_{-1}))$, and for all $j \notin S$, let $R_j'' \in \mathcal{R}_s$ be such that $p(R_j'') = p(R_1)$ and $r_j(\varphi(R_1, R'_{-1})) < r_j''(\varphi(R_1, R'_{-1}))$. An argument similar to Case 1 shows that $\varphi(R_1, R'_{-1}) = \varphi(R_1, R''_{-1})$, so $p(R_1) = p(R_2'') = \ldots = p(R_n'') < \varphi(R_1, R''_{-1})$, contradicting Claim 1 (unanimity). Repeating the argument shows that $\varphi(R_1, R'_{-1}) = \varphi(R_1, R_2, R'_{-12}) = \ldots = \varphi(R_S, R'_{-S}) = \varphi(R_i', R_{-i})$. Claim 2.1 is established

If (ii) holds, an argument similar to the proof of Claim 2.1 shows that there is a contradiction to Claim 1. Claim 2 is established.

Finally, a solution is own peak only if and only if it is peak only. □

Sprumont (1995) adapted the proof of Proposition 2 to show the following stronger result due to Barberà, Gül, and Stacchetti: if the range of a strategy-proof solution is a closed interval, then it is peak only (or

continuous). This result can be strengthened by showing that the converse is also true. By the Theorem, a solution satisfying strategy-proofness and continuity can be represented as an augmented median-voter solution $V^a \in \mathcal{V}$. Let $R \in \mathcal{R}_s^n$, by repeated applications of peak monotonicity, $V^a(R) \in [V^a(\underline{R}), V^a(\overline{R})] = [a_N, a_\emptyset]$. Let $x \in [a_N, a_\emptyset]$ and $R \in \mathcal{R}_s^n$ be such that $p(R_i) = x$ for all $i \in N$, then $V^a(R) = x$. Therefore, the range of V^a is the closed interval $[a_N, a_\emptyset]$. Altogether, we have the following stronger version of Proposition 2:

Proposition 2′: *A strategy-proof solution φ satisfies continuity if and only if its range is the closed interval $[\varphi(\underline{R}), \varphi(\overline{R})]$.*

Barberà and Jackson showed a very close result of which a strategy-proof solution satisfies peak only if and only if the restriction of any single-peaked preference to its range is single-peaked. Proposition 2′ is more precise about the range of these solutions.[8]

It is now easy to show that the generalized Condorcet-winner solutions are the only solutions satisfying anonymity, voter sovereignty, and strategy-proofness.

Proof of the Corollary: Let φ be a solution satisfying anonymity, voter sovereignty, and strategy-proofness. By Proposition 2 and Theorem 1, φ is an augmented median-voter solution. Let $a_S = \varphi(\underline{R}_S, \overline{R}_{-S})$ for all $S \subseteq N$. First, since voter sovereignty and strategy-proofness together imply unanimity, then $a_N = 0$ and $a_\emptyset = M$. Second, by anonymity, $a_S = a_{S'}$ for all $S, S' \subset N$ such that $|S| = |S'| > 0$. Let $a_{|S|} = a_S$ for all $\emptyset \subset S \subset N$, then

$$\varphi(R) = \text{med}\{p(R_1), \ldots, p(R_n), a_1, \ldots, a_{n-1}\}.$$

Conversely, it is well-known that a generalized Condorcet-winner solution satisfies the three properties. □

Remark 4: Since the range of an augmented median-voter solution V^a is $[a_N, a_\emptyset]$, it satisfies voter sovereignty if and only if $a_N = 0$ and $a_\emptyset = M$. If $a_N = 0$ and $a_\emptyset = M$, then $V^a(R) \in [\min_{i \in N}\{p(R_i)\}, \max_{i \in N}\{p(R_i)\}]$ for all $R \in \mathcal{R}_s^n$. Since Pareto efficiency implies voter sovereignty, we also have that V^a satisfies Pareto efficiency (and *a fortiori* unanimity) if and only if

[8] Proposition 2′ also shows that the converse of Proposition 2 is not necessarily true.

$a_N = 0$ and $a_\emptyset = M$. It is clear that V^a satisfies anonymity if and only if $a_S = a_{S'}$ for all $S, S' \subset N$ such that $|S| = |S'| > 0$. The parameters therefore correspond naturally to these properties.

Remark 5: Barberà, Gül, and Stacchetti considered a generalized l-dimensional model, $l \geq 1$. They showed that a solution satisfies strategy-proofness and peak only if and only if it can be decomposed into l one-dimensional solution(s) that each satisfies the same two properties. Then they concentrated on the one-dimensional case. They characterized the class of solutions satisfying strategy-proofness and peak only as the class of "committee solutions."[9] Let us describe a *left committee solution* in terms of a list of parameters $a \in [0, M]^{2^n}$ satisfying (1) as follows:

$$\forall S \subseteq N, \text{ the coalition } S \text{ is } \textit{left winning at } x \text{ if } x \in [a_S, M] \text{ and} \tag{4}$$

$$\forall R \in \mathcal{R}_s^n, \; F^a(R) = \min\{x \in [0, M] | \{i \in N | p(R_i) \leq x\} \text{ is winning at } x\}. \tag{5}$$

It can be shown that the committee solution F^a is equivalent to the augmented median-voter solution V^a.[10] (See also Barberà, Massó, and Neme, 1993; and Serizawa, 1993.)

