Title	Stability and dissipativity analysis of static neural networks with time delay
Author(s)	Wu, ZG; Lam, J; Su, H; Chu, J
Citation	IEEE Transactions on Neural Networks, 2012, v. 23 n. 2, p. 199-210
Issued Date	2012
URL	http://hdl.handle.net/10722/164241
Rights	IEEE Transactions on Neural Networks. Copyright © IEEE.

Stability and Dissipativity Analysis of Static Neural Networks with Time Delay

Zheng-Guang Wu, James Lam, *Senior Member, IEEE*, Hongye Su, and Jian Chu

Abstract—This paper is concerned with the problems of stability and dissipativity analysis for static neural networks (NNs) with time delay. Some improved delay-dependent stability criteria are established for static NNs with time-varying or time-invariant delay using the delay partitioning technique. Based on these criteria, several delay-dependent sufficient conditions are given to guarantee the dissipativity of static NNs with time delay. All the given results in this paper are not only dependent upon the time delay but also upon the number of delay partitions. Some examples are given to illustrate the effectiveness and reduced conservatism of the proposed results.

Index Terms—Dissipativity, stability, static neural networks, time delay.

I. INTRODUCTION

IN RECENT years, there has been increasing research interest on neural networks (NNs) for their successful application in different areas, such as pattern recognition, associate memory, and combinatorial optimization [1]. According to whether the neuron states or local field states of the neurons are chosen as basic variables to describe the evolution rule of an NN, NNs can be classified as *static* NNs or *local field* NNs [2]. The two models can be transferred equivalently from one to the other under some assumptions. However, in many applications, these assumptions cannot always be satisfied [3]. Thus, it is necessary and important to study them separately. The detailed relationship between static NNs and local field NNs can be found in [1], [2], and [4].

Recently, much attention has been paid to this paper on the stability problem of time-delay systems, because time delays are inherent features of many physical processes such as chemical reactions, nuclear reactors, and biological systems, and may lead to instability or significantly deteriorated performances for the corresponding closed-loop systems ([5]–[19] and references therein). It should be pointed out that time-delay NNs have been widely applied in many areas, such

Z.-G. Wu, H. Su, and J. Chu are with the National Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou 310027, China (e-mail: nashwzhg@126.com; hysu@iipc.zju.edu.cn; chuj@iipc.zju.edu.cn).

J. Lam is with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong (e-mail: james.lam@hku.hk).

as signal and image processing, industrial automation, system identification, and artificial intelligence. Many important theoretical results have been obtained for the stability problem of all kinds of local field NNs with time delay ([20]–[32] and references therein). In contrast, the corresponding results for static NNs with time delay are relatively few.

It should be pointed out that the neurons of static NNs form layered network configurations through only feedforward interlayered synaptic connections in terms of the neural signal flow. In general, an individual neuron aggregates its weighted inputs and yields an output through a nonlinear activation function with a threshold. As a tool for scientific computing and engineering application, an obvious characteristic of static NNs is its capability for implementing a nonlinear mapping from many neural inputs to many neural outputs [1]. The static NN model plays an important role in many types of problems, for example, the linear variational inequality problem that contains linear and convex quadratic programming problems and linear complementary problems as special cases. The stability problem for static NNs with a constant delay has been investigated in [3], and a sufficient condition is proposed for ascertaining the global asymptotic stability of the unique equilibrium of the NNs based on the linear matrix inequality (LMI) approach. However, the condition proposed in [3] is delay-independent and thus appears to be somewhat conservative, especially when the time delay is comparatively small. Based on the delay partitioning approach and Finsler's Lemma, some delay-dependent stability criteria have been established in [33] to guarantee the global asymptotic stability of static NNs with a constant delay. It should be pointed out that the idea of delay partitioning was independently proposed in [34] and [35], and has been widely applied to obtain the less conservative delay-dependent results for various kinds of time-delay systems, [24], [27], [36], [37]. For time-varying delay, the stability problem has been considered for static NNs in [38] and [39], where both delay-independent and delay-dependent criteria were obtained in terms of LMIs. However, the range of time-varying delays considered in [38] and [39] is from zero to an upper bound. In practice, a time-varying interval delay is often encountered, that is, the range of delay varies in an interval for which the lower bound is not restricted to zero. In this case, the stability criteria in [38] and [39] are conservative because they do not take into account the information of the lower bound of the delay. In [40], the stability problem for static NNs with time-varying interval delay has been studied, and a new type of delay-range-dependent condition was proposed using the free-weighting

matrix technique to obtain a tighter upper bound on the derivative of the Lyapunov–Krasovskii functional, which has less conservatism than those of [38] and [39]. The condition of [40] can be viewed as an extension of existing result on local field NNs with time delay proposed in [6] to static NNs with time delay. It is worth pointing out that the free-weighting matrix technique was originally proposed in [41] and further developed in [6], [7], and [42]. However, there still exists room for further improvement because some useful terms are ignored in the Lyapunov–Krasovskii functional employed in [40], which may lead to conservatism to some extent. Thus, it is important and necessary to further study the stability of static NNs with time delay, which is the motivation for this paper.

On the other hand, the theory of dissipative systems plays an important role in system and control areas, and thus has been attracting a great deal of attention. It has been shown that the dissipative theory gives a framework for the design and analysis of control systems using an input–output description based on energy-related considerations [43]. It serves as a powerful or even indispensable tool in characterizing important system behaviors such as stability and passivity, and has close connections with passivity theorem, the bounded real lemma, the Kalman–Yakubovich lemma, and the circle criterion. With the advancement of the LMI approach, many interesting and important results on dissipativity analysis and synthesis have been reported for different kinds of dynamic systems. For example, the problem of reliable dissipative control has been investigated in [44] for a type of stochastic hybrid systems in terms of the LMI approach, and linear state feedback controllers and impulsive controllers are designed such that, for all admissible uncertainties as well as actuator failure occurring among a prespecified subset of actuators, the stochastic hybrid system is stochastically robustly stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-dissipative. In [46], delay-dependent sufficient conditions have been established in terms of LMIs, which ensure singular systems with time delay to be admissible and dissipative. Very recently, the problem of delay-dependent dissipativity analysis has been investigated for local field NNs with distributed delay in [45], and a sufficient condition was given to guarantee the local field NN is dissipative. It should be pointed out that, up to now, very little attention has been paid to the dissipativity analysis of static NNs with time delay.

In recent years, the most common way to obtain less conservative results for time-delay systems is to make use of the free-weighting matrix method together with the delay partitioning technique [28], [42], [47]. It should be pointed out that an obvious drawback of this method is that very many free-weighting matrices are introduced with the partitioning getting thinner. However, some of them have no effect on reducing the conservatism of the obtained results. On the contrary, they mathematically complicate the system analysis and consequently lead to a significant increase in computational complexity [16]. Naturally, it is meaningful to consider how to overcome the aforementioned disadvantages of the combination of the free-weighting matrix method and the delay partitioning technique. Thus, another purpose of this paper to introduce fewer free-weighting matrices to achieve

some less conservative conditions to guarantee the delay-dependent stability and dissipativity of the addressed NNs.

