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Asset Allocation Under Regime-Switching Models

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Abstract—We discuss an optimal asset allocation problem in a wide class of discrete-time regime-switching models including the hidden Markovian regime-switching (HMRS) model, the interactive hidden Markovian regime-switching (IHMRS) model and the self-exciting threshold autoregressive (SETAR) model. In the optimal asset allocation problem, the object of the investor is to select an optimal portfolio strategy so as to maximize the expected utility of wealth over a finite investment horizon. We solve the optimal portfolio problem using a dynamic programming approach in a discrete-time set up. Numerical results are provided to illustrate the practical implementation of the models and the impacts of different types of regime switching on optimal portfolio strategies.

Keywords-Asset Allocation; Regime-Switching Models; IHMM; HMM; SETAR Model; Stochastic Dynamical System.

I. INTRODUCTION

The optimal asset allocation problem is one of the key problems in modern finance. In [1] and [2], Merton provide a very simple and intuitive solution to the optimal asset allocation problem under the assumptions of the lognormality of the returns from the risky asset and the power utility. After Merton's pioneering work, numerous authors studied the optimal asset allocation problem in different continuous-time stochastic models which can better incorporate empirical features of asset price dynamics than the geometric Brownian motion assumption underlying the Merton's model. For example, in [3], the authors discussed the optimal asset allocation problem in jump-diffusion models, [4] for stochastic volatility models, and [5] for continuous-time regime-switching models. It seems that the literature on the optimal asset allocation problem mainly focus on continuous-time asset price models. There is a relatively small amount of work on the problem in a discrete-time framework. Samuelson [6] pioneers the optimal asset allocation problem in a discrete-time setting. His framework is similar to a discrete-time version of the model adopted by Merton [1]. Song et al. [7] explored an optimal asset allocation problem in a stochastic nonlinear dynamical world, where price dynamics were described by the self-exciting threshold autoregressive (SETAR) model pioneered by Howell Tong, see Tong [8] and the Smooth Threshold Autoregressive (STAR) model first introduced in [9].

In this paper we discuss an optimal asset allocation problem in a wide class of discrete-time regime-switching models. The rationale for using these models is to incorporate the impact of regime shifts on financial returns attributed to structural changes in market or economic conditions. Regime-switching models provide a natural and convenient way to incorporate such impacts. There are different types of regime shifts in the regime-switching models. The first type of regime shifts describe transitions in regimes using a hidden Markov model (HMM), and this leads to a hidden Markovian regime-switching (HMRS) model. For an excellent account of the HMM, interested readers may refer to Elliott et al. [10]. The second type of regime shifts describe transitions in regimes using an Interactive Hidden Markov Model (IHMM), and this leads to an interactive hidden Markovian regime-switching (IHMRS) model. This type of model has been introduced and extensively investigated in Ching and Ng [11] and [12], [13]. The key feature of an IHMM is that transitions in hidden regimes depend on observation processes. This feature is absent in the traditional HMM. Hidden Markov models (HMMs) have many applications in diverse fields including management science, economics and finance, see [14], [12] and [13]. The third type of regime shifts is self-exciting and is dictated by the observation process itself. This is the self-exciting threshold autoregressive model pioneered by Tong, see [8]. Here we shall consider the problem of maximizing the expected utility of wealth over a finite investment horizon under the above three types of discrete-time regime-switching models. As in [7], we use a discrete-time dynamic programming approach here.

The rest of the paper is organized as follows. In Section II, we present the three-type of regime-switching models, namely the HMRS model, IHMRS model and the SETAR model. In Section III, we describe a general framework for the optimal asset allocation, and a discrete-time dynamic programming approach is presented to discuss the optimal asset allocation problem. The results of the numerical experiments are then presented in Section IV. We then summarize the main results in the last section.

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II. THE REGIME SWITCHING MODELS

In this section we present the three types of regime-switching models, namely the HMRS model, the IHMRS model and the SETAR model.

A. Hidden Regimes

Firstly, we focus on the HMRS model and the IHMRS model whose hidden regimes evolve over time according to a hidden Markov chain and an interactive hidden Markov chain.

We fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Here, we consider a discrete-time financial model with time index set $\mathcal{T} := \{0, 1, 2, \ldots\}$ and with two investment assets, namely, a risk-free bond B and a risky asset S. For each $t \in \mathcal{T}$, $\mathcal{F}_t$ represents the information set containing all market information up to and including time t. For each $t \in \mathcal{T}$, let ξ_t represents the noise term in the return process from the risky asset S at time t. It is assumed that ξ_t is known given $\mathcal{F}_t$ and that $\{\xi_t\}_{t \in \mathcal{T}}$ is a sequence of independent and identically distributed (i.i.d.) and $\xi_t \in N(0,1)$, for each $t \in \mathcal{T}$, where $N(0,1)$ is the standard normal distribution.

First, we let r be the constant continuously compounded risk-free interest rate of the risk-free asset B. For each $t \in \mathcal{T}$, let B_t and S_t denote the prices of B and S at time t, respectively. We then suppose that the price dynamics of B are governed by

$$B_t = B_{t-1}(1+r), \quad t = 1, 2, \ldots. \tag{1}$$

Let $Y_t := \ln(\frac{S_t}{S_{t-1}})$, which represents the log return from S in the period $[t-1, t]$. Then, we assume that, under $\mathcal{P}$, the dynamics of the log returns $\{Y_t\}_{t \in \mathcal{T}}$ from S satisfy the following k-regime Markovian regime-switching model:

$$Y_t = \sum_{i=1}^{k} \left(\mu^{(i)} + \sum_{j=1}^{p_i} \beta_j^{(i)} Y_{t-j} + \sigma_i \xi_t \right) I_{\{O_t = e_i\}}, t = 1, 2, \ldots \tag{2}$$

where

1) The index i represents a state of the world or regime of the model. For each $i = 1, 2, \ldots, k$, The parameter p_i is the autoregressive order in the ith regime of the model.

2) I_A is the indicator function of the event A. It determines in which regime the process of log returns falls and O_t is an observable state which is modeled by a HMM or IHMM. Here O_t presents the changes of the regimes or the economic conditions. And $O_t = e_i$ represents that at time t the observable state is in state i, where e_i is the unit vector with the ith entry being one.

3) σ_i^2 is the conditional variance of Y_t given $\mathcal{F}_{t-1}$ in the ith regime of the model.

4) The regime of the model at each time t depends on the observable state O_t. In particular, the regime at each time t is determined by the value of $I_{\{O_t = e_i\}}$.

In the Markovian regime-switching models, the dynamic of financial returns switches over time according to the states of external economic factors, which might be unobservable and governed by a discrete-time, finite-state, hidden Markov chain. To simplify our discussion, we consider the following 2-regime Markovian regime-switching model for the optimal asset allocation problem.

$$
\begin{aligned}
Y_t = & \left(\mu^{(1)} + \beta^{(1)} Y_{t-1} + \sigma_1 \xi_t \right) I_{\{O_t = e_1\}} \\
& + \left(\mu^{(2)} + \beta^{(2)} Y_{t-1} + \sigma_2 \xi_t \right) I_{\{O_t = e_2\}}.
\end{aligned}
$$

We assume the transitions of regimes in our Markovian regime-switching models are governed by the HMM and IHMM.

B. The Interactive Hidden Markov Model and the Hidden Markov Model

The idea of Interactive Hidden Markov model (IHMM) was first introduced by Ching and Ng in [11]. The key feature of an IHMM is that the transitions of the hidden states are affected by the observable states only and vice versa. This is different from the traditional HMM. In the HMM, the transitions of hidden states are independent with the observable states while the observable states can be determined by the hidden states. Here we assume that there are m hidden states and n observable states. We use vectors H_t $(t = 1, 2, \ldots, T)$ to denote the hidden state at time t, where T is the length of a sequence. And $H_t = e_k$ represents that at time t the hidden state is in state k, where e_k is the unit vector with the kth entry being one and $e_k \in R^m$. Similarly, we use vectors $O_t = e_j$ to denote that the observable state is in state j at time t and $e_j \in R^n$. The hidden states and the observable states will affect each other in a IHMM. And we assume the following relationship for a IHMM:

