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A New Method for Robust Beamforming Using Iterative Second-Order Cone Programming

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***Abstract*—This paper addresses the problem of beamforming for antenna arrays in the presence of mismatches between the true and nominal steering vectors. A new method for robust beamforming is proposed by minimizing the array output power while controlling the array mainlobe response. Due to the presence of the non-convex response constraints, a new approach based on iteratively linearizing the non-convex constraints is proposed to reformulate the non-convex problem to a series of second-order cone programming (SOCP) subproblems, each of which can be optimally solved by well-established convex optimization techniques. Simulation results show that the proposed method offers better performance than conventional methods tested.**

I. INTRODUCTION

During last decades, adaptive beamforming using antenna arrays have been successfully applied to many engineering fields including wireless communications, radar, radio astronomy, sonar, navigation, tracking, rescue and other emergency assistance devices [1]. One of the most popular beamformers is the Capon beamformer, which is developed based on an ideal antenna array with exactly known array manifold. Unfortunately, it is known that antenna arrays in real systems may suffer from kinds of uncertainties or mismatches, which considerably degrade the performance of the Capon beamformer [2]–[8].

Numerous methods have been proposed to improve the robustness of the traditional Capon beamformer during the last decades. It has been shown in [2] and [3] that additional linear or derivative constraints can be imposed achieve this purpose. More recently, additional quadratic constraints on the steering vector with known and constant radius have been considered in [4]–[7]. In these methods, the actual steering vector is assumed to lie inside a spherical uncertainty set centered at the nominal steering vector. Hence, the robust beamformer can be obtained by solving the beamforming problem constrained by a spherical uncertainty set. In some applications, it is desirable to control the response of the mainlobe to account for uncertainties in direction-of-arrival (DOA). Though the uncertainty set of larger size can be used in the conventional methods to handle large tolerable mismatch, the degradation of performance may still be unavoidable [8]. To address the problem of beamforming in the presence of steering vector mismatch, an iterative approach is developed in [9] to estimate the difference between the actual and presumed steering vectors and to use this difference to correct the erroneous presumed steering vector.

In this paper, the uncertainty set is imposed in the mainlobe region around the nominal look direction. This differs from the conventional methods that the set is imposed only in the nominal look direction. Hence, the mainlobe can be better and more flexibly controlled. However, the constraints due to these uncertainty sets are non-convex and the problem cannot be directly solved using conventional optimization techniques. To tackle this problem, a new method based on iterative second-order cone programming (SOCP) is proposed. Its basic idea is to linearize the non-convex constraints in a neighborhood of the complex array weight vector in each iteration. Therefore, the original non-convex problem is relaxed to a series of SOCP subproblems which can be optimally solved. Simulation results show that the proposed robust beamformer can achieve better performance than conventional robust beamformers tested.

II. CAPON BEAMFORMING

Consider an antenna array with M sensors impinged by K narrowband source signals $\{s_k(t)\}_{k=0}^{K-1}$ from far-field. In particular, let the first source, i.e., $s_0(t)$, be the signal of interest (SOI), and the other $K-1$ sources be undesired interferences. The array output $\boldsymbol{x}(t)$ observed at the tth snapshot can be written as

$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta_0)s_0(t) + \sum_{k=1}^{K-1}\boldsymbol{a}(\theta_k)s_k(t) + \boldsymbol{n}(t), \tag{1}$$

where θ_0 is the DOA of the SOI, $\{\theta_k\}_{k=1}^{K-1}$ are DOAs of interferences, $\boldsymbol{a}(\theta)$ is the steering vector corresponding to the angle θ, and $\boldsymbol{n}(t)$ is the sensor noise vector.

