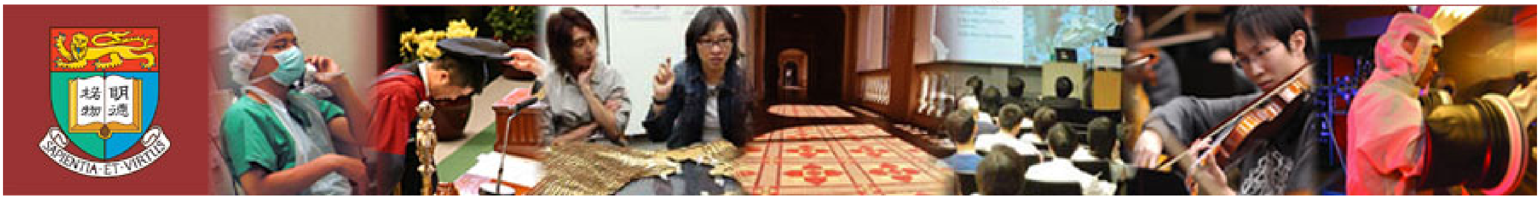

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Characterization of Wave Physics Using the Rigorous Helmholtz Decomposition Based on the Surface Integral Equation

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***Abstract*—Helmholtz decomposition (HD) is a fundamental tool of vector calculus and plays an important role in electromagnetics. In this work, arbitrary vector field defined on the open or closed surface is decomposed into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field by using the surface integral equation method. Unlike the popular loop-tree decomposition that is only a quasi-HD suitable for the circuit physics in the low frequency regime, the HD developed in this paper is rigorous and can capture both circuit and wave physics from very low frequency to high frequency regimes. The work could provide insightful physical interpretations for complex electromagnetic phenomena.**

I. Introduction

In physics and mathematics, Helmholtz decomposition (HD) states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be decomposed into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field [1]. This theorem is of great importance to electrostatics, because Maxwell's equations for electric and magnetic fields are exactly this type. In the low frequency regime, the loop-tree decomposition [2], which is commonly used to solve the low-frequency breakdown problem, can properly capture the circuit physics where electric and magnetic fields almost decouple with each other. However, the loop-tree decomposition and its derivatives (loop-star decomposition and tree-cotree decomposition) are only the quasi-HD due to imperfect curl-free nature of the tree basis function.

In this work, we explore the universal feature of HD that rigorously works from low frequency to high frequency regimes. Revelent numerical implementation is developed for arbitrary vector fields defined on either open or closed surfaces using the surface integral equation (SIE) method. Different from the conventional loop-tree decomposition, the HD in this paper is not only rigorous, but also suitable for the wave physics at higher frequencies. The work provides the useful physical insight and detailed numerical solution for HD, which is not only important in electromagnetics but also in hydrodynamics and other areas.

II. Theory

The helicity Hamiltonian of Maxwell's equations, which is a topological measure of how much the field rotates about itself in a given volume [3], can be defined as

$$G(\mathbf{H},\mathbf{E}) = \frac{1}{2}\left(\frac{1}{\varepsilon}\mathbf{H}\cdot\nabla\times\mathbf{H} + \frac{1}{\mu}\mathbf{E}\cdot\nabla\times\mathbf{E}\right) \tag{1}$$

The corresponding canonical Euler-Hamilton equations are of the form

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{\partial G}{\partial \mathbf{E}}, \qquad \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial G}{\partial \mathbf{H}} \tag{2}$$

This exactly retrieves Maxwell's equations by the aid of variation principle. Although Helmholtz's original paper mostly concerned with hydrodynamics and he was more interested in vortex motion expressed as the helicity functional, the HD itself is a great finding and can be extended to a vector field defined on the open or closed surface by using surface divergence and gradient operators. The splitting on the surface [4] is now

$$\mathbf{f}_S = \nabla_S\phi + \mathbf{u}_n\times\nabla_S\psi \tag{3}$$

where $\mathbf{u}_n$ is the unit vector along the outward normal direction. Moreover, ϕ and ψ corresponds to $\mathbf{J}_{irr}$ and $\mathbf{J}_{sol}$ respectively. They can be uniquely determined by the following expressions

$$\nabla_S^2\phi = \nabla_S\cdot\mathbf{f}_S, \qquad \nabla_S^2\psi = \nabla_S\cdot(\mathbf{f}_S\times\mathbf{u}_n) \tag{4}$$

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_S \frac{\nabla'\cdot\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}dS' \tag{5}$$

By using the electric-field integral-equation (EFIE), the irrotational and solenoidal fields can be trivially calculated. The irrotational part is calculated first. Then the solenoidal part is obtained by subtracting the irrotational one from the total field.