Example 4: Let N, R, and a be the same as in Example 2. When $x < 5$, either $\{2\}$ or N is a winning coalition, but only voter 1's peak is less than x. When $x \geq 5$, all coalitions are winning and $p(R_1) < p(R_2) \leq x$. Therefore, $F^a(R) = 5$. Again, it can be verified that $F^a = V^a$.

Remark 6: In an independent paper, Danilov considered a model in which the set of alternatives has a "tree structure." He (and Barberà, Gül, and Stacchetti indirectly) provided another characterization of the class of solutions satisfying strategy-proofness and peak only. The essence of this

[9] Barberà, Gül, and Stacchetti called these solutions generalized median voter schemes. To avoid confusion with the augmented median-voter solutions, we follow Sprumont (1995) to call them committee solutions.

[10] Similarly, we can describe a *right committee solution* in terms of a list of parameters $a' \in [0, M]^{2^n}$ satisfying (1') as follows:

$$\forall S \subseteq N, \text{ the coalition } S \text{ is } \textit{right winning at } x \text{ if } x \in [0, a'_S] \text{ and}$$

$$\forall R \in \mathcal{R}_s^n, \; G^{a'}(R) = \max\{x \in [0, M] | \{i \in N | p(R_i) \leq x\} \text{ is winning at } x\}.$$

Of course, $G^{a'} = U^{a'}$ (in Remark 2), thus $G^{a'} = U^{a'} = V^a = F^a$ if and only if $a'_S = a_{-S}$ for all $S \subseteq N$.

result is best understood in the one-dimensional case. The class of solutions can be obtained by recursive substitutions of the median formula (3):

$$\begin{aligned}\forall R \in \mathcal{R}_s^n,\ \varphi(R) &= \text{med}\{p(R_i), \varphi(\underline{R}_i, R_{-i}), \varphi(\overline{R}_i, R_{-i})\} \\ &= \text{med}\{p(R_i), \text{med}\{p(R_j), \varphi(\underline{R}_i, \underline{R}_j, R_{-ij}), \varphi(\underline{R}_i, \overline{R}_j, R_{-ij})\}, \\ &\qquad \text{med}\{p(R_j), \varphi(\overline{R}_i, \underline{R}_j, R_{-ij}), \varphi(\overline{R}_i, \overline{R}_j, R_{-ij})\}\} \\ &= \ldots\end{aligned}$$

The substitutions stop until all profiles are such that the peaks of the preferences are either 0 or M. In other words, a recursive median solution can be described in terms of the outcomes of these extreme economies, or equivalently, the parameters of an augmented median-voter solution.

Example 5: Let N, R be the same as in Example 1, and $\varphi(\underline{R}_S, \overline{R}_{-S}) = a_S$ (in Example 2) for all $S \subseteq N$. Then $\varphi(R) = \text{med}\{p(R_1), \varphi(\underline{R}_1, R_2), \varphi(\overline{R}_1, R_2)\} = \text{med}\{2, \text{med}\{p(R_2), \varphi(\underline{R}_1, \underline{R}_2), \varphi(\underline{R}_1, \overline{R}_2\}, \text{med}\{p(R_2), \varphi(\overline{R}_1, \underline{R}_2), \varphi(\overline{R}_1, \overline{R}_2\}\} = \text{med}\{2, \text{med}\{5, 1, 5\}, \text{med}\{5, 3, 5\}\} = 5$. Indeed, $\varphi = V^a$.

Danilov interpreted a recursive median solution as a solution which is obtained by taking medians of dictatorial solutions and constant solutions. Similarly, the median formula (3) can be interpreted as a solution φ satisfying strategy-proofness and continuity is the median of a dictatorial solution and the two solutions $\varphi(\underline{R}_i, \cdot)$ and $\varphi(\overline{R}_i, \cdot)$ which have to satisfy the same two properties. In fact, He showed a more general result of which the median of any three solutions satisfying strategy-proofness and peak only is a solution satisfying the same two properties.

Remark 7: Bossert and Weymark (1993ab) considered the problem of constructing social preference for a society in which agents have monotonic and "linear" preferences. They were interested in social welfare functions satisfying anonymity, unanimity, and *binary independence* (which requires that if the ranking of any two alternatives remains unchanged after a change in the preferences, their social ranking remain unchanged). They showed that their model is formally equivalent to Moulin's (1980) model and obtained results that are counterparts to Moulin's (1980). Consequently, our proofs can be adapted to their model and their results can be obtained as by-products of our analysis. Our approach provides an additional characterization of the class of social welfare functions satisfying binary independence.

4 Conclusions

We show that the augmented median-voter solutions are the only solutions satisfying strategy-proofness and continuity. This result establishes a formal connection between strategy-proofness and a generalized notion of median voter. We also show that several conditions are equivalent to strategy-proofness and continuity together. A solution φ satisfies strategy-proofness and any one of the following four conditions: (i) continuity; (ii) peak only; (iii) peak monotonicity; or (iv) the range is the closed interval $[\varphi(\underline{R}), \varphi(\overline{R})]$; if and only if it is uncompromising. Hence, the class of augmented median-voter solutions is the same as the class of minmax solutions.