The beamformer output $y(t)$ is a linear combination of the array observation at each sensor, i.e.,

$$y(t) = \boldsymbol{w}^H\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{w}$ is the beamformer weight vector and $(\cdot)^H$ denotes the Hermitian transpose. Hence, the array output power is given by

$$E\{|y(t)|^2\} = E\{|\boldsymbol{w}^H\boldsymbol{x}(t)|^2\} = \boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w}, \tag{3}$$

where $E\{\cdot\}$ denotes the mathematical expectation. $\boldsymbol{R}_x = E\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\}$ is the array covariance matrix, which is usually estimated using a number of snapshots in real systems as $\hat{\boldsymbol{R}}_x = N^{-1}\sum_{t=1}^{N}\boldsymbol{x}(t)\boldsymbol{x}^H(t)$, where N is the total number of snapshots. We know that the Capon beamformer is obtained by minimizing the output power subject to a constraint of unity array response at the DOA of the SOI. That is

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w}\\&\text{s.t.}\quad \boldsymbol{w}^H\boldsymbol{a}(\theta_0)=1.\end{aligned} \tag{4}$$

The constraint $\boldsymbol{w}^H\boldsymbol{a}(\theta_0)=1$ prevents the gain in the DOA of the SOI from being reduced, and the solution of (4) can be easily determined using Lagrange multiplier method as:

$$\boldsymbol{w}_{Capon} = \frac{\boldsymbol{R}_x^{-1}\boldsymbol{a}(\theta_0)}{\boldsymbol{a}^H(\theta_0)\boldsymbol{R}_x^{-1}\boldsymbol{a}(\theta_0)}. \tag{5}$$

However, it is known that the performance of the Capon beamformer in (5) is sensitive to the mismatch between the nominal and true steering vectors due to the uncertainty in the DOA of the SOI as well as other array imperfections. Motivated by this problem, a new robust beamformer is proposed in this paper to tackle possible uncertainties.

III. PROPOSED ROBUST BEAMFORMING

Given an array beamformer weight vector $\boldsymbol{w}$, the array response at the angle θ is given by $\boldsymbol{w}^H\boldsymbol{a}(\theta)$. As previously mentioned, when there is no uncertainty in the steering vector, the equality constraint $\boldsymbol{w}^H\boldsymbol{a}(\theta_0)=1$ or say

$$\boldsymbol{w}^H\boldsymbol{R}_a(\theta_0)\boldsymbol{w}=1 \tag{6}$$

can be used to prevent the gain in the look direction from being reduced. In (6), $\boldsymbol{R}_a(\theta)$ is the outer product matrix of the steering vector $\boldsymbol{a}(\theta)$, i.e.,

$$\boldsymbol{R}_a(\theta)=\boldsymbol{a}(\theta)\boldsymbol{a}^H(\theta). \tag{7}$$

To deal with possible uncertainties, it has been shown in [10] that, the inequality constraint $|\boldsymbol{w}^H(\boldsymbol{a}(\theta_0)+\boldsymbol{e})|\geq 1$ can be imposed to improve the robustness, where $\boldsymbol{e}$ is the uncertainty in the steering vector $\boldsymbol{a}(\theta_0)$ and it is bounded by some known constant. To achieve a better control on the mainlobe response, especially when there is large uncertainty in the steering vector, we consider in this paper the following constraint instead:

$$\boldsymbol{w}^H(\boldsymbol{R}_a(\theta)+\Delta)\boldsymbol{w}\geq 1,\quad \theta\in\Omega \tag{8}$$

where $\Omega=[\theta_L,\theta_U]$, θ_L and θ_U are the lower and upper DOA bounds between which the SOI impinges with a high probability. Δ denotes the uncertainty in the outer product matrix $\boldsymbol{R}_a(\theta)$, which may arise from DOA mismatch and sensor gain/phase mismatch, and following the common practice, its norm is assumed to be bounded by some known constant ε:

$$\|\Delta\|\leq\varepsilon \tag{9}$$

Consequently, the proposed beamformer is given by

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w}\\&\text{s.t.}\ \ \boldsymbol{w}^H(\boldsymbol{R}_a(\theta)+\Delta)\boldsymbol{w}\geq 1,\ \ \theta\in\Omega\ \ \text{for all}\ \ \|\Delta\|\leq\varepsilon.\end{aligned} \tag{10}$$

Using similar derivation described in [4], the constraint in (10) can be further rewritten as

$$\boldsymbol{w}^H(\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I})\boldsymbol{w}\geq 1,\quad \theta\in\Omega, \tag{11}$$

where $\boldsymbol{I}$ is an $M\times M$ identity matrix. It is worth noting that the bound value of the uncertainty Δ, i.e., ε, should be appropriately chosen so as to guarantee that $\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I}$ is definite. As a result, the problem in (10) is reformulated as follows:

