



<b>Title</b>	<b>Valuing contingent exotic options: a discounted density approach</b>
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<b>Citation</b>	<b>The 2012 International Conference on Actuarial Science and Risk Management (ASRM 2012), Xiamen, China, 24-26 June 2012.</b>
<b>Issued Date</b>	<b>2012</b>
<b>URL</b>	<b><a href="http://hdl.handle.net/10722/165735">http://hdl.handle.net/10722/165735</a></b>
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# Valuing contingent exotic options: a discounted density approach

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Based on a paper with Hans Gerber and Elias Shiu

## Brownian motion (Wiener process)

- ▶  $X(t) = \mu t + \sigma W(t)$
- ▶  $\{W(t)\}$ : standard Wiener process
- ▶ notation:  $D = \frac{\sigma^2}{2}$
- ▶ running minimum:  $m(t) = \min_{0 \leq s \leq t} X(s)$
- ▶ running maximum:  $M(t) = \max_{0 \leq s \leq t} X(s)$

## 2 Exponential stopping of Brownian motion

- ▶  $\tau$ : exponential random variable  
independent of  $\{X(t)\}$   
 $f_{\tau}(t) = \lambda e^{-\lambda t}, \quad t > 0$
- ▶ We are interested in  $X(\tau), M(\tau), \dots$
- ▶  $\delta$ : force of interest used for discounting

## Three discounted density functions

$$\blacktriangleright f_{X(\tau)}^\delta(x) = \int_0^\infty e^{-\delta t} f_{X(t)}(x) f_\tau(t) dt$$

$$\blacktriangleright f_{M(\tau)}^\delta(m) = \int_0^\infty e^{-\delta t} f_{M(t)}(m) f_\tau(t) dt$$

$$\blacktriangleright f_{X(\tau), M(\tau)}^\delta(x, m) = \int_0^\infty e^{-\delta t} f_{X(t), M(t)}(x, m) f_\tau(t) dt$$

$\alpha < 0$  and  $\beta > 0$  solutions of the quadratic equation  
 $D\xi^2 + \mu\xi - (\lambda + \delta) = 0$

▶ 1).  $f_{X(\tau), M(\tau)}^\delta(x, m) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m} = \frac{\lambda}{D} e^{\alpha(m-x) - \beta m}$ ,  
 $-\infty < x \leq m, m \geq 0$

▶ 2).  $f_{M(\tau)}^\delta(m) = \frac{\lambda}{\lambda + \delta} \beta e^{-\beta m}$ ,  $m \geq 0$

▶ 3).  $f_{X(\tau)}^\delta(x) = \begin{cases} \frac{\lambda}{D(\beta - \alpha)} e^{-\alpha x}, & \text{if } x < 0, \\ \frac{\lambda}{D(\beta - \alpha)} e^{-\beta x}, & \text{if } x > 0. \end{cases}$

$$f_{X(\tau), M(\tau)-X(\tau)}^\delta(x, z) = \frac{\lambda}{D} e^{-\beta x - (\beta - \alpha)z}, \quad z \geq \max(-x, 0).$$

$$f_{M(\tau), M(\tau)-X(\tau)}^\delta(y, z) = \frac{\lambda}{D} e^{-\beta y + \alpha z}, \quad y \geq 0, z \geq 0.$$

$$f_{M(\tau)-X(\tau)}^\delta(z) = \frac{\lambda}{\beta D} e^{\alpha z} = \frac{\lambda}{\lambda + \delta} (-\alpha) e^{\alpha z}, \quad z \geq 0.$$