$$H_t = \sum_{i=1}^{h} \lambda_{t-i+1} P_{t-i+1} O_{t-i+1} \quad, \quad O_t = \sum_{i=1}^{k} \mu_{t-i} M_{t-i} H_{t-i}. \tag{3}$$

where h and k are the orders of the hidden states and observable states respectively. While the matrices P_i and M_i are the i-step transition probability matrices and we have

$$0 \leq \lambda_i, \mu_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{h} \lambda_i = \sum_{i=1}^{n} \mu_i = 1.$$

For a HMM, we have the following relationship correspondingly:

$$H_t = \sum_{i=1}^{h} \lambda_{t-i+1} P_{t-i+1} H_{t-i+1} \quad, \quad O_t = \sum_{i=1}^{k} \mu_{t-i} M_{t-i} H_{t-i}. \tag{4}$$

In this paper we consider IHMM with $k = 1, h = 2$. Then the model is given by

$$H_t = \lambda P O_t + (1-\lambda) Q O_{t-1} \quad, \quad O_t = M H_{t-1}.$$

where $0 \leq \lambda$.
Also for the HMM, we assume that:

$$H_t = \lambda P H_t + (1-\lambda) Q H_{t-1} \quad, \quad O_t = M H_{t-1}.$$

C. The SETAR Model

We also consider the SETAR model for describing the dynamics of financial returns. In the SETAR model, the dynamics of financial returns switches over time according to the past values of the financial returns, which are observed by market participants. We assume that, under $\mathcal{P}$, the dynamics of the log returns $\{Y_t\}_{t \in \mathcal{T}}$ from S satisfy the following k-regime SETAR($k; p_1, p_2, \ldots, p_k$):

$$Y_t = \sum_{i=1}^{k} \left(\mu^{(i)} + \sum_{j=1}^{p_i} \beta_j^{(i)} Y_{t-j} + \sigma_i \xi_t \right) I_{\{r_{i-1} < Y_{t-d} \le r_i\}}$$
$$t = 1, 2, \ldots, \tag{5}$$

where

1) The index i represents a state of the world or regime of the model.
2) d is the delay parameter, which is a positive integer.
3) The threshold parameters satisfy the constraint $-\infty = r_0 < r_1 < \ldots < r_k < \infty$.
4) The regime of the model at each time t depends on the observable history of the log returns $\{Y_t\}_{t \in \mathcal{T}}$. In particular, the regime at each time t is determined by the value of Y_{t-d}. Hence the term a self-exciting threshold autoregressive model.

Here we consider the following SETAR(2;1,1) model for the financial returns:

$$Y_t = \left(\mu^{(1)} + \beta^{(1)} Y_{t-1} + \sigma_1 \xi_t \right)(1 - I_{r_1}(Y_{t-1}))$$
$$+ \left(\mu^{(2)} + \beta^{(2)} Y_{t-1} + \sigma_2 \xi_t \right) I_{r_1}(Y_{t-1}),$$

where $I_{r_1}(y)$ is an indicator function with value 1 (0) when $y > r_1$ ($y \le r_1$).

III. THE ASSET ALLOCATION PROBLEMS AND THEIR SOLUTIONS

In this section, we consider an investor who wishes to allocate his/her wealth rationally among two primary assets: the risk-free asset B and the risky asset S. The price process of the bond B is given by (1). The dynamic of the log returns Y_t from S satisfy the Markovian regime-switching models or the SETAR model. The objective of the investor is to maximize the expected utility of his/her wealth over a finite-time horizon $[0, T]$. Here we represent the risk preference of the investor via constant relative risk aversion (CRRA) utility function with the following form:

$$U(W) = \frac{W^\gamma}{\gamma}, \quad 0 < \gamma \le 1, \quad \text{and} \quad U(W) = \ln(W), \quad \gamma = 0.$$

where W is the wealth of the investor and γ represents an index of risk preference.