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w}\\&\text{s.t.}\ \ \boldsymbol{w}^H(\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I})\boldsymbol{w}\geq 1,\ \ \theta\in\Omega.\end{aligned} \tag{12}$$

It can be seen that the problem in (12) is not convex due to the presence of the non-convex constraint $\boldsymbol{w}^H(\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I})\boldsymbol{w}\geq 1$, and traditional convex optimization techniques are not directly applicable.

In this paper, we propose to solve the non-convex problem in (12) using iterative SOCP, which has been successfully applied to filter design [11] and power pattern synthesis [12] problems. First of all, we define

$$\boldsymbol{z}=\begin{bmatrix}\mathrm{Re}\{\boldsymbol{w}\}\\ \mathrm{Im}\{\boldsymbol{w}\}\end{bmatrix},$$

$$\boldsymbol{Q}_\theta=\begin{bmatrix}\mathrm{Re}\{\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I}\} & -\mathrm{Im}\{\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I}\}\\ \mathrm{Im}\{\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I}\} & \mathrm{Re}\{\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I}\}\end{bmatrix}, \text{ and}$$

$$\boldsymbol{R}=\begin{bmatrix}\mathrm{Re}\{\boldsymbol{R}_x\} & -\mathrm{Im}\{\boldsymbol{R}_x\}\\ \mathrm{Im}\{\boldsymbol{R}_x\} & \mathrm{Re}\{\boldsymbol{R}_x\}\end{bmatrix}=\boldsymbol{C}^T\boldsymbol{C},$$

where $(\cdot)^T$ denotes the transpose operation, $\boldsymbol{Q}_\theta=\boldsymbol{Q}_\theta^T$, and $\boldsymbol{R}=\boldsymbol{C}^T\boldsymbol{C}$ is Cholesky factorization of $\boldsymbol{R}$. Using these quantities, the left hand term in (11) and the objective function in (12) can be respectively written as

$$\boldsymbol{w}^H(\boldsymbol{R}_a(\theta)-\varepsilon\boldsymbol{I})\boldsymbol{w}=\boldsymbol{z}^T\boldsymbol{Q}_\theta\boldsymbol{z}\triangleq H_\theta(\boldsymbol{z}), \text{ and}$$

$$\boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w}=\boldsymbol{z}^T\boldsymbol{R}\boldsymbol{z}=\boldsymbol{z}^T\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{z}=\|\boldsymbol{C}\boldsymbol{z}\|^2.$$

Consequently, the problem in (12) can be reformulated by introducing a slack variable τ as follows:

$$\begin{aligned}&\min_{\boldsymbol{z},\tau}\ \tau\\&\text{s.t.}\ \|\boldsymbol{C}\boldsymbol{z}\|\leq\tau\\&\quad\ H_\theta(\boldsymbol{z})\geq 1,\quad \theta\in\Omega.\end{aligned} \tag{13}$$

In what follows, we shall describe the proposed iterative optimization approach for solving the non-convex problem in (13). Suppose that the algorithm starts with an initial guess

z_0 and arrives at a point z_k after k iterations. At a sufficiently small neighborhood of z_k, the smooth function $H_\theta(z)$ can be approximated by the linear approximation as:

$$H_\theta(z_k+\delta) \approx H_\theta(z_k) + g_\theta^T(z_k)\delta \,, \tag{14}$$

where $g_\theta(z)$ is the gradient of $H_\theta(z)$ with respect to z and δ is the linear update vector to be determined to satisfy (11) under the approximation in (14), and

$$g_\theta(z_k) = 2Q_\theta z_k \,. \tag{15}$$

In order to determine δ, we substitute $z = z_k + \delta$ into (13) and obtain the following problem:

$$\begin{aligned} &\min_{\delta,\tau} \ \tau \\ &\text{s.t.} \ \| Cz_k + C\delta \| \le \tau \\ &\quad H_\theta(z_k) + g_\theta^T(z_k)\delta \ge 1, \quad \theta \in \Omega \\ &\quad \| \delta \| \le \delta_{\max} \,, \end{aligned} \tag{16}$$