In this paper, the problems of stability and dissipativity analysis are investigated for static NNs with time delay using the delay partitioning technique. First, several improved delay-dependent stability criteria are established for static NNs with time-varying or time-invariant delay. Then, several delay-dependent sufficient conditions are given to guarantee the dissipativity of these networks. All the results gien in this paper are delay-dependent as well as partition-dependent. The effectiveness as well as the reduced conservatism of the derived results is demonstrated by several illustrative examples.

Notation: The notations used throughout this paper are fairly standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices. The notation $X > Y$ $(X \geq Y)$, where X and Y are symmetric matrices, means that $X - Y$ is positive definite (positive semidefinite). I and 0 represent the identity matrix and a zero matrix, respectively. The superscript "T" represents the transpose, diag$\{\cdots\}$ stands for a block-diagonal matrix, and $L_2[0, \infty)$ stands for the space of square integrable functions on $[0, \infty)$. For an arbitrary matrix B and two symmetric matrices A and C

$$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$$

denotes a symmetric matrix, where "$*$" denotes the term that is induced by symmetry. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

II. PRELIMINARIES

Consider the following static NN with time delay:

$$\begin{cases} \dot{z}(t) = -Az(t) + f(Wz(t - d(t)) + J) \\ z(t) = \phi(t), t \in [-\max\{d(t)\}, 0] \end{cases} \tag{1}$$

where $A = \text{diag}\{a_1, a_2, \ldots, a_n\} > 0$, $W = [\hat{W}_1^T \ \hat{W}_2^T \cdots \hat{W}_n^T]^T$ is the delayed connection weight matrix, $J = [J_1 \ J_2 \ \cdots \ J_n]^T$ represents the external inputs, $z(t) = [z_1(t) \ z_2(t) \cdots z_n(t)]^T$ is the state vector associated with the n neurons, $f(Wz(t)) = [f_1(\hat{W}_1 z(t)) f_2(\hat{W}_2 z(t)) \cdots f_n(\hat{W}_n z(t))]^T$ is the activation function of neurons, $\phi(t)$ is the initial condition, and $d(t)$ is the time delay and satisfies the following three cases:

C1) time-varying delay: $0 < d_1 \leq d(t) \leq d_2, \dot{d}(t) \leq \mu$;

C2) time-varying delay: $0 < d_1 \leq d(t) \leq d_2$;

C3) time-invariant delay: $d(t) \equiv d > 0$;

where d_1, d_2, μ, and d are known real constants, and $d_2 > d_1$.

It is assumed that each activation function $f_i(\cdot)$ in (1) is bounded and satisfies

$$0 \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i, \quad s_1 \neq s_2 \in \mathbb{R} \tag{2}$$

where $l_i > 0$ are known real constants. This assumption guarantees that there is an equilibrium point u^* of the NN (1). Let $x(t) = z(t) - u^*$, $g(Wx(t)) = f(W(x(t) + u^*) + J) -$

$f(Wu^* + J)$, and $\varphi(t) = \phi(t) - u^*$. Then NN (1) can be expressed as

$$
\begin{cases}
\dot{x}(t) = -Ax(t) + g(Wx(t - d(t))) \\
x(t) = \varphi(t), \ t \in [-\max\{d(t)\}, 0].
\end{cases} \tag{3}
$$

It can be shown that the activation function $g_i(\cdot)$ is bounded and satisfies

$$
0 \leq \frac{g_i(s)}{s} \leq l_i, \quad g_i(0) = 0. \tag{4}
$$

Remark 1: The NN (3) is called a *static* NN. It is interesting that, when the delayed connection weight matrix W is nonsingular and $WA = AW$, by defining $\hat{y}(t) = Wx(t)$, NN (3) can be transformed into

$$
\dot{\hat{y}}(t) = -A\hat{y}(t) + Wg(\hat{y}(t - d(t)))
$$

which is called a *local field* NN, which has been extensively studied in the literature. However, most static NNs do not satisfy the transformation condition. It is thus necessary and important to study such NNs.

One aim in this paper is to investigate the stability of the static NN (3) and establish some new delay-dependent stability criteria that are better than the existing ones.

When an external disturbance appears in the NN (3), we have the following NN:

$$
\begin{cases}
\dot{x}(t) = -Ax(t) + g(Wx(t - d(t))) + \omega(t) \\
y(t) = g(Wx(t)) \\
x(t) = \varphi(t), \ t \in [-\max\{d(t)\}, 0]
\end{cases} \tag{5}
$$

where $y(t)$ is the output of the NN, and $\omega(t)$ is the disturbance input which cannot be fully measured and is not completely known beforehand. In this paper, it is assumed that $\omega(t) \in L_2[0, \infty)$, which implies that it is a function of finite energy.

We are now in a position to introduce the definition on dissipativity. Let the energy supply function of the NN (5) be defined by

$$
G(\omega, y, \tau) = \langle y, \mathcal{Q}y \rangle_\tau + 2\langle y, \mathcal{S}\omega \rangle_\tau + \langle \omega, \mathcal{R}\omega \rangle_\tau \quad \forall \tau \geq 0 \tag{6}
$$

where $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{R}$ are real matrices with $\mathcal{Q}$, $\mathcal{R}$ symmetric, and $\langle a, b \rangle_\tau = \int_0^\tau a^T b \, dt$. Without loss of generality, it is assumed that $\mathcal{Q} \leq 0$ and that $-\mathcal{Q} = \mathcal{Q}_-^T \mathcal{Q}_-$ for some $\mathcal{Q}_-$.

Definition 1: NN (5) is said to be strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative if, for some scalar $\gamma > 0$, the inequality

$$
G(\omega, y, \tau) \geq \gamma \langle \omega, \omega \rangle_\tau \quad \forall \tau \geq 0 \tag{7}
$$

holds under zero initial condition for any nonzero disturbance $\omega \in L_2[0, \infty)$.

Another aim of this paper is to establish delay-dependent condition such that the NN (5) is globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative.

Before moving on, the following results are required.