We suppose that the investor makes his/her investment decision at the beginning of each time period. Let $t_0 > 0$, then, at each time point $t = t_0, t_0 + 1, \ldots, T - 1$, the investor decides the proportion π_t of his/her wealth to be invested in the risky asset S. W_t represents the total wealth of the investor at time t. In the asset allocation problem, the objective of the investor is to choose π_t to maximize the expected discounted utility of his/her wealth over the planning horizon, for each $t = t_0, t_0 + 1, \ldots, T - 1$. We suppose that the investor does not consume his/her wealth

in the planning horizon $[0, T]$. Let $R_t := \frac{S_t}{S_{t-1}}$. Then one can state the asset allocation problem of the investor as follows.

$$\max_{\{\pi_t\}} J(t, W_t, \pi_t) := E\left[\sum_{i=1}^{t+1} (1 + r)^{-i} U(W_i) | \mathcal{F}_t \right].$$

subject to the constraint:

$$W_{t+1} = W_t[(1 - \pi_t)(1 + r) + \pi_t R_{t+1}]$$

with a given initial wealth $W_{t_0} = w$. This is a recursive asset allocation problem, in which the investor updates his asset allocation decision when new information comes. Initially, the investor decides the proportion π_{t_0} of his/her wealth W_{t_0} invested in the risky asset and invests the rest of his/her wealth in the risk-free asset. At time $t_0 + 1$, the value of the return from the risky asset R_{t_0+1} is realized and W_{t_0+1} is known exactly, the investor then use this piece of information to make his/her asset allocation decision π_{t_0+1} at time $t_0 + 1$, and so on.

We shall derive a forward recursion formula for the solution of the optimal asset allocation problem. At time $t = t_0$, we have

$$
\begin{aligned}
J(t_0, W_{t_0}, \pi_{t_0}) &= \sum_{i=1}^{t_0} (1 + r)^{-i} U(W_i) \\
&\quad + E[(1 + r)^{-t_0-1} U(W_{t_0+1}) | \mathcal{F}_{t_0}] \\
&= \sum_{i=1}^{t_0} \frac{(1 + r)^{-i} W_i^\gamma}{\gamma} + \frac{(1 + r)^{-t_0-1} W_{t_0}^\gamma}{\gamma} \\
&\quad \cdot E\{[(1 - \pi_{t_0})(1 + r) + \pi_{t_0} R_{t_0+1}]^\gamma | \mathcal{F}_{t_0}\}.
\end{aligned}
$$

Now our goal is to find π_{t_0} so as to maximize $J(t_0, W_{t_0}, \pi_{t_0})$. That is, we consider the maximization of the next period's expected utility given the current and past information. This is a single-period optimization problem. Differentiating $J(t_0, W_{t_0}, \pi_{t_0})$ with respect to π_{t_0} and setting the derivative equal to zero, we get the following first-order condition for the optimal asset allocation problem at time t_0:

$$E\{[(1 - \pi_{t_0})(1 + r) + \pi_{t_0} R_{t_0+1}]^{\gamma-1}$$
$$\cdot [R_{t_0+1} - (1 + r)] | \mathcal{F}_{t_0}\} = 0.$$

from which we can solve for the optimal asset allocation $\hat{\pi}_{t_0}$ at time t_0.

For other time periods, say $t = t_0 + 1, t_0 + 2, \cdots, T - 1$, we determine the optimal asset allocation strategies $(\hat{\pi}_{t_0+1}, \ldots, \hat{\pi}_{T-1})$ by solving the similar recursive formula:

$$E\{[(1 - \pi_t)(1 + r) + \pi_t R_{t+1}]^{\gamma-1}$$
$$\cdot [R_{t+1} - (1 + r)] | \mathcal{F}_t\} = 0, t = t_0 + 1, \ldots, T - 1,$$