where the additional quadratic constraint $\| \delta \| \le \delta_{\max}$ is imposed to guarantee the linear approximation in (14) is sufficiently accurate. So far, it can be seen that the problem in (16) is convex and can be solved using SOCP by discretizing the considered region Ω as employed in [13]. More precisely, we first discretize the region Ω to a finite set Θ with N_Θ elements as

$$\Theta = \{\theta_u \mid u = 1,2,\ldots,N_\Theta\} \,. \tag{17}$$

Then, the problem in (16) can be reformulated as the following SOCP problem:

$$\begin{aligned} &\min_{\delta,\tau} \ \tau \\ &\text{s.t.} \ \| Cz_k + C\delta \| \le \tau \\ &\quad H_{\theta_u}(z_k) + g_{\theta_u}^T(z_k)\delta \ge 1, \ u = 1,2,\ldots,N_\Theta \\ &\quad \| \delta \| \le \delta_{\max} \,, \end{aligned} \tag{18}$$

in which δ can be optimally solved. Once δ is available, the new iteration can be updated using the optimally obtained δ as $z_{k+1} = z_k + \delta$. This process is repeated until the relative change of two successive solutions is sufficiently small.

It should be noted that when the robust region Ω only contains one single point, i.e., $\Omega = \theta_L = \theta_U = \theta_0$, then the problem in (10) is reduced to the following problem

$$\begin{aligned} &\min_{w} \ w^H R_x w \\ &\text{s.t.} \ w^H (R_a(\theta_0) + \Delta) w \ge 1, \quad \text{for all} \ \| \Delta \| \le \varepsilon, \end{aligned} \tag{19}$$

which is equivalent to the problem in (29) of [4] for point source case. Hence, the proposed method can be considered as a generalization of the method in [4]. Also, it has been shown in [4] that the constraint $w^H(R_a(\theta_0)+\Delta)w \ge 1$ in (19) can be equivalently replaced by $w^H(R_a(\theta_0)+\Delta)w = 1$ and further by $w^H(R_a(\theta_0)-\varepsilon I)w = 1$. As a result, the optimal solution for the above problem can be obtained in closed-form using Lagrange multiplier method.

As described earlier, the proposed robust beamformers are obtained through an iterative procedure. Hence, it is important to choose a reasonably good initial guess w_0 (or z_0) to obtain a satisfactory solution. Since the proposed method can be regarded as a generalized case of the problem in (19), its solution, denoted by $\overline{w}$, is utilized to determine the initial guess. More precisely, we first obtain $\overline{w}$ using the closed-form formula proposed in [4], and thus it satisfies $\overline{w}^H(R_a(\theta_0)-\varepsilon I)\overline{w} = 1$. Then, we choose w_0 as $w_0 = \overline{w} / \rho^{-0.5}$, where $\rho = \min_{\theta\in\Omega}\{\overline{w}^H(R_a(\theta)-\varepsilon I)\overline{w}\}$. As a result, the so obtained initial guess w_0 satisfies the constraint in (12), i.e., $w_0^H(R_a(\theta)-\varepsilon I)w_0 \ge 1, \ \theta \in \Omega$. Extensive computer simulations and those will be presented in Section IV suggest that such choice of the initial guess always converge to satisfactory solution of the non-convex problem in (12). In each case, the algorithm converges within a few iterations. Therefore, the total complexity of the proposed algorithm mainly depends on the complexity of solving each subproblem using convex optimization. Fortunately, with the dramatic increase in computing power and advanced coding techniques, it is suggested in [14] that nowadays convex optimization can almost be carried out in real-time for a modest-size problem, and the common thought on its long computational time is no longer justified.