$$m(t) = -\max\{-X(s); 0 \leq s \leq t\}$$

If

$$E[e^{-\delta\tau} g(X(\tau), M(\tau))] = h(\alpha, \beta),$$

then we can translate it to

$$E[e^{-\delta\tau} g(-X(\tau), -m(\tau))] = h(-\beta, -\alpha).$$



$$\begin{aligned}
f_{X(\tau), m(\tau)}^\delta(x, y) &= \frac{\lambda}{D} e^{-\beta x + (\beta - \alpha)y}, & y \leq \min(x, 0), \\
f_{X(\tau), X(\tau) - m(\tau)}^\delta(x, z) &= \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)z}, & z \geq \max(x, 0), \\
f_{m(\tau), X(\tau) - m(\tau)}^\delta(y, z) &= \frac{\lambda}{D} e^{-\alpha y - \beta z}, & y \leq 0, z \geq 0, \\
f_{m(\tau)}^\delta(y) = \frac{\lambda}{\beta D} e^{-\alpha y} &= \frac{\lambda}{\lambda + \delta} (-\alpha) e^{-\alpha y}, & y \leq 0, \\
f_{X(\tau) - m(\tau)}^\delta(z) = \frac{\lambda}{-\alpha D} e^{-\beta z} &= \frac{\lambda}{\lambda + \delta} \beta e^{-\beta z}, & z \geq 0.
\end{aligned}$$

## Factorization formula

If  $\tau$  is exponential with mean  $1/\lambda$ , then the following factorization formula holds,

$$E[e^{-\delta\tau} g_{\tau}(X)] = E[e^{-\delta\tau}] \times E[g_{\tau^*}(X)],$$

where  $\tau^*$  is an exponential random variable with mean  $1/(\lambda + \delta)$  and independent of  $X$ .

Remarks (i)  $E[e^{-\delta\tau}] = \frac{\lambda}{\lambda + \delta}$ .

(ii) The condition  $\delta > 0$  can be replaced by the condition  $\delta > -\lambda$ .

### 3. Financial applications

- ▶  $S(t)$ : stock price
- ▶  $S(t) = S(0)e^{X(t)} = S(0)e^{\mu t + \sigma W(t)}$ ,  $t \geq 0$
- ▶ a contingent option provides a payoff at time  $\tau$
- ▶ Example:  $\tau$ : time of death  
GMDB (Guaranteed Minimum Death Benefits)

### 3. Lookback options

- ▶ Many equity-indexed annuities credit interest using a high water mark method or a low water mark method

## Out-of-the-money fixed strike lookback call option

- ▶ Payoff:

$$[S(0)e^{M(\tau)} - K]_+$$

- ▶ Time-0 value

$$\begin{aligned} \int_k^\infty [S(0)e^y - K] f_{M(\tau)}^\delta(y) dy &= \frac{\lambda}{\lambda + \delta} [S(0) \frac{\beta e^{-(\beta-1)k}}{\beta - 1} - Ke^{-\beta k}] \\ &= \frac{\lambda}{\lambda + \delta} \frac{K}{\beta - 1} \left[ \frac{S(0)}{K} \right]^\beta. \end{aligned}$$

- ▶ Another expression for the option value

$$\frac{\lambda}{D - \alpha\beta(\beta - 1)} \left[ \frac{S(0)}{K} \right]^\beta.$$

## In-the-money fixed strike lookback call option

- ▶ Payoff

$$\max(H, S(0)e^{M(\tau)}) - K.$$

- ▶ Rewriting as

$$H - K + [S(0)e^{M(\tau)} - H]_+$$

- ▶ Time-0 value

$$\frac{\lambda}{\lambda + \delta} \left\{ H - K + \frac{H}{\beta - 1} \left[ \frac{S(0)}{H} \right]^\beta \right\}.$$

## Floating strike lookback put option

- ▶ Payoff

$$\max(H, \max_{0 \leq t \leq \tau} S(t)) - S(\tau), \quad (1)$$

where  $H \geq S(0)$ .

- ▶ Time-0 value

$$\frac{\lambda}{\lambda + \delta} \left\{ H + \frac{H}{\beta - 1} \left[ \frac{S(0)}{H} \right]^\beta \right\} - E[e^{-\delta\tau} S(\tau)].$$

## Floating strike lookback put option

- ▶ Special case:  $H = S(0)$ , the time-0 value

$$\begin{aligned} & \frac{\lambda}{\lambda + \delta} \frac{\beta}{\beta - 1} S(0) - E[e^{-\delta\tau} S(\tau)] \\ &= \frac{1 - \alpha}{-\alpha} E[e^{-\delta\tau} S(\tau)] - E[e^{-\delta\tau} S(\tau)] \\ &= \frac{1}{-\alpha} E[e^{-\delta\tau} S(\tau)]. \end{aligned} \tag{2}$$