III. Numerical Examples

A. 3D plate with fictitious current

A fictitious vector current defined on a 3D infinitely-thin plate is decomposed by the SIE method. We compare the results with the pseudo-spectral method (PSM) using the discrete sine transform [5] with very dense mesh. Fig. 1(a) shows the original current on the plate. Fig. 1(b) shows the convergence of the relative error as a function of mesh points per wavelength (PPW). Fig. 1(c) and (d) illustrate the curl-free

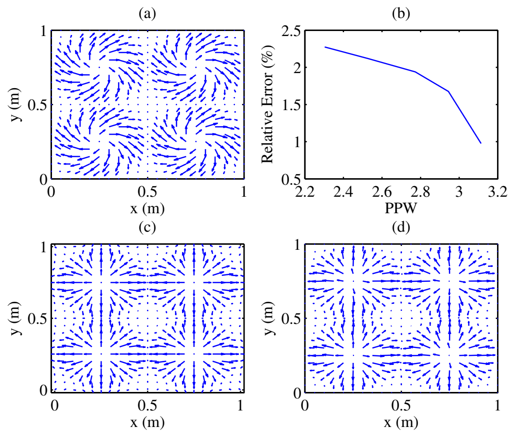

Fig. 1. HD based on the PSM and SIE method. (a) original current; (b) relative error compared with the PSM (log-log scales); (c) curl-free current calculated by the PSM; (d) curl-free current calculated by the SIE method.

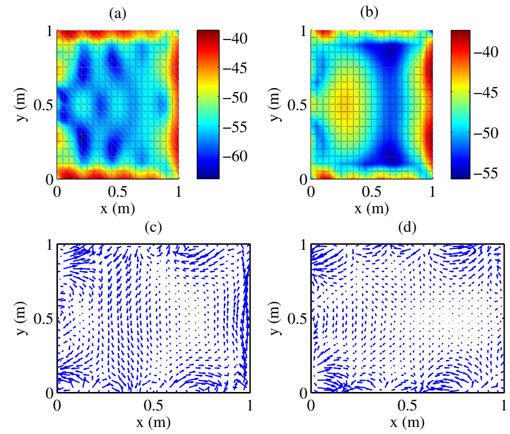

Fig. 3. The electromagnetic coupling between two PEC plates; (a) current for the single-plate case; (b) current for the double-plate case; (c),(d) divergence-free and curl-free currents for the double-plate case, respectively.

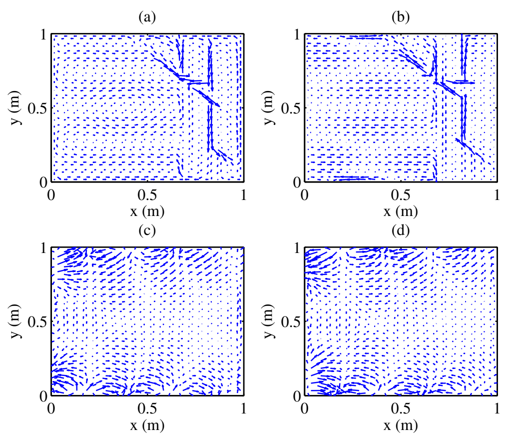

Fig. 2. Comparisons with the loop-tree decomposition. (a) loop current; (b)tree current; (c) divergence-free current; (d) curl-free current.

currents on the plate calculated by PSM and the proposed SIE method respectively.

B. Comparisons with the loop-tree decomposition

Fig. 2 shows the current results on a 3D perfectly electric conductor (PEC) plate by the loop-tree decomposition and SIE method. The plate is illuminated by an $-\hat{x}$-traveling plane wave at 650MHz. The polarization is directed at 45 degree with respect to z-axis. Fig. 3(a) shows the original current distribution. The size of the plate is around two wavelengths. Hence, the curl-free and divergence-free currents obtained by the rigorous HD show significant standing-wave patterns (Fig. 2(c) and (d)). However, the loop-tree current cannot capture this wave physics properly (Fig. 2(a) and (b)).

C. Coupling between two PEC plates

In comparison to the case B, Fig. 3 shows the changed currents induced by the electromagnetic coupling between the two PEC plates illuminated by the same plane wave as that of case B. Fig. 3(a) and (b) depict the current distributions of the same PEC plate for single-plate and double-plate cases, respectively. The current distribution is significantly modified after the PEC plate is coupled to another one. The curl-free and divergence-free currents as presented in Fig. 3(c) and (d) capture the coupling wave physics and the incremental feature compared to the uncoupled case as shown in Fig. 2(c) and (d).

IV. Conclusion

The rigorous HD for arbitrary vector fields defined on surfaces is successfully implemented based on the SIE method. It provides a precise decomposition for EM fields and can capture both circuit and wave physics. It will be employed to understand complex electromagnetic and other physical phenomena.

ACKNOWLEDGEMENT

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