Comment on "Connection between entanglement and the speed of quantum evolution" and on "Entanglement and the lower bounds on the speed of quantum evolution"

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Batle *et al.* [Phys. Rev. A **72**, 032337 (2005)] and Borrás *et al.* [Phys. Rev. A **74**, 022326 (2006)] studied the connection between entanglement and speed of quantum evolution for certain low-dimensional bipartite quantum states. However, their studies did not cover all possible cases. And the relation between entanglement and the maximum possible quantum evolution speed for these uncovered cases can be very different from the ones that they have studied.

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Batle *et al.* [1] studied those pure two-qubit states that can evolve to their orthogonal subspaces under the action of the local time-independent Hamiltonian $H_A \otimes H_B$ in which the spectra of the two-level Hamiltonians H_A and H_B equal $\{0,\epsilon\}$ for some $\epsilon > 0$. They denoted the energy eigenstates of H_A and H_B by $|0\rangle$ and $|1\rangle$ so that $H_i|0\rangle = 0$ and $H_i|1\rangle = \epsilon|1\rangle$ for $i = A,B$. By writing a normalized pure two-qubit state $|\psi\rangle$ of two distinguishable particles in the form $c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$, Batle *et al.* deduced that the time t at which $|\psi\rangle$ evolves to its orthogonal subspace is given by the quadratic equation [1],

$$|c_0|^2 + (|c_1|^2 + |c_2|^2)z + |c_3|^2 z^2 = 0, \tag{1}$$

where $z = \exp(i\epsilon t/\hbar)$. In Ref. [1], Batle *et al.* were interested in those states that must evolve to their orthogonal subspaces at some time.

Note that there are two cases to consider, namely, the generic case in which the leading coefficient of the previously mentioned quadratic equation, $|c_3|^2$, is nonzero and the singular case in which $|c_3|^2 = 0$.

Batle *et al.* only considered the generic case in Ref. [1]. In this case, the condition that $|\psi\rangle$ must evolve to its orthogonal subspace at some time implies $|c_0|^2 = |c_3|^2 > 1/4$. By solving Eq. (1), Batle *et al.* obtained an expression for the time τ when $|\psi\rangle$ first evolved to its orthogonal subspace. Combined with the literature results that τ is lower bounded by

$$T_{\min} \equiv \min\left(\frac{\pi\hbar}{2E},\frac{\pi\hbar}{2\Delta E}\right), \tag{2}$$

where E and ΔE are the average energy and the standard deviation of the energy of the state, respectively, Batle *et al.* showed that [1]

$$\tau \geqslant T_{\min} = T_{\min 1} \equiv \frac{\pi\hbar}{2\sqrt{2}\epsilon|c_0|} = \frac{\pi\hbar}{2\sqrt{2}\epsilon|c_3|}. \tag{3}$$

Since the concurrence C of the state is given by the equation,

$$C^2 = 4\left||c_0|^2 - e^{i\phi}\sqrt{\delta(1-\delta)}2|c_0|^2\cos\alpha\right|^2, \tag{4}$$

for some parameters $\phi \in \mathbb{R}$ and $\delta \in [0,1]$. By means of the observation that $|c_0|^2 \leqslant (1+|C|)/4$ for a fixed concurrence C, they deduced from Eqs. (3) and (4) that

$$\frac{\tau}{T_{\min}} \geqslant \frac{\sqrt{2(1+|C|)}}{\pi}\cos^{-1}\left(\frac{|C|-1}{|C|+1}\right) \geqslant 1. \tag{5}$$

Most importantly, they concluded from Eq. (5) that $\tau = T_{\min}$ if and only if the state was maximally entangled [1].

Batle *et al.*, however, did not consider the singular case in which $|c_3|^2 = 0$ in Ref. [1]. For the singular case, Eq. (1) becomes a linear equation. [Should $|c_3|^2 = |c_1|^2 + |c_2|^2 = 0$ so that of Eq. (1) becomes a constant, the state $|\psi\rangle$ can never evolve to its orthogonal subspace. So, the case in which Eq. (1) is a linear equation is the only unanalyzed situation.] The condition that the state $|\psi\rangle$ with $|c_3|^2 = 0$ can evolve to its orthogonal subspace is $|c_0|^2 = |c_1|^2 + |c_2|^2 = 1/2$. And by solving Eq. (1), we arrive at $\tau = \pi\hbar/\epsilon = \pi\hbar/2E = \pi\hbar/2\Delta E = T_{\min}$. Most importantly, τ cannot be expressed as a function of the concurrence $C = 2|c_1||c_2|$ of the state and $T_{\min 1}$ is not well defined as $|c_0|^2 \neq |c_3|^2$. In fact, for each C, there is a state that attains the least evolution time $T_{\min}$. An example is the family of states $(|00\rangle + \sqrt{x}|01\rangle + \sqrt{1-x}|10\rangle)/\sqrt{2}$ for $x \in [0,1]$ whose concurrence $C = 2\sqrt{x(1-x)}$. This family of states violates the first inequality in Eq. (5) provided that $0 < x < 1$. It also shows that partially entangled states can attain the evolution time lower-bound $T_{\min}$. Thus, the relation between entanglement and the time needed to evolve to the orthogonal subspace for the singular case can be very different from the generic case.

Since Batle *et al.* did not discuss similar singular cases for bosonic two-qubit pure states and fermionic two-qutrit pure states of indistinguishable particles in Ref. [1], their analysis in these two situations is also incomplete. In fact, the case of $x = 1/2$ in the previously mentioned family of states is an example of a nonmaximally entangled bosonic two-qubit pure state with the least possible evolution time. Their followup paper [2] dealing with the extensions to the cases of mixed states and evolution into nonorthogonal states is also incomplete as it suffers the same problem of effectively restricting the analysis only to generic situations because they sample the initial states according to the Haar measure in their Monte Carlo simulations.

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[1] J. Batle, M. Casas, A. Plastino, and A. R. Plastino, Phys. Rev. A **72**, 032337 (2005).

[2] A. Borrás, M. Casas, A. R. Plastino, and A. Plastino, Phys. Rev. A **74**, 022326 (2006).