- ▶ This result can be reformulated as

$$E[e^{-\delta\tau} \max_{0 \leq t \leq \tau} S(t)] = \left( \frac{1}{-\alpha} + 1 \right) E[e^{-\delta\tau} S(\tau)].$$



## Floating strike lookback put option

Milevsky and Posner (2001) have evaluated (1) with a risk-neutral stock price process and  $H = S(0)$ . They also assume that the stock pays dividends continuously at a rate proportional to its price. With  $l$  denoting the dividend yield rate,  $\delta = r$ , and  $\mu = r - D - l$ , the RHS of (2) is

$$\frac{2D}{(r - D - l) + \sqrt{(r - D - l)^2 + 4D(\lambda + r)}} \times S(0) \frac{\lambda}{\lambda + l}.$$

Although it seems rather different from formula (38) on page 117 of Milevsky and Posner (2001), they are the same.

# Fractional floating strike lookback put option

► Payoff

$$[\gamma \max_{0 \leq t \leq \tau} S(t) - S(\tau)]_+ = S(0)[\gamma e^{M(\tau)} - e^{X(\tau)}]_+.$$

► Notice

$$[\gamma e^{M(\tau)} - e^{X(\tau)}]_+ = e^{M(\tau)}[\gamma - e^{X(\tau) - M(\tau)}]_+$$

## Fractional floating strike lookback put option

► Hence

$$\begin{aligned} & \mathbb{E}[e^{-\delta\tau} e^{M(\tau)} [\gamma - e^{X(\tau) - M(\tau)}]_+] \\ &= \int_0^\infty \int_0^\infty e^y [\gamma - e^{-z}]_+ f_{M(\tau), M(\tau) - X(\tau)}^\delta(y, z) dy dz \\ &= \frac{\lambda}{D} \left[ \int_0^\infty e^y e^{-\beta y} dy \right] \left[ \int_0^\infty [\gamma - e^{-z}]_+ e^{\alpha z} dz \right] \\ &= \frac{\lambda}{D} \frac{1}{\beta - 1} \frac{\gamma^{1-\alpha}}{-\alpha(1-\alpha)} \\ &= \gamma^{1-\alpha} \frac{\lambda}{\lambda + \delta} \frac{\beta}{(1-\alpha)(\beta - 1)} \\ &= \gamma^{1-\alpha} \frac{1}{-\alpha} \mathbb{E}[e^{-\delta\tau} e^{X(\tau)}]. \end{aligned}$$

## Fractional floating strike lookback put option

- ▶ This can be rewritten as

$$\mathbb{E}[e^{-\delta\tau}[\gamma e^{M(\tau)} - e^{X(\tau)}]_{+}] = \gamma^{1-\alpha} \mathbb{E}[e^{-\delta\tau}(e^{M(\tau)} - e^{X(\tau)})].$$

- ▶ Time-0 value

$$\mathbb{E}[e^{-\delta\tau}[\gamma \max_{0 \leq t \leq \tau} S(t) - S(\tau)]_{+}] = \frac{\gamma^{1-\alpha}}{-\alpha} \mathbb{E}[e^{-\delta\tau} S(\tau)],$$

# Out-of-the-money fixed strike lookback put option

- ▶ Payoff

$$[K - S(0)e^{m(\tau)}]_+,$$

- ▶ Time-0 value

$$\int_{-\infty}^k [K - S(0)e^y] f_{m(\tau)}^\delta(y) dy = \frac{\lambda}{\lambda + \delta} \frac{K}{1 - \alpha} \left[ \frac{K}{S(0)} \right]^{-\alpha}.$$

## In-the-money fixed strike lookback put option

- ▶ Payoff

$$K - \min(H, S(0)e^{m(\tau)}) = K - H + [H - S(0)e^{m(\tau)}]_+,$$

- ▶ Time-0 value

$$\frac{\lambda}{\lambda + \delta} \left\{ K - H + \frac{H}{1 - \alpha} \left[ \frac{H}{S(0)} \right]^{-\alpha} \right\}.$$

## Floating strike lookback call option

- ▶ Payoff

$$S(\tau) - \min(H, \min_{0 \leq t \leq \tau} S(t)),$$

where  $0 < H \leq S(0)$ .