Lemma 1 (Jensen Inequality) [13], [48]: For any matrix $W > 0$, scalars γ_1 and γ_2 satisfying $\gamma_2 > \gamma_1$, a vector

function $\omega: [\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n$ such that the integrations $\int_{\gamma_1}^{\gamma_2} \omega(\alpha)^T W\omega(\alpha) \, d\alpha$ and $\int_{\gamma_1}^{\gamma_2} \omega(\alpha) \, d\alpha$ are well defined, then

$$
(\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega(\alpha)^T W\omega(\alpha) \, d\alpha
$$

$$
\geq \left[\int_{\gamma_1}^{\gamma_2} \omega(\alpha) \, d\alpha \right]^T W \left[\int_{\gamma_1}^{\gamma_2} \omega(\alpha) \, d\alpha \right]. \tag{8}
$$

Lemma 2 [49]: For any matrix $\begin{bmatrix} M & S \\ * & M \end{bmatrix} \geq 0$, scalars d_1, d_2, $d(t)$ satisfying $d_1 \leq d(t) \leq d_2$, vector function $\dot{x}(t + \cdot): [-d_2, -d_1] \rightarrow \mathbb{R}^n$ such that the integrations $\int_{t-d_2}^{t-d_1} \dot{x}(\alpha)^T M\dot{x}(\alpha) \, d\alpha$, $\int_{t-d(t)}^{t-d_1} \dot{x}(\alpha) \, d\alpha$, and $\int_{t-d_2}^{t-d(t)} \dot{x}(\alpha) \, d\alpha$ are well defined, then

$$
-\hat{d} \int_{t-d_2}^{t-d_1} \dot{x}(\alpha)^T M\dot{x}(\alpha) \, d\alpha \leq \varpi(t)^T \Omega \varpi(t) \tag{9}
$$

where $\hat{d} = d_2 - d_1$ and

$$
\varpi(t) = \begin{bmatrix} x(t - d_1)^T & x(t - d(t))^T & x(t - d_2)^T \end{bmatrix}^T
$$

$$
\Omega = \begin{bmatrix} -M & M - S & S \\ * & -2M + S + S^T & -S + M \\ * & * & -M \end{bmatrix}.
$$

Remark 2: Lemma 2 is a special case of [49, Th. 1], which is presented in a form more convenient for the present application.

III. STABILITY ANALYSIS

In this section, the delay partitioning technique will be developed to investigate the stability problem for the NN (3). For convenience of presentation, we denote

$$
\eta_1(t) = \begin{bmatrix} x(t) \\ x\left(t - \frac{1}{m}d_1\right) \\ x\left(t - \frac{2}{m}d_1\right) \\ \vdots \\ x\left(t - \frac{m-1}{m}d_1\right) \end{bmatrix}, \ \eta_2(t) = \begin{bmatrix} g(Wx(t)) \\ g\left(Wx\left(t - \frac{1}{m}d_1\right)\right) \\ g\left(Wx\left(t - \frac{2}{m}d_1\right)\right) \\ \vdots \\ g(Wx(t - \frac{m-1}{m}d_1)) \end{bmatrix}
$$

$$
\theta_1(t) = \begin{bmatrix} \eta_1(t) \\ x(t - d_1) \end{bmatrix}, \quad \theta_2(t) = \begin{bmatrix} \eta_2(t) \\ g(Wx(t - d_1)) \end{bmatrix}
$$

$$
L = \text{diag}\{l_1, l_2, \ldots, l_n\}
$$

$$
W_1 = \begin{bmatrix} I_{mn} & 0_{mn \times n} \end{bmatrix}
$$

$$
W_2 = \begin{bmatrix} 0_{mn \times n} & I_{mn} \end{bmatrix}
$$

$$
e_\iota = \begin{bmatrix} 0_{n \times (\iota - 1)n} & I_n & 0_{n \times (m+1-\iota)n} \end{bmatrix}, \quad \iota = 1, 2, \ldots, m + 1
$$

and $\hat{\eta}_1(t) = \eta_1(t)|_{d_1 = d}$, $\hat{\eta}_2(t) = \eta_2(t)|_{d_1 = d}$, $\hat{\theta}_1(t) = \theta_1(t)|_{d_1 = d}$, $\hat{\theta}_2(t) = \theta_2(t)|_{d_1 = d}$, and $\tilde{d} = d_2 - d_1$.

Theorem 1: Given an integer $m > 0$, the NN (3) with C1 is globally asymptotically stable if there exist matrices $P > 0$, $Z_i > 0 \ (i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} > 0$, $\begin{bmatrix} Z_{m+1} & U_3 \\ * & Z_{m+1} \end{bmatrix} \geq 0$, and diagonal matrices

$S = \text{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \text{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m+3)$ such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & 0 \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} \\ * & * & * & \Xi_{44} & \Xi_{45} & 0 \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0 \qquad (10)$$

where

$$\Xi_{11} = -e_1^T P A e_1 - e_1^T A P e_1 + W_1^T Q_1 W_1 - W_2^T Q_1 W_2$$
$$+ \left(\frac{d_1}{m}\right)^2 e_1^T A \left(\sum_{i=1}^m Z_i\right) A e_1 + e_1^T Y_1 e_1 + e_1^T Y_3 e_1$$
$$- \sum_{i=1}^m (e_i - e_{i+1})^T Z_i (e_i - e_{i+1}) + \tilde{d}^2 e_1^T A Z_{m+1} A e_1$$
$$- e_{m+1}^T Z_{m+1} e_{m+1} - e_1^T W^T L R W A e_1$$
$$- e_1^T A W^T R L W e_1$$

$$\Xi_{12} = e_{m+1}^T Z_{m+1} - e_{m+1}^T U_3$$
$$\Xi_{13} = e_{m+1}^T U_3$$
$$\Xi_{14} = -e_1^T A W^T S e_1 + W_1^T V W_1 - W_2^T V W_2 + e_1^T U_1 e_1$$
$$+ e_1^T U_2 e_1 + \sum_{i=1}^{m+1} e_i^T W^T L D_i e_i + e_1^T A W^T R e_1$$
$$\Xi_{15} = e_1^T P - \left(\frac{d_1}{m}\right)^2 e_1^T A \left(\sum_{i=1}^m Z_i\right) - \tilde{d}^2 e_1^T A Z_{m+1}$$
$$+ e_1^T W^T L R W$$
$$\Xi_{22} = -(1-\mu) Y_1 - 2 Z_{m+1} + U_3 + U_3^T$$
$$\Xi_{23} = Z_{m+1} - U_3$$
$$\Xi_{25} = -(1-\mu) U_1 + W^T L D_{m+2}$$
$$\Xi_{33} = -Y_3 - Z_{m+1}$$
$$\Xi_{36} = -U_2 + W^T L D_{m+3}$$
$$\Xi_{44} = W_1^T Q_2 W_1 - W_2^T Q_2 W_2 + e_1^T Y_2 e_1 + e_1^T Y_4 e_1$$
$$- 2 \sum_{i=1}^{m+1} e_i^T D_i e_i$$
$$\Xi_{45} = e_1^T S W - e_1^T R W$$
$$\Xi_{55} = \left(\frac{d_1}{m}\right)^2 \sum_{i=1}^m Z_i - (1-\mu) Y_2 + \tilde{d}^2 Z_{m+1} - 2 D_{m+2}$$
$$\Xi_{66} = -Y_4 - 2 D_{m+3}.$$