We show how to relate a minmax solution X^b and the same augmented median-voter solution V^a in terms of their parameters. It turns out that there is no loss of generality to focus on the parameters a. Interestingly, whether the solution V^a satisfies voter sovereignty, unanimity, and Pareto efficiency depends only on the parameters a_N and $a_\emptyset$; and whether it satisfies anonymity depends only on the other $2^n - 2$ parameters. We prefer to interpret the parameters in terms of these properties. In addition, this interpretation of the parameters gives a simple proof of a stronger characterization of the generalized Condorcet-winner solutions.

There are three alternative characterizations of the class of solutions satisfying strategy-proofness and peak only (or continuity). Border and Jordan used a maxmin formula to describe it; Barberà, Gül, and Stacchetti characterized it as the class of committee solutions; and Danilov described it in terms of a recursive median formula. It is interesting that these three classes of solutions can also be described in terms of the parameters of the augmented median-voter solutions.

References

[1] Alcalde, José and Salvador Barberà (1994), "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory 4:* 417–435.

[2] Barberà, Salvador, Faruk Gül, and Ennio Stacchetti (1993), "Generalized Median Voter Schemes and Committees," *Journal of Economic Theory 61:* 262–289.

[3] —— and Matthew Jackson (1994), "A Characterization of Strategy-Proof Social Choice Functions for Economies with Pure Public Goods," *Social Choice and Welfare 11:* 241–252.

[4] ——, Jordi Massó, and Alejandro Neme (1993), "Voting under Constraints," Universitat Autònoma de Barcelona mimeo.

[5] Black, Duncan (1948), "On the Rationale of Group Decision-Making," *Journal of Political Economy 56:* 23–34.

[6] Border, Kim and J. S. Jordan (1983), "Straightforward Elections, Unanimity and Phantom Voters." *Review of Economic Studies 50:* 153–170.

[7] Bossert, Walter and John Weymark (1993a), "Generalized Median Social Welfare Functions," *Social Choice and Welfare 10:* 17–33.

[8] —— and —— (1993b), "An Alternative Characterization of Paretian Generalized Median Social Welfare Functions," in W. Erwin Diewert, Klaus Spremann, and Frank Stehling (eds), *Mathematical Modelling in Economics,* Springer-Verlag.

[9] Danilov, Vladimir (1994), "The Structure of Non-manipulable Social Choice Rules on a Tree," *Mathematical Social Sciences 27:* 123–131.

[10] Foley, Duncan (1967), "Resource Allocation and the Public Sector," *Yale Economic Essays 7:* 45–98.

[11] Gibbard, Alan (1973), "Manipulation of Voting Scheme: A General Result," *Econometrica 41:* 587–602.

[12] Kim, K. H. and F. W. Roush (1984), "Nonmanipulability in Two Dimensions," *Mathematical Social Sciences 8:* 29–43.

[13] Moulin, Hervé (1980), "On Strategy-Proofness and Single Peakedness," *Public Choice 35:* 437–455.

[14] —— (1984), "Generalized Condorcet-Winners for Single Peaked and Single-Plateau Preferences." *Social Choice and Welfare 1:* 127–147.

[15] Muller, Eitan and Mark Satterthwaite (1985), "Strategy-Proofness: The Existence of Dominant-Strategy Mechanisms," in Leonid Hurwicz, David Schmeidler, and Hugo Sonnenschein (eds), *Social Goals and Social Organization*. Cambridge: Cambridge University Press.

[16] Peters, Hans, Hans van der Stel, and Ton Storcken (1991), "On Uncompromisingness and Strategy-Proofness," mimeo.

[17] Roth, Alvin E. (1982), "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research 7:* 617–628.

[18] Satterthwaite, Mark (1975), "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory 10:* 187–217.

[19] Serizawa, Shigehiro (1993), *Nonmanipulable Schemes in Public Goods Economies*, University of Rochester dissertation.

[20] Sönmez, Tayfun (1994), "Strategy-Proofness in Many-to-One Matching Problems," University of Rochester mimeo.

[21] Sprumont, Yves (1991), "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Rule," *Econometrica 59:* 509–519.

[22] —— (1995), "Strategyproof Collective Choice in Economic and Political Environments," *Canadian Journal of Economics 28:* 68-107.

[23] Thomson, William (1993a), "Strategy-Proof Allocation Rules," University of Rochester mimeo.

[24] —— (1993b), "The Replacement Principle in Public Good Economies with Single-Peaked Preferences," *Economics Letters 42:* 31–36.

[25] —— and Stephen Ching (1992), "Population-Monotonic Solutions in Public Good Economies with Single-Peaked Preferences," University of Rochester mimeo, *Social Choice and Welfare*, forthcoming.

[26] Zhou, Lin (1991), "Impossibility of Strategy-Proof Mechanisms in Economies with Pure Public Good," *Review of Economics Studies, 58:* 107–119.