- ▶ Time-0 value

$$E[e^{-\delta\tau} S(\tau)] + \frac{\lambda}{\lambda + \delta} \left\{ -H + \frac{H}{1 - \alpha} \left[ \frac{H}{S(0)} \right]^{-\alpha} \right\}.$$

- ▶ In the special case where  $H = S(0)$ , the time-0 value

$$\begin{aligned} & E[e^{-\delta\tau} S(\tau)] - \frac{\lambda}{\lambda + \delta} \frac{-\alpha}{1 - \alpha} S(0) \\ &= E[e^{-\delta\tau} S(\tau)] - \frac{\beta - 1}{\beta} E[e^{-\delta\tau} S(\tau)] \\ &= \frac{1}{\beta} E[e^{-\delta\tau} S(\tau)]. \end{aligned}$$

# Fractional floating strike lookback call option

► Payoff

$$\begin{aligned} [S(\tau) - \gamma \min_{0 \leq t \leq \tau} S(t)]_+ &= S(0)[e^{X(\tau)} - \gamma e^{m(\tau)}]_+ \\ &= S(0)e^{m(\tau)}[e^{X(\tau)-m(\tau)} - \gamma]_+ \end{aligned}$$



## Fractional floating strike lookback call option

- ▶ Its expected discounted value is  $S(0)$  times the following expectation

$$\begin{aligned} & \mathbb{E}[e^{-\delta\tau} e^{m(\tau)} [e^{X(\tau)-m(\tau)} - \gamma]_+] \\ &= \frac{\lambda}{D} \left[ \int_{-\infty}^0 e^y e^{-\alpha y} dy \right] \left[ \int_0^{\infty} [e^z - \gamma]_+ e^{-\beta z} dz \right] \\ &= \frac{\lambda}{D} \frac{1}{1-\alpha} \frac{\gamma^{1-\beta}}{\beta(\beta-1)} \\ &= \frac{1}{\gamma^{\beta-1}} \frac{\lambda}{\lambda + \delta} \frac{-\alpha}{(1-\alpha)(\beta-1)} \\ &= \frac{1}{\gamma^{\beta-1}} \frac{1}{\beta} \mathbb{E}[e^{-\delta\tau} e^{X(\tau)}]. \end{aligned}$$

## Fractional floating strike lookback call option

- ▶ This can be rewritten as

$$\mathbb{E}[e^{-\delta\tau}[e^{X(\tau)} - \gamma e^{m(\tau)}]_{+}] = \gamma^{-(\beta-1)} \mathbb{E}[e^{-\delta\tau}(e^{X(\tau)} - e^{m(\tau)})].$$

- ▶ We have

$$\mathbb{E}[e^{-\delta\tau}[S(\tau) - \gamma \min_{0 \leq t \leq \tau} S(t)]_{+}] = \frac{1}{\beta\gamma^{\beta-1}} \mathbb{E}[e^{-\delta\tau} S(\tau)].$$

## High-low option

- ▶ Payoff

$$\max(\bar{H}, \max_{0 \leq t \leq \tau} S(t)) - \min(\underline{H}, \min_{0 \leq t \leq \tau} S(t)),$$

where  $0 < \underline{H} \leq S(0) \leq \bar{H}$ .

- ▶ Time-0 value

$$\frac{\lambda}{\lambda + \delta} \left\{ \bar{H} + \frac{\bar{H}}{\beta - 1} \left[ \frac{S(0)}{\bar{H}} \right]^{\beta} - \underline{H} + \frac{\underline{H}}{1 - \alpha} \left[ \frac{\underline{H}}{S(0)} \right]^{-\alpha} \right\}.$$

- ▶ In the special case where  $\underline{H} = S(0) = \bar{H}$ , time-0 value

$$S(0) \frac{\lambda}{\lambda + \delta} \frac{\beta - \alpha}{(\beta - 1)(1 - \alpha)} = \frac{\beta - \alpha}{-\alpha\beta} \mathbb{E}[e^{-\delta\tau} S(\tau)].$$

- ▶ This can be rewritten as

$$\left( \frac{1}{-\alpha} + \frac{1}{\beta} \right) \mathbb{E}[e^{-\delta\tau} S(\tau)],$$

## Barrier options

- ▶ A barrier option is an option whose payoff depends on whether or not the price of the underlying asset has reached a predetermined level or *barrier*.
- ▶ Knock-out options are those which go out of existence if the asset price reaches the barrier, and *knock-in options* are those which come into existence if the barrier is reached.