Proof: Construct the following Lyapunov–Krasovskii functional candidate for NN (3):

$$V(x(t)) = \sum_{i=1}^6 V_i(x(t)) \qquad (11)$$

where

$$V_1(x(t)) = x(t)^T P x(t) + 2 \sum_{i=1}^n s_i \int_0^{\hat{W}_i x(t)} g_i(s)\, ds$$
$$+ 2 \sum_{i=1}^n r_i \int_0^{\hat{W}_i x(t)} (l_i s - g_i(s))\, ds$$

$$V_2(x(t)) = \int_{t-\frac{d_1}{m}}^t \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix} ds$$

$$V_3(x(t)) = \frac{d_1}{m} \sum_{i=1}^m \int_{-\frac{i}{m}d_1}^{-\frac{i-1}{m}d_1} \int_{t+\alpha}^t \dot{x}(s)^T Z_i \dot{x}(s)\, ds\, d\alpha$$

$$V_4(x(t)) = \int_{t-d(t)}^t \begin{bmatrix} x(s) \\ g(Wx(s)) \end{bmatrix}^T \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} \begin{bmatrix} x(s) \\ g(Wx(s)) \end{bmatrix} ds$$

$$V_5(x(t)) = \int_{t-d_2}^t \begin{bmatrix} x(s) \\ g(Wx(s)) \end{bmatrix}^T \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} x(s) \\ g(Wx(s)) \end{bmatrix} ds$$

$$V_6(x(t)) = \tilde{d} \int_{-d_2}^{-d_1} \int_{t+\alpha}^t \dot{x}(s)^T Z_{m+1} \dot{x}(s)\, ds\, d\alpha.$$

Evaluating the derivative of $V(x(t))$ along the solution of NN (3), we obtain

$$\dot{V}_1(x(t)) = 2x(t)^T P \dot{x}(t) + 2g(Wx(t))^T S W \dot{x}(t)$$
$$+ 2(LWx(t) - g(Wx(t)))^T R W \dot{x}(t)$$
$$= 2\theta_1(t)^T e_1^T P(-Ae_1\theta_1(t) + g(Wx(t-d(t))))$$
$$+ 2\theta_2(t)^T e_1^T S W(-Ae_1\theta_1(t) + g(Wx(t-d(t))))$$
$$+ 2(LWe_1\theta_1(t) - e_1\theta_2(t))^T R W$$
$$\times (-Ae_1\theta_1(t) + g(Wx(t-d(t))))$$
$$= -2\theta_1(t)^T e_1^T P A e_1\theta_1(t) - 2\theta_2(t)^T e_1^T S W A e_1\theta_1(t)$$
$$+ 2\theta_1(t)^T e_1^T P g(Wx(t-d(t)))$$
$$+ 2\theta_2(t)^T e_1^T S W g(Wx(t-d(t)))$$
$$- 2\theta_1(t)^T e_1^T W^T L R W A e_1\theta_1(t)$$
$$+ 2\theta_2(t)^T e_1^T R W A e_1\theta_1(t)$$
$$+ 2\theta_1(t)^T e_1^T W^T L R W g(Wx(t-d(t)))$$
$$- 2\theta_2(t)^T e_1^T R W g(Wx(t-d(t))) \qquad (12)$$

$$\dot{V}_2(x(t)) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$$
$$- \begin{bmatrix} \eta_1\left(t-\frac{d_1}{m}\right) \\ \eta_2\left(t-\frac{d_1}{m}\right) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1\left(t-\frac{d_1}{m}\right) \\ \eta_2\left(t-\frac{d_1}{m}\right) \end{bmatrix}$$
$$= \begin{bmatrix} W_1\theta_1(t) \\ W_1\theta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_1\theta_1(t) \\ W_1\theta_2(t) \end{bmatrix}$$
$$- \begin{bmatrix} W_2\theta_1(t) \\ W_2\theta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_2\theta_1(t) \\ W_2\theta_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}^T$$
$$\times \begin{bmatrix} W_1^T Q_1 W_1 - W_2^T Q_1 W_2 & W_1^T V W_1 - W_2^T V W_2 \\ * & W_1^T Q_2 W_1 - W_2^T Q_2 W_2 \end{bmatrix}$$
$$\times \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} \qquad (13)$$

$$\dot{V}_3(x(t)) = \left(\frac{d_1}{m}\right)^2 \dot{x}(t)^T \left(\sum_{i=1}^m Z_i\right) \dot{x}(t)$$
$$- \frac{d_1}{m} \sum_{i=1}^m \int_{t-\frac{i}{m}d_1}^{t-\frac{i-1}{m}d_1} \dot{x}(s)^T Z_i \dot{x}(s)\, ds$$
$$\leq \left(\frac{d_1}{m}\right)^2 \theta_1(t)^T e_1^T A \left(\sum_{i=1}^m Z_i\right) A e_1\theta_1(t)$$

$$-2\left(\frac{d_1}{m}\right)^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) g(Wx(t-d(t)))$$

$$+\left(\frac{d_1}{m}\right)^2 g(Wx(t-d(t)))^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) g(Wx(t-d(t)))$$

$$-\sum_{i=1}^{m} \int_{t-\frac{i}{m}d_1}^{t-\frac{i-1}{m}d_1} \dot{x}(s)^{\mathrm{T}} ds\, Z_i \int_{t-\frac{i}{m}d_1}^{t-\frac{i-1}{m}d_1} \dot{x}(s)\, ds$$

$$=\left(\frac{d_1}{m}\right)^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) A e_1 \theta_1(t)$$

$$-2\left(\frac{d_1}{m}\right)^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) g(Wx(t-d(t)))$$

$$+\left(\frac{d_1}{m}\right)^2 g(Wx(t-d(t)))^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) g(Wx(t-d(t)))$$

$$-\sum_{i=1}^{m} \theta_1(t)^{\mathrm{T}} (e_i - e_{i+1})^{\mathrm{T}} Z_i (e_i - e_{i+1}) \theta_1(t) \quad (14)$$

where Lemma 1 is applied

$$\dot{V}_4(x(t)) \leq \begin{bmatrix} x(t) \\ g(Wx(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} \begin{bmatrix} x(t) \\ g(Wx(t)) \end{bmatrix}$$

$$-(1-\mu) \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix}$$

$$\times \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix}$$

$$= \begin{bmatrix} e_1\theta_1(t) \\ e_1\theta_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} \begin{bmatrix} e_1\theta_1(t) \\ e_1\theta_2(t) \end{bmatrix}$$

$$-(1-\mu) \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix}$$

$$\times \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} e_1^{\mathrm{T}} Y_1 e_1 & e_1^{\mathrm{T}} U_1 e_1 \\ * & e_1^{\mathrm{T}} Y_2 e_1 \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}$$

$$-(1-\mu) \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix}$$

$$\times \begin{bmatrix} x(t-d(t)) \\ g(Wx(t-d(t))) \end{bmatrix} \quad (15)$$

$$\dot{V}_5(x(t)) = \begin{bmatrix} x(t) \\ g(Wx(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} x(t) \\ g(Wx(t)) \end{bmatrix}$$

$$-\begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix}$$

$$= \begin{bmatrix} e_1\theta_1(t) \\ e_1\theta_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} e_1\theta_1(t) \\ e_1\theta_2(t) \end{bmatrix}$$

$$-\begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} e_1^{\mathrm{T}} Y_3 e_1 & e_1^{\mathrm{T}} U_2 e_1 \\ * & e_1^{\mathrm{T}} Y_4 e_1 \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}$$