IV. Simulation Results

A ULA with $M = 10$ sensor elements separated by half wavelength is considered to evaluate the performance of the proposed method. Two equal power interferences with an interference-to-noise ratio (INR) of 30dB impinge on the array from far-field at angles $\theta_1 = -40^\circ$ and $\theta_2 = 80^\circ$. The SOI impinges on the array from far-field at 3°, whereas the nominal direction is $\theta_0 = 0^\circ$. Hence, there is a 3° look direction mismatch. The common settings for the following examples are: the number of snapshots $N = 100$; the maximum norm of the linear update vector $\delta_{\max} = 10^{-2}$; the parameter $\varepsilon = 5$; the robust region for the proposed method is $\Omega = [-2^\circ, 2^\circ]$; and the region Ω is uniformly discretized with step size of 0.1°. The CVX Matlab Toolbox [15] is employed to solve the SOCP optimization problems.

First, the performance of the proposed method is firstly tested with a 10 dB signal-to-noise ratio (SNR) of the SOI. The resultant normalized beampattern of the proposed beamformer is depicted as solid line in Fig. 1. The conventional robust beamformer (general-rank) [4] and robust Capon beamformer [5], and the Capon beamformer in (5) are also shown for comparison. The parameter ε_0 for the robust Capon beamformer is chosen to be $\varepsilon_0 = \| a(3^\circ) - a(0^\circ) \|^2 = 6.8227$ according to a 3° look direction mismatch. We can see that all robust methods tested

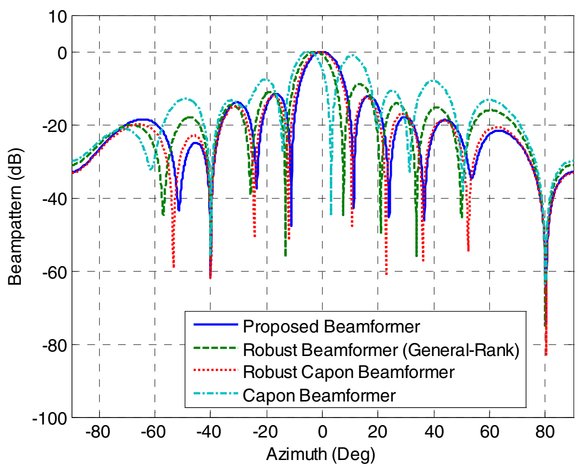

Fig.1. Normalized beampatterns of various beamformers.

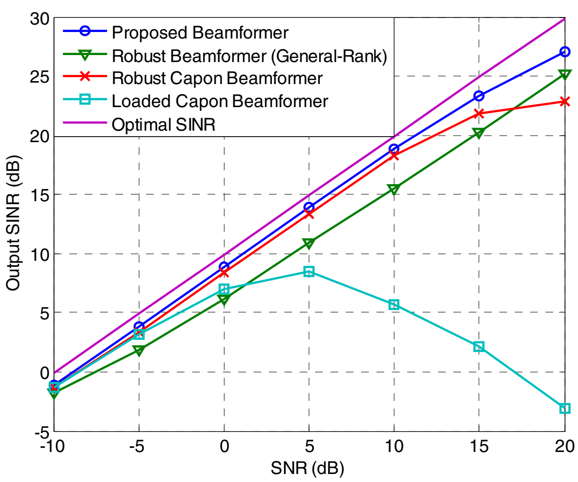

Fig.2. Output SINRs of various beamformers.

can satisfactorily reject the interferences and successfully maintain the SOI, while the Capon beamformer wrongly rejects the SOI. Next, we compare the output signal-to-interference-plus-noise ratio (SINR) of the proposed beamformer with the two aforementioned robust beamformers and the diagonally loaded Capon beamformer with a loading factor of 10 dB with respect to the noise power. The SNR of the SOI varies from –10dB to 20dB. The output SINRs of the tested beamformers are shown in Fig. 2, where each point is averaged from 100 Monte-Carlo experiments. It can be seen that the proposed beamformer outperforms others at a higher output SNR due to a better controlled mainlobe. Finally, it can be shown that the proposed method is able to offer almost identical solution of (19) as the closed-form solution in [4], when Ω only contains one single point. However, the results are omitted due to page limitation.

V. Conclusions

A new method for designing robust beamformers with better control of mainlobe response has been presented. An iterative SOCP method has been presented to solve the resultant non-convex problem. Simulation results show that the proposed method is an attractive alternative compared with conventional robust beamformers.

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