## Parity relation



Knock-out option + Knock-in option = Ordinary option.

▶ Notation:  $L$  denotes the barrier and  $\ell = \ln[L/S(0)]$

## Up-and-out and up-and-in options ( $L > S(0)$ ( $\ell > 0$ ))

- ▶ Payoffs

$$I_{([\max_{0 \leq t \leq \tau} S(t)] < L)} b(S(\tau)) = I_{(M(\tau) < \ell)} b(S(0)) e^{X(\tau)}$$



$$I_{([\max_{0 \leq t \leq \tau} S(t)] \geq L)} b(S(\tau)) = I_{(M(\tau) \geq \ell)} b(S(0)) e^{X(\tau)}$$

## The expected discounted values

► Up-and-out

$$\begin{aligned} & \int_0^{\infty} \left[ \int_{-\infty}^y I_{(y < \ell)} b(S(0)e^x) f_{X(\tau), M(\tau)}^{\delta}(x, y) dx \right] dy \\ &= \frac{\lambda}{D} \int_0^{\ell} \left[ \int_{-\infty}^y b(S(0)e^x) e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy \end{aligned}$$

► Up-and-in

$$\frac{\lambda}{D} \int_{\ell}^{\infty} \left[ \int_{-\infty}^y b(S(0)e^x) e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy;$$

## Down-and-out and down-and-in options

$(0 < L < S(0) \text{ } (\ell < 0))$

► Payoffs

$$I_{([\min_{0 \leq t \leq \tau} S(t)] > L)} b(S(\tau)) = I_{(m(\tau) > \ell)} b(S(0)e^{X(\tau)})$$



$$I_{([\min_{0 \leq t \leq \tau} S(t)] \leq L)} b(S(\tau)) = I_{(m(\tau) \leq \ell)} b(S(0)e^{X(\tau)})$$



## The expected discounted values



$$\frac{\lambda}{D} \int_{\ell}^0 \left[ \int_y^{\infty} b(S(0)e^x) e^{-\beta x} dx \right] e^{(\beta-\alpha)y} dy$$



$$\frac{\lambda}{D} \int_{-\infty}^{\ell} \left[ \int_y^{\infty} b(S(0)e^x) e^{-\beta x} dx \right] e^{(\beta-\alpha)y} dy,$$

## Notation

$$A_1(n) = \frac{\lambda}{D} \frac{S(0)^n}{(n - \alpha)(\beta - n)},$$

$$A_2(n) = \frac{\lambda}{D} \frac{L^n}{(n - \alpha)(\beta - n)} \left[ \frac{S(0)}{L} \right]^\beta,$$

$$A_3(n) = \frac{\lambda}{D} \frac{L^n}{(n - \alpha)(\beta - n)} \left[ \frac{L}{S(0)} \right]^{-\alpha},$$

$$A_4 = \frac{\lambda}{D} \frac{K^n}{(n - \alpha)(\beta - \alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha} = \frac{\kappa K^n}{n - \alpha} \left[ \frac{K}{S(0)} \right]^{-\alpha},$$