$$-\begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} \begin{bmatrix} x(t-d_2) \\ g(Wx(t-d_2)) \end{bmatrix} \quad (16)$$

$$\dot{V}_6(x(t)) = \tilde{d}^2 \dot{x}(t)^{\mathrm{T}} Z_{m+1} \dot{x}(t) - \tilde{d} \int_{t-d_2}^{t-d_1} \dot{x}(s)^{\mathrm{T}} Z_{m+1} \dot{x}(s)\, ds$$

$$\leq \tilde{d}^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A Z_{m+1} A e_1 \theta_1(t)$$

$$-2\tilde{d}^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A Z_{m+1} g(Wx(t-d(t)))$$

$$+\tilde{d}^2 g(Wx(t-d(t)))^{\mathrm{T}} Z_{m+1} g(Wx(t-d(t)))$$

$$-\begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix}^{\mathrm{T}}$$

$$\times \begin{bmatrix} Z_{m+1} & -Z_{m+1}+U_3 & -U_3 \\ * & 2Z_{m+1}-U_3-U_3^{\mathrm{T}} & U_3-Z_{m+1} \\ * & * & Z_{m+1} \end{bmatrix}$$

$$\times \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix}$$

$$\leq \tilde{d}^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A Z_{m+1} A e_1 \theta_1(t)$$

$$-2\tilde{d}^2 \theta_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A Z_{m+1} g(Wx(t-d(t)))$$

$$+\tilde{d}^2 g(Wx(t-d(t)))^{\mathrm{T}} Z_{m+1} g(Wx(t-d(t)))$$

$$-\begin{bmatrix} \theta_1(t) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} e_{m+1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}}$$

$$\times \begin{bmatrix} Z_{m+1} & -Z_{m+1}+U_3 & -U_3 \\ * & 2Z_{m+1}-U_3-U_3^{\mathrm{T}} & U_3-Z_{m+1} \\ * & * & Z_{m+1} \end{bmatrix}$$

$$\times \begin{bmatrix} e_{m+1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix} \quad (17)$$

where Lemma 2 is used. On the other hand, by (4), we have

$$2\sum_{i=1}^{m+1} g\left(Wx\left(t-\frac{i-1}{m}d_1\right)\right)^{\mathrm{T}} D_i$$

$$\times \left(LWx\left(t-\frac{i-1}{m}d_1\right) - g\left(Wx\left(t-\frac{i-1}{m}d_1\right)\right)\right) \geq 0. \quad (18)$$

That is

$$\Upsilon_1 = 2\sum_{i=1}^{m+1} \theta_2(t)^{\mathrm{T}} e_i^{\mathrm{T}} D_i \left(LWe_i\theta_1(t) - e_i\theta_2(t)\right) \geq 0. \quad (19)$$

We can also get from (4)

$$\Upsilon_2 = 2g(Wx(t-d(t)))^{\mathrm{T}} D_{m+2}$$

$$\times \left(LWx(t-d(t)) - g(Wx(t-d(t)))\right) \geq 0 \quad (20)$$

and

$$\Upsilon_3 = 2g(Wx(t-d_2))^{\mathrm{T}} D_{m+3}$$

$$\times \left(LWx(t-d_2) - g(Wx(t-d_2))\right) \geq 0. \quad (21)$$

Thus, we have from (12)–(17) and (19)–(21)

$$\dot{V}(x(t)) \leq \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t))$$

$$+\dot{V}_5(x(t)) + \dot{V}_6(x(t)) + \Upsilon_1 + \Upsilon_2 + \Upsilon_3$$

$$\leq \rho(t)^{\mathrm{T}} \Xi \rho(t) \quad (22)$$

where

$$\rho(t) = \begin{bmatrix} \rho_1(t)^{\mathrm{T}} & \rho_2(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\rho_1(t) = \begin{bmatrix} \theta_1(t)^{\mathrm{T}} & x(t-d(t))^{\mathrm{T}} & x(t-d_2)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\rho_2(t) = \begin{bmatrix} \theta_2(t)^{\mathrm{T}} & g(Wx(t-d(t)))^{\mathrm{T}} & g(Wx(t-d_2))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Clearly, if $\Xi < 0$, $\dot{V}(x(t)) < 0$ holds, which implies that NN (3) is globally asymptotically stable. This completes the proof. ∎

Remark 3: In terms of the delay partitioning technique, a new delay-range-dependent and delay-derivative-dependent sufficient condition is proposed in Theorem 1 for the global asymptotical stability of NN (3) with time-varying interval delay. When $r_i = 0$, $V = Q_2 = 0$, $Q_1 = I_m \otimes \hat{Q}_1$, $Z_i = (m/d_1)\hat{Z}_1$, $U_1 = Y_4 = U_2 = 0$, and $Z_{m+1} = d_{12}^{-1}(\hat{Z}_1 + \hat{Z}_2)$, the Lyapunov–Krasovskii functional candidate (11) reduces to that of [40]. Thus, our Lyapunov–Krasovskii functional candidate is much more general than that of [40], and Theorem 1 in this paper has less conservatism than [40, Th. 2]. Moreover, the conservatism reduction of Theorem 1 in this paper becomes more obvious with the partitioning getting thinner (i.e., m becoming larger), which will be demonstrated in Section V.

Remark 4: It is worth pointing out that, if we utilize the free-weighting matrix method together with the delay partitioning technique to deal with the delay-dependent stability problem of NN (3), a great many free-weighting matrices will be introduced with increasing number of partitions, which will make the obtained result rather complicated and consequently lead to computational burden [50]. But in this paper, we make use of integral inequalities (8) and (9) instead of the free-weighting matrix method. An obvious and important merit of such an approach is that only one matrix is introduced no matter how large the partitioning is, and thus we can achieve a condition with fewer decision variables.

As for C2, we can get the following stability criterion for NN (3) via the following Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_5(x(t)) + V_6(x(t)) \tag{23}$$

where $V_1(x(t))$, $V_2(x(t))$, $V_3(x(t))$, $V_5(x(t))$, and $V_6(x(t))$ follow the same definitions as those in (11).

Theorem 2: Given an integer $m > 0$, NN (3) with C2 is globally asymptotically stable if there exist matrices $P > 0$, $Z_i > 0$ $(i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} > 0$, $\begin{bmatrix} Z_{m+1} & U_3 \\ * & Z_{m+1} \end{bmatrix} \geq 0$, and diagonal matrices $S = \mathrm{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \mathrm{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m + 3)$ such that $(10)|_{Y_1 = Y_2 = U_1 = 0}$ holds.

Proof: By using a similar method as employed in Theorem 1, we can easily obtain Theorem 2. This completes the proof. ∎

Remark 5: Theorem 2 proposes a delay-range-dependent and delay-derivative-independent sufficient condition for the global asymptotical stability of NN (3) with time-varying interval delay. It should be pointed out that, when $\mu \geq 1$, $\begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix}$ will no longer be helpful to improve the conservatism

of Theorem 1, since $-(1 - \mu)\begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} \geq 0$. Therefore, Theorem 1 with $\mu \geq 1$ is equivalent to Theorem 2.