Now, according to the above recursive formula, we present the solution to the optimal asset allocation problem under the Markovian regime-switching model described in (2). In this case, the optimal asset allocation decisions $(\hat{\pi}_{t_0}, \hat{\pi}_{t_0+1}, \ldots, \hat{\pi}_{T-1})$ can be obtained from solving the

following recursive integral equation:

$$\sum_{i=1}^{k} \left\{ \int_{\Re} \left[(1 - \pi_t)(1 + r) \right. \right.$$
$$\left. + \pi_t \exp\left(\mu^{(i)} + \sum_{j=1}^{p} \beta_j^{(i)} Y_{t+1-j} + \sigma_i y \right) \right]^{\gamma-1}$$
$$\cdot \left[\exp\left(\mu^{(i)} + \sum_{j=1}^{p} \beta_j^{(i)} Y_{t+1-j} + \sigma_i y \right) \right.$$
$$\left. \left. - (1 + r) \right] \phi(y) dy \right\} I_{\{r_{i-1} \leq Y_t \leq r_i\}} = 0 \,,$$

$t = t_0, \ldots, T - 1$. Here $\phi(\cdot)$ denotes the probability density function of a standard normal distribution.

Then, for the SETAR model described in (5), the optimal asset allocation decisions $(\hat{\pi}_{t_0}, \hat{\pi}_{t_0+1}, \ldots, \hat{\pi}_{T-1})$ satisfies the following recursive equation:

$$\sum_{i=1}^{k} \left\{ \int_{\Re} \left[(1 - \pi_t)(1 + r) \right. \right.$$
$$\left. + \pi_t \exp\left(\mu^{(i)} + \sum_{j=1}^{p} \beta_j^{(i)} Y_{t+1-j} + \sigma_i y \right) \right]^{\gamma-1}$$
$$\cdot \left[\exp\left(\mu^{(i)} + \sum_{j=1}^{p} \beta_j^{(i)} Y_{t+1-j} + \sigma_i y \right) \right.$$
$$\left. \left. - (1 + r) \right] \phi(y) dy \right\} I_{\{r_{i-1} \leq Y_t \leq r_i\}} = 0 \,,$$
$$t = t_0, \ldots, T - 1.$$

IV. NUMERICAL EXPERIMENTS AND DISCUSSIONS

In this section, we conduct numerical experiments to illustrate the practical implementation of the proposed models. We shall compare the temporal behaviors of the optimal portfolio strategies obtained from the above models. All computations in this section were done by MATLAB codes.

Given the same observed data sequence, we can apply the algorithm presented in [15] which employ the non-negative matrix factorization (NMF) techniques for IHMRS model, the Baum-Welch algorithm presented in [16] for HMRS model to determine the parameters λ_i, μ_i, P_i and M_i in (3) and (4). With these parameters we can predict the observable data sequence to govern the changes of regimes for the financial returns and then solve the optimal asset allocation problem correspondingly. The function $J(t, W_t, \pi_t)$ is a differentiable, concave, function of π_t defined on the interval $[0, 1]$ with fixed t and W_t. We employ Newton's method to solve the optimal allocation problem numerically. We set $T = 100$ and repeat this process until a sufficiently accurate value is attained. The proportion π_t takes a value between 0 and 1. Consequently, if the approximation of the solution obtained by Newton's method is greater than 1, we record "1" as the optimal allocation, also we record "0" if the approximation solution is less than 0. We shall consider
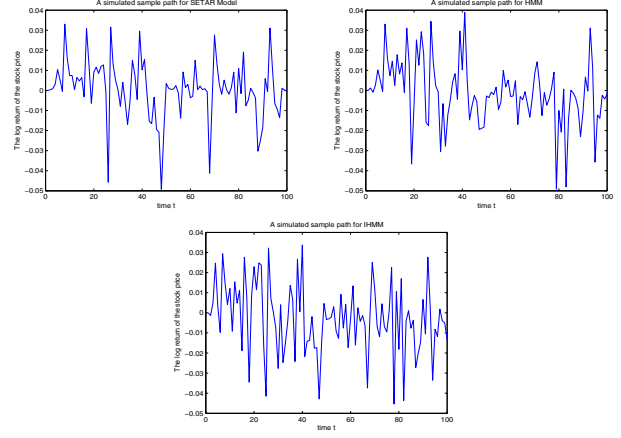