## Notation

$$\begin{aligned}A_5 &= \frac{\lambda}{D} \frac{K^{n-\alpha} L^\alpha}{(n-\alpha)(\beta-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta = \frac{\kappa K^{n-\alpha} L^\alpha}{n-\alpha} \left[ \frac{S(0)}{L} \right]^\beta, \\A_6 &= \frac{\lambda}{D} \frac{K^n}{(\beta-n)(\beta-\alpha)} \left[ \frac{S(0)}{K} \right]^\beta = \frac{\kappa K^n}{\beta-n} \left[ \frac{S(0)}{K} \right]^\beta, \\A_7 &= \frac{\lambda}{D} \frac{K^{-(\beta-n)} L^\beta}{(\beta-n)(\beta-\alpha)} \left[ \frac{L}{S(0)} \right]^{-\alpha} = \frac{\kappa K^{-(\beta-n)} L^\beta}{\beta-n} \left[ \frac{L}{S(0)} \right]^{-\alpha}, \\A_8 &= \frac{\lambda}{D} \frac{K}{-\alpha(1-\alpha)(\beta-\alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha} = \frac{\kappa K}{-\alpha(1-\alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha},\end{aligned}$$

## Notation

$$\begin{aligned}A_9 &= \frac{\lambda}{D} \frac{K^{1-\alpha} L^\alpha}{-\alpha(1-\alpha)(\beta-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta \\ &= \frac{\kappa K^{1-\alpha} L^\alpha}{-\alpha(1-\alpha)} \left[ \frac{S(0)}{L} \right]^\beta,\end{aligned}$$

$$A_{10} = \frac{\lambda}{D} \frac{K}{\beta(\beta-1)(\beta-\alpha)} \left[ \frac{S(0)}{K} \right]^\beta = \frac{\kappa K}{\beta(\beta-1)} \left[ \frac{S(0)}{K} \right]^\beta,$$

$$\begin{aligned}A_{11} &= \frac{\lambda}{D} \frac{K^{-(\beta-1)} L^\beta}{\beta(\beta-1)(\beta-\alpha)} \left[ \frac{L}{S(0)} \right]^{-\alpha} \\ &= \frac{\kappa K^{-(\beta-1)} L^\beta}{\beta(\beta-1)} \left[ \frac{L}{S(0)} \right]^{-\alpha}.\end{aligned}$$

## Up-and-out all-or-nothing call option

The option value is

$$\begin{cases} 0, & \text{if } L < K, \\ \frac{\lambda}{D} \int_0^L [\int_k^y S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy, & \text{if } L \geq K \text{ and } S(0) > K, \\ \frac{\lambda}{D} \int_k^L [\int_k^y S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy, & \text{if } L \geq K \text{ and } S(0) \leq K \end{cases}$$
$$= \begin{cases} 0, & \text{if } L < K, \\ A_1(n) - A_2(n) - A_4 + A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_6 - A_2(n) + A_5, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases}$$

## Up-and-out all-or-nothing put option

The option value is

$$\begin{cases} \frac{\lambda}{D} \int_0^{\ell} [\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy, & \text{if } L < K, \\ \frac{\lambda}{D} \int_0^{\ell} [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy, & \text{if } L \geq K \& S(0) > K \\ \frac{\lambda}{D} \left\{ \int_0^k [\int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy \right. \\ \quad \left. + \int_k^{\ell} [\int_{-\infty}^k S(0)^n e^{nx} e^{-\alpha x} dx] e^{-(\beta-\alpha)y} dy \right\}, & \text{if } L \geq K \& S(0) \leq K \end{cases}$$
$$= \begin{cases} A_1(n) - A_2(n), & \text{if } L < K, \\ A_4 - A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_1(n) - A_5 - A_6, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases}$$

## up-and-out option with payoff $S(\tau)^n$

$$\frac{\lambda}{D} \int_0^\ell \left[ \int_{-\infty}^y S(0)^n e^{nx} e^{-\alpha x} dx \right] e^{-(\beta-\alpha)y} dy = A_1(n) - A_2(n).$$

This is the sum of the value of the up-and-out all-or-nothing put option and the value of the up-and-out all-or-nothing call option.