As for C3, we can get the stability criterion for NN (3) via the following Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \tag{24}$$

where $V_1(x(t))$ follows the same definition as that in (11) and

$$V_2(x(t)) = \int_{t-\frac{d}{m}}^{t} \begin{bmatrix} \hat{\eta}_1(s) \\ \hat{\eta}_2(s) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(s) \\ \hat{\eta}_2(s) \end{bmatrix} ds$$

$$V_3(x(t)) = \frac{d}{m} \sum_{i=1}^{m} \int_{-\frac{i}{m}d}^{-\frac{i-1}{m}d} \int_{t+\alpha}^{t} \dot{x}(s)^{\mathrm{T}} Z_i \dot{x}(s)\, ds\, d\alpha.$$

Theorem 3: Given an integer $m > 0$, NN (3) with C3 is globally asymptotically stable if there exist matrices $P > 0$, $Z_i > 0$ $(i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, and diagonal matrices $S = \mathrm{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \mathrm{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m + 1)$ such that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0 \tag{25}$$

where

$$\begin{aligned}
\Psi_{11} ={}& -e_1^{\mathrm{T}} P A e_1 - e_1^{\mathrm{T}} A P e_1 + W_1^{\mathrm{T}} Q_1 W_1 - W_2^{\mathrm{T}} Q_1 W_2 \\
&+ \left(\frac{d}{m}\right)^2 e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) A e_1 \\
&- \sum_{i=1}^{m} (e_i - e_{i+1})^{\mathrm{T}} Z_i (e_i - e_{i+1}) \\
&- e_1^{\mathrm{T}} W^{\mathrm{T}} L R W A e_1 - e_1^{\mathrm{T}} A W^{\mathrm{T}} R L W e_1 \\
\Psi_{12} ={}& -e_1^{\mathrm{T}} A W^{\mathrm{T}} S e_1 + W_1^{\mathrm{T}} V W_1 - W_2^{\mathrm{T}} V W_2 \\
&+ \sum_{i=1}^{m+1} e_i^{\mathrm{T}} W^{\mathrm{T}} L D_i e_i + e_1^{\mathrm{T}} A W^{\mathrm{T}} R e_1 \\
&- \left(\frac{d}{m}\right)^2 e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) e_{m+1} \\
&+ e_1^{\mathrm{T}} P e_{m+1} + e_1^{\mathrm{T}} W^{\mathrm{T}} L R W e_{m+1} \\
\Psi_{22} ={}& W_1^{\mathrm{T}} Q_2 W_1 - W_2^{\mathrm{T}} Q_2 W_2 + e_{m+1}^{\mathrm{T}} W^{\mathrm{T}} S e_1 \\
&- 2 \sum_{i=1}^{m+1} e_i^{\mathrm{T}} D_i e_i + e_1^{\mathrm{T}} S W e_{m+1} - e_{m+1}^{\mathrm{T}} W^{\mathrm{T}} R e_1 \\
&+ \left(\frac{d}{m}\right)^2 e_{m+1}^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) e_{m+1} - e_1^{\mathrm{T}} R W e_{m+1}.
\end{aligned}$$

Proof: Evaluating the derivative of $V(x(t))$ along the solution of NN (3), we obtain

$$\begin{aligned}
\dot{V}_1 &(x(t)) \\
={}& 2x(t)^{\mathrm{T}} P \dot{x}(t) + 2g(Wx(t))^{\mathrm{T}} S W \dot{x}(t) \\
&+ 2(LWx(t) - g(Wx(t)))^{\mathrm{T}} R W \dot{x}(t) \\
={}& -2\hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} P A e_1 \hat{\theta}_1(t) - 2\hat{\theta}_2(t)^{\mathrm{T}} e_1^{\mathrm{T}} S W A e_1 \hat{\theta}_1(t) \\
&+ 2\hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} P e_{m+1} \hat{\theta}_2(t) \\
&+ 2\hat{\theta}_2(t)^{\mathrm{T}} e_1^{\mathrm{T}} S W e_{m+1} \hat{\theta}_2(t) \\
&- 2\hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} W^{\mathrm{T}} L R W A e_1 \hat{\theta}_1(t) \\
&+ 2\hat{\theta}_2(t)^{\mathrm{T}} e_1^{\mathrm{T}} R W A e_1 \hat{\theta}_1(t)
\end{aligned}$$

$$+ 2\hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} W^{\mathrm{T}} L R W e_{m+1} \hat{\theta}_2(t)$$
$$- 2\hat{\theta}_2(t)^{\mathrm{T}} e_1^{\mathrm{T}} R W e_{m+1} \hat{\theta}_2(t) \tag{26}$$

$$\dot{V}_2(x(t)) = \begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix}$$
$$- \begin{bmatrix} \hat{\eta}_1(t - \frac{d}{m}) \\ \hat{\eta}_2(t - \frac{d}{m}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(t - \frac{d}{m}) \\ \hat{\eta}_2(t - \frac{d}{m}) \end{bmatrix}$$
$$= \begin{bmatrix} \hat{\theta}_1(t) \\ \hat{\theta}_2(t) \end{bmatrix}^{\mathrm{T}}$$
$$\times \begin{bmatrix} W_1^{\mathrm{T}} Q_1 W_1 - W_2^{\mathrm{T}} Q_1 W_2 & W_1^{\mathrm{T}} V W_1 - W_2^{\mathrm{T}} V W_2 \\ * & W_1^{\mathrm{T}} Q_2 W_1 - W_2^{\mathrm{T}} Q_2 W_2 \end{bmatrix}$$
$$\times \begin{bmatrix} \hat{\theta}_1(t) \\ \hat{\theta}_2(t) \end{bmatrix} \tag{27}$$

$$\dot{V}_3(x(t)) = \left(\frac{d}{m}\right)^2 \dot{x}(t)^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) \dot{x}(t)$$
$$- \frac{d}{m} \sum_{i=1}^{m} \int_{t - \frac{i}{m} d}^{t - \frac{i-1}{m} d} \dot{x}(s)^{\mathrm{T}} Z_i \dot{x}(s)\, ds$$
$$\leq \left(\frac{d}{m}\right)^2 \hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) A e_1 \hat{\theta}_1(t)$$
$$- 2\left(\frac{d}{m}\right)^2 \hat{\theta}_1(t)^{\mathrm{T}} e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right) e_{m+1} \hat{\theta}_2(t)$$
$$+ \left(\frac{d}{m}\right)^2 \hat{\theta}_2(t)^{\mathrm{T}} e_{m+1}^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) e_{m+1} \hat{\theta}_2(t)$$
$$- \sum_{i=1}^{m} \hat{\theta}_1(t)^{\mathrm{T}} (e_i - e_{i+1})^{\mathrm{T}} Z_i (e_i - e_{i+1}) \hat{\theta}_1(t) \tag{28}$$

where Lemma 1 is applied. Thus, we have from (26)–(28) and (19)$|_{d_1 = d}$

$$\dot{V}(x(t)) \leq \begin{bmatrix} \hat{\theta}_1(t) \\ \hat{\theta}_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1(t) \\ \hat{\theta}_2(t) \end{bmatrix}. \tag{29}$$