Figure 1. The simulated sample path for SETAR model, HMRS model and IHMRS model
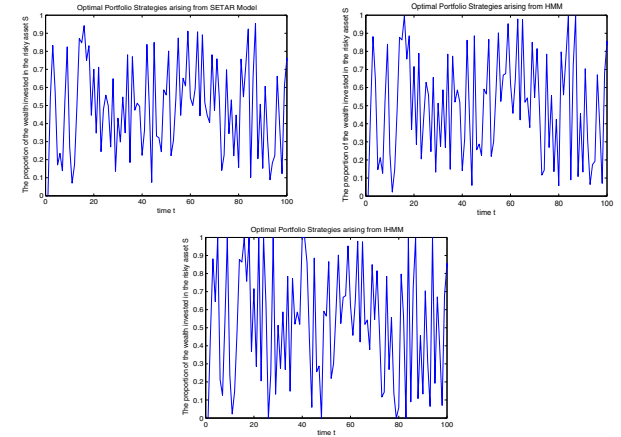

Figure 2. Optimal portfolio strategies arising from SETAR model, HMRS model and IHMRS model

some specimen values of the model parameters and assume that the risk-free interest rate $r = 0.0003$; $\mu^{(1)} = 0.0004$; $\mu^{(2)} = 0.0014$; $\sigma^{(1)} = 0.03$; $\sigma^{(2)} = 0.007$; $\beta^{(1)} = 0.1$; $\beta^{(2)} = 0.3$; $r_1 = 0$ and $\delta = 0.1$ and $\gamma = 0.5$.

Figures 1 depict a simulated sample path for each of the SETAR model(model I), the HMRS model (model II) and the IHMRS model (model III). From Figures 1, we see that the simulated returns from SETAR model are less volatile than those from the Markovian regime-switching models since the changes of regimes in Markovian regime-switching model are influenced by economic conditions in the market directly. The IHMRS model seems giving the most volatile and extreme simulated returns. This can be explained by the fact that the structural changes in the model dynamics in the IHMRS model are abrupt while those in the HMRS model are gradual since the interactivity is incorporated in the IHMM model.

Figures 2 depict plots of the optimal portfolio strategies

arising from Models I-III. These optimal strategies are the optimal proportions invested in the risky asset over time. From Figures 2, we see that the endogenous time series of optimal strategies arising from the SETAR model is the least extreme one. The economic agent reacts rationally to the variations of financial returns. Hence, among the Markovian regime-switching models, the endogenous time series of optimal portfolio strategies from the IHMRS model is the most volatile and extreme one.

From the above numerical results, we see that choice of a time series model for financial returns may lead to quite different optimal asset allocation strategies. It's very crucial to select an appropriate parametric form of the time series model to solve the asset allocation problem. The Markovian regime-switching models can describe abrupt structural changes in model dynamics of financial returns. These structural changes may be attributed to changes in economic conditions. Consequently, if the goal of a fund manager is to develop a asset allocation policy which takes into account the adverse effect of the market and economic catastrophes on financial returns, the manager may consider the Markovian regime-switching models for financial returns in developing the asset allocation policy, since the changes of regimes in the SETAR model are decided by the past values of the returns. If one wishes to incorporate the feedback effect, the IHMRS model seems more appropriate than the HMRS model. If the price dynamic of the asset react to the changes of economic conditions gradually and highly depend on it's historical data, for example the stock of a commodity firm and some defensive securities, then the manager should consider the SETAR model for financial returns.

V. Conclusions

We discussed the optimal asset allocation problem in a wide class of discrete-time regime switching models, where the hidden regimes are described by a hidden Markov chain, an interactive hidden Markov chain and self-exciting model. A discrete-time dynamic programming approach was used to discuss the optimal asset allocation problem in these three types of regime switching models. Numerical results revealed that different from the SETAR model, changes in the model regimes in the Markovian regime-switching model are more volatile since we takes into account the adverse effect of the market and economic catastrophes on financial returns. The structural changes in the model dynamics in the IHMRS model are abrupt while those in the HMRS model are gradual since the interactivity is incorporated in the IHMM model.

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