## Up-and-out call option

The value is

$$\begin{cases} 0, & \text{if } L < K, \\ A_1(1) - A_2(1) - A_1(0)K \\ \quad + A_2(0)K + A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_2(0)K + A_{10} - A_2(1) - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases}$$



## Up-and-out put option

The value is

$$\begin{cases} A_1(0)K - A_2(0)K - A_1(1) + A_2(1), & \text{if } L < K, \\ A_8 - A_9, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_1(0)K - A_1(1) + A_{10} - A_9, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases}$$

## Double barrier option

Payoff:

$$\pi(S(\tau))I\{a < m(\tau), M(\tau) < b\}$$

## Double barrier option

Using a formula from B & S (2002), we have

$$\begin{aligned} & E[b(S(0)e^{X(\tau)})I\{a < m(\tau), M(\tau) < b\}] \\ = & \frac{\kappa}{\Xi - \Xi^{-1}} \left[ \Xi \int_a^0 b(S(0)e^z)e^{-\alpha z} dz + \Xi^{-1} \int_a^0 b(S(0)e^z)e^{-\beta z} dz \right. \\ & + \Xi \int_0^b b(S(0)e^z)e^{-\beta z} dz + \Xi^{-1} \int_0^b b(S(0)e^z)e^{-\alpha z} dz \\ & \left. - \Upsilon^{-1} \int_a^b b(S(0)e^z)e^{-\alpha z} dz - \Upsilon \int_a^b b(S(0)e^z)e^{-\beta z} dz \right], \end{aligned}$$

where  $\Xi = e^{\frac{1}{2}(b-a)(\beta-\alpha)}$  and  $\Upsilon = e^{\frac{1}{2}(a+b)(\beta-\alpha)}$ .

## Double barrier option

When  $\delta > 0$ , we have

$$\begin{aligned} & E[e^{-\delta\tau} b(S(0)e^{X(\tau)}) I\{a < m(\tau), M(\tau) < b\}] \\ &= \lambda \int_0^\infty e^{-(\lambda+\delta)t} b(S(0)e^{X(t)}) I\{a < m(t), M(t) < b\} dt \\ &= \frac{\kappa}{\Xi - \Xi^{-1}} \left[ \Xi \int_a^0 b(S(0)e^z) e^{-\alpha z} dz + \Xi^{-1} \int_a^0 b(S(0)e^z) e^{-\beta z} dz \right. \\ &\quad \left. + \Xi \int_0^b b(S(0)e^z) e^{-\beta z} dz + \Xi^{-1} \int_0^b b(S(0)e^z) e^{-\alpha z} dz \right. \\ &\quad \left. - \Upsilon^{-1} \int_a^b b(S(0)e^z) e^{-\alpha z} dz - \Upsilon \int_a^b b(S(0)e^z) e^{-\beta z} dz \right]. \end{aligned}$$

## Several stocks



$$\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_n(t))'$$

$n$ -dimensional Brownian motion.

- ▶  $\boldsymbol{\mu}$  the mean vector
- ▶  $\mathbf{C}$  the covariance matrix of  $\mathbf{X}(1)$



$$g_t(\mathbf{X})$$

a real-valued functional of the process up to time  $t$ .

- ▶  $\mathbf{h}$  an  $n$ -dimensional vector of real numbers

$$E[e^{-\delta\tau} e^{\mathbf{h}'\mathbf{X}(\tau)} g_{\tau}(\mathbf{X})] = E[e^{-\delta(\mathbf{h})\tau} g_{\tau}(\mathbf{X}); \mathbf{h}], \quad (3)$$

where

$$\begin{aligned} \delta(\mathbf{h}) &= \delta - \ln[M_{\mathbf{X}(1)}(\mathbf{h})] \\ &= \delta - \mathbf{h}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h}. \end{aligned}$$

## Proof of (3)

- ▶ Conditioning on  $\tau = t$ , the LHS (3) is

$$\int_0^{\infty} e^{-\delta t} \mathbb{E}[e^{\mathbf{h}'\mathbf{X}(t)} g_t(\mathbf{X})] f_{\tau}(t) dt.$$

- ▶ By the factorization formula in the method of Esscher transforms, the expectation inside the integrand can be written as the product of two expectations,

$$\begin{aligned} & \mathbb{E}[e^{\mathbf{h}'\mathbf{X}(t)}] \times \mathbb{E}[g_t(\mathbf{X}); \mathbf{h}] \\ &= [M_{\mathbf{X}(1)}(\mathbf{h})]^t \times \mathbb{E}[g_t(\mathbf{X}); \mathbf{h}]. \end{aligned}$$