Clearly, if (25) holds, $\dot{V}(x(t)) < 0$ holds, which implies NN (3) is globally asymptotically stable. This completes the proof. ∎

Remark 6: Theorem 3 presents a new delay-dependent stability criterion for NN (3) with constant delay by employing the Lyapunov–Krasovskii functional candidate (24). When $r_i = 0$, $V = 0$, $Q_1 = \mathrm{diag}\{\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_m\}$, $Q_2 = I_m \otimes \hat{Q}_{m+1}$, and $Z_i = (m/d)\hat{Z}$, the Lyapunov–Krasovskii functional candidate (24) reduces to that applied in [33, Corollary 4]. Thus, our Lyapunov–Krasovskii functional candidate is much more general than that of [33], and Theorem 3 in this paper improves [33, Corollary 4].

IV. DISSIPATIVITY ANALYSIS

In this section, we will give the results on dissipativity analysis for NN (5) based on the stability conditions that were proposed in Section III.

Theorem 4: Given a scalar $\gamma > 0$ and an integer $m > 0$, NN (5) with C1 is globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative if there exist matrices $P > 0$,

$Z_i > 0$ $(i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_1 & U_1 \\ * & Y_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} > 0$, $\begin{bmatrix} Z_{m+1} & U_3 \\ * & Z_{m+1} \end{bmatrix} \geq 0$, and diagonal matrices $S = \mathrm{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \mathrm{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m+3)$ such that

$$\hat{\Xi} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & 0 & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & 0 \\ * & * & * & \hat{\Xi}_{44} & \Xi_{45} & 0 & \Xi_{47} \\ * & * & * & * & \Xi_{55} & 0 & \Xi_{57} \\ * & * & * & * & * & \Xi_{66} & 0 \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0 \tag{30}$$

where Ξ_{11}, Ξ_{12}, Ξ_{13}, Ξ_{14}, Ξ_{15}, Ξ_{22}, Ξ_{23}, Ξ_{25}, Ξ_{33}, Ξ_{36}, Ξ_{45}, Ξ_{55}, and Ξ_{66} follow the same definitions as those in Theorem 1, and

$$\hat{\Xi}_{44} = W_1^{\mathrm{T}} Q_2 W_1 - W_2^{\mathrm{T}} Q_2 W_2 + e_1^{\mathrm{T}} Y_2 e_1 + e_1^{\mathrm{T}} Y_4 e_1$$
$$- 2\sum_{i=1}^{m+1} e_i^{\mathrm{T}} D_i e_i - e_1^{\mathrm{T}} \mathcal{Q} e_1$$

$$\Xi_{17} = e_1^{\mathrm{T}} P + e_1^{\mathrm{T}} W^{\mathrm{T}} L R W - \left(\frac{d_1}{m}\right)^2 e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right)$$
$$- \tilde{d}^2 e_1^{\mathrm{T}} A Z_{m+1}$$

$$\Xi_{47} = e_1^{\mathrm{T}} S W - e_1^{\mathrm{T}} R W - e_1^{\mathrm{T}} \mathcal{S}$$

$$\Xi_{57} = \left(\frac{d_1}{m}\right)^2 \left(\sum_{i=1}^{m} Z_i\right) + \tilde{d}^2 Z_{m+1}$$

$$\Xi_{77} = \left(\frac{d_1}{m}\right)^2 \left(\sum_{i=1}^{m} Z_i\right) + \tilde{d}^2 Z_{m+1} - \mathcal{R} + \gamma I.$$

Proof: It is clear that (30) holds implies (10) holds. Therefore, NN (5) with $\omega(t) = 0$ is stable according to Theorem 1. To prove the dissipativity performance, we consider the Lyapunov–Krasovskii functional candidate (11) and the following index for NN (5):

$$J_{\tau, \gamma} = \int_0^{\tau} \left(y(t)^{\mathrm{T}} \mathcal{Q} y(t) + 2 y(t)^{\mathrm{T}} \mathcal{S} \omega(t) + \omega(t)^{\mathrm{T}} (\mathcal{R} - \gamma I) \omega(t) \right) dt. \tag{31}$$

Applying a similar analysis method employed in the proof of Theorem 1, we have

$$\int_0^{\tau} \dot{V}(x(t))\, dt - J_{\tau, \gamma} \leq \int_0^{\tau} \bar{\theta}(t)^{\mathrm{T}} \hat{\Xi} \bar{\theta}(t)\, dt \tag{32}$$

where

$$\bar{\theta}(t) = \begin{bmatrix} \rho(t) \\ \omega(t) \end{bmatrix}$$

and $\rho(t)$ follows the same definition as that in (22). We can get from (30)

$$\int_0^{\tau} \dot{V}(x(t))\, dt \leq J_{\tau, \gamma} \tag{33}$$

which implies

$$V(x(\tau)) - V(x(0)) \leq J_{\tau, \gamma}. \tag{34}$$

Thus, (7) holds under zero initial condition. Therefore, NN (5) is strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative. This completes the proof. ∎

Remark 7: Based on Theorem 1, a delay-dependent sufficient condition is proposed in Theorem 4 to ensure NN (5) with time-varying interval delay to be globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative. It should be pointed out that, for given m, d_1, d_2, and μ, by setting $\delta = -\gamma$ and minimizing δ subject to (30), we can obtain the optimal dissipativity performance γ^* (by $\gamma^* = -\delta$).

By using a similar analysis method employed in Theorem 4, we have the following results on dissipativity analysis of NN (5) with C2 or C3, respectively.

Theorem 5: Given a scalar $\gamma > 0$ and an integer $m > 0$, NN (5) with C2 is globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative if there exist matrices $P > 0$, $Z_i > 0$ $(i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} Y_3 & U_2 \\ * & Y_4 \end{bmatrix} > 0$, $\begin{bmatrix} Z_{m+1} & U_3 \\ * & Z_{m+1} \end{bmatrix} \geq 0$, and diagonal matrices $S = \text{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \text{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m + 3)$ such that $(30)|_{Y_1=Y_2=U_1=0}$ holds.

Theorem 6: Given a scalar $\gamma > 0$ and an integer $m > 0$, NN (5) with C3 is globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative if there exist matrices $P > 0$, $Z_i > 0$ $(i = 1, 2, \ldots, m)$, $\begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} > 0$, and diagonal matrices $S = \text{diag}\{s_1, s_2, \ldots, s_n\} > 0$, $R = \text{diag}\{r_1, r_2, \ldots, r_n\} > 0$, $D_i > 0$ $(i = 1, 2, \ldots, m + 1)$ such that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \hat{\Psi}_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} < 0 \tag{35}$$

where Ψ_{11} and Ψ_{12} follow the same definitions as those in Theorem 3, and

$$\hat{\Psi}_{22} = W_1^{\mathrm{T}} Q_2 W_1 - W_2^{\mathrm{T}} Q_2 W_2 + e_{m+1}^{\mathrm{T}} W^{\mathrm{T}} S e_1 - e_1^{\mathrm{T}} \mathcal{Q} e_1$$
$$- 2 \sum_{i=1}^{m+1} e_i^{\mathrm{T}} D_i e_i + e_1^{\mathrm{T}} S W e_{m+1} - e_{m+1}^{\mathrm{T}} W^{\mathrm{T}} R e_1$$
$$+ \left(\frac{d}{m}\right)^2 e_{m+1}^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right) e_{m+1} - e_1^{\mathrm{T}} R W e_{m+1}$$

$$\Psi_{13} = e_1^{\mathrm{T}} P + e_1^{\mathrm{T}} W^{\mathrm{T}} L R W - \left(\frac{d_1}{m}\right)^2 e_1^{\mathrm{T}} A \left(\sum_{i=1}^{m} Z_i\right)$$

$$\Psi_{23} = e_1^{\mathrm{T}} S W - e_1^{\mathrm{T}} R W - e_1^{\mathrm{T}} S + \left(\frac{d}{m}\right)^2 e_{m+1}^{\mathrm{T}} \left(\sum_{i=1}^{m} Z_i\right)$$

$$\Psi_{33} = \left(\frac{d}{m}\right)^2 \left(\sum_{i=1}^{m} Z_i\right) - \mathcal{R} + \gamma I.$$

V. NUMERICAL EXAMPLES

In this section, we will make use of some numerical examples to illustrate the advantages of the proposed results in this paper.

TABLE I
ADMISSIBLE DELAY UPPER BOUND d_2

μ	0.05	0.5
[40]	0.5341	0.3388
Theorem 1 ($m = 1$)	0.5342	0.3390
Theorem 1 ($m = 2$)	0.5434	0.3523
Theorem 1 ($m = 3$)	0.5451	0.3543
Theorem 1 ($m = 4$)	0.5457	0.3550

TABLE II
ADMISSIBLE DELAY UPPER BOUND d_2

[40]	0.5142
Theorem 2 ($m = 1$)	0.5163
Theorem 2 ($m = 2$)	0.5265
Theorem 2 ($m = 3$)	0.5282
Theorem 2 ($m = 4$)	0.5287

Example 1: Consider the NN (3) with

$$A = \begin{bmatrix} 7.3458 & 0 & 0 \\ 0 & 6.9987 & 0 \\ 0 & 0 & 5.5949 \end{bmatrix}$$

$$W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7920 & -2.6334 & -20.1300 \end{bmatrix}$$

$$L = \lambda \begin{bmatrix} 0.3680 & 0 & 0 \\ 0 & 0.1795 & 0 \\ 0 & 0 & 0.2876 \end{bmatrix}$$

where the parameter $\lambda > 0$ in matrix L can take different values for comparison purpose.

1) We assume that the involved time-delay in the underlying NN satisfies C1, and choose $\lambda = 1.1229$ and $d_1 = 0.2$. The admissible delay upper bound d_2 for various μ values computed by [40, Th. 1] proposed in this paper can be found in Table I. It is obvious that our method gives a less conservative result than that in [40]. It can also be seen from Table I that, with the partitioning becoming thinner, the conservatism of our method decreases.

2) We assume that the involved time delay in the underlying NN satisfies C2, and choose $\lambda = 0.86$ and $d_1 = 0.3$. The admissible delay upper bound d_2 computed by [40] is 0.5142, which can be found in Table II. Table II also gives the admissible delay upper bound d_2 computed by Theorem 2 proposed in this paper. It is clear that our approach can obtain a larger d_2 than the approach of [40].

3) We assume that the involved time delay in the underlying NN satisfies C3, and choose $\lambda = 1$. Table III gives the admissible delay upper bound d for different m values via the approach introduced in [33, Th. 3] proposed in this paper. It is obvious that our method improves that in [33].

TABLE III
ADMISSIBLE DELAY UPPER BOUND d

m	2	5	10
[33]	1.7683	1.9025	1.9221
Theorem 3	1.7733	1.9099	1.9299

TABLE IV
OPTIMAL DISSIPATIVITY PERFORMANCE γ^*

μ	0.3	0.5	0.7	0.9	≥ 1.0
Theorem 4 ($m = 1$)	1.3292	1.2910	1.2094	1.0237	0.8328
Theorem 4 ($m = 2$)	1.3355	1.2966	1.2137	1.0260	0.8338
Theorem 4 ($m = 3$)	1.3367	1.2976	1.2145	1.0264	0.8340
Theorem 4 ($m = 4$)	1.3371	1.2980	1.2148	1.0266	0.8341

TABLE V
OPTIMAL DISSIPATIVITY PERFORMANCE γ^*

d_2	0.30	0.35	0.40	0.45	0.50
Theorem 5 ($m = 1$)	0.8328	0.7392	0.5952	0.3959	0.0901
Theorem 5 ($m = 2$)	0.8338	0.7421	0.5998	0.4027	0.1033
Theorem 5 ($m = 3$)	0.8340	0.7425	0.6004	0.4036	0.1052
Theorem 5 ($m = 4$)	0.8341	0.7427	0.6006	0.4039	0.1058

Example 2: Consider the NN (5) with

$$A = \begin{bmatrix} 7.0214 & 0 \\ 0 & 7.4367 \end{bmatrix}, \quad W = \begin{bmatrix} -6.4993 & -12.0275 \\ -0.6867 & 5.6614 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

In this example, we choose

$$\mathcal{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

We first assume that the involved time delay in the underlying NN satisfies C1. Choosing $d_1 = 0.1$ and $d_2 = 0.3$, we can get the optimal dissipativity performance γ^* for different m and μ, which is listed in Table IV. It is obvious from Table IV that the optimal dissipativity performance γ^* depends on m and μ. Specifically, when m is fixed, the larger $\mu (\leq 1)$ corresponds to the smaller γ^*, when $\mu (\leq 1)$ is fixed, the larger m corresponds to the larger γ^*. Furthermore, when $\mu \geq 1$, the conservatism of Theorem 4 is dependent on m and independent of μ.

Next, we assume that the involved time delay in the underlying NN satisfies C2, and choose $d_1 = 0.1$. Table V gives the optimal dissipativity performance γ^* for different m and d_2 values. It can be found from Table V that the optimal dissipativity performance γ^* depends on m and d_2. Specifically, when m is fixed, a larger d_2 corresponds to a smaller γ^*, when d_2 is fixed, a larger m corresponds to a larger γ^*.

From Tables IV and V, we can see that, when $d_1 = 0.1$ and $d_2 = 0.3$, Theorem 4 with $\mu \geq 1.0$ and Theorem 5 give the same optimal dissipativity performance γ^*, that is, for the same delay upper bound d_2 and lower bound d_1, the conservatism of Theorem 4 with $\mu \geq 1.0$ is the same as that of Theorem 5.