- ▶ Hence

$$\int_0^{\infty} e^{-\delta t} \mathbb{E}[e^{\mathbf{h}'\mathbf{X}(t)} g_t(\mathbf{X})] f_{\tau}(t) dt = \int_0^{\infty} e^{-\delta(\mathbf{h})t} \mathbb{E}[g_t(\mathbf{X}); \mathbf{h}] f_{\tau}(t) dt.$$

## Application of (3)

- ▶  $\mathbf{k}$   $n$ -dimensional vector of real numbers
- ▶  $q_t(\mathbf{k}'\mathbf{X})$  real-valued functional of the process up to time  $t$
- ▶

$$E[e^{-\delta\tau} e^{\mathbf{h}'\mathbf{X}(\tau)} q_\tau(\mathbf{k}'\mathbf{X})] = E[e^{-\delta(\mathbf{h})\tau} q_\tau(\mathbf{k}'\mathbf{X}); \mathbf{h}].$$

- ▶ The quadratic equation becomes

$$\begin{aligned} & \frac{1}{2}\text{Var}[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi^2 + E[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi - [\lambda + \delta(\mathbf{h})] \\ &= \frac{1}{2}\mathbf{k}'\mathbf{C}\mathbf{k}\xi^2 + \mathbf{k}'(\boldsymbol{\mu} + \mathbf{C}\mathbf{h})\xi - (\lambda + \delta - \mathbf{h}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h}) \end{aligned}$$



## Special case: $n = 2$

- ▶  $S_1(t) = S_1(0)e^{X_1(t)}$  and  $S_2(t) = S_2(0)e^{X_2(t)}$
- ▶  $\mu = (\mu_1, \mu_2)'$
- ▶

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

# Margrabe option

Payoff:

$$[S_1(\tau) - S_2(\tau)]_+. \quad (4)$$

If we rewrite (4) as

$$e^{X_2(\tau)} [S_1(0)e^{X_1(\tau)-X_2(\tau)} - S_2(0)]_+,$$

$$E[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] = \frac{\kappa^* S_2(0)}{\beta^*(\beta^* - 1)} \left[ \frac{S_1(0)}{S_2(0)} \right]^{\beta^*}.$$

Here,

$$\kappa^* = \frac{\lambda}{D^*(\beta^* - \alpha^*)},$$

$$D^* = \frac{1}{2} \text{Var}[X_1(1) - X_2(1)] = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2),$$

and  $\alpha^* < 0$  and  $\beta^* > 0$  are the zeros of

$$\begin{aligned} & D^* \xi^2 + (\mu_1 - \mu_2 + \rho\sigma_1\sigma_2 - \sigma_2^2)\xi - (\lambda + \delta - \mu_2 - \frac{1}{2}\sigma_2^2) \\ & = \ln[M_{\mathbf{X}(1)}((\xi, 1 - \xi)')] - (\lambda + \delta). \end{aligned}$$

- ▶ If we write (4) as

$$e^{X_1(\tau)}[S_1(0) - S_2(0)e^{X_2(\tau) - X_1(\tau)}]_+,$$



$$\begin{aligned} & \mathbb{E}[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] \\ &= \frac{\kappa^{**} S_1(0)}{-\alpha^{**}(1 - \alpha^{**})} \left[ \frac{S_1(0)}{S_2(0)} \right]^{-\alpha^{**}}. \end{aligned}$$

Here,

$$\begin{aligned} \kappa^{**} &= \frac{\lambda}{D^{**}(\beta^{**} - \alpha^{**})}, \\ D^{**} &= \frac{1}{2} \text{Var}[X_2(1) - X_1(1)] = D^*, \end{aligned}$$

and  $\alpha^{**} < 0$  and  $\beta^{**} > 0$  are the zeros of

$$\ln[M_{\mathbf{X}(1)}((1 - \xi, \xi)')] - (\lambda + \delta).$$

Hence

$$\alpha^* = 1 - \beta^{**}$$

and

$$\beta^* = 1 - \alpha^{**}.$$

Thus,  $\kappa^* = \kappa^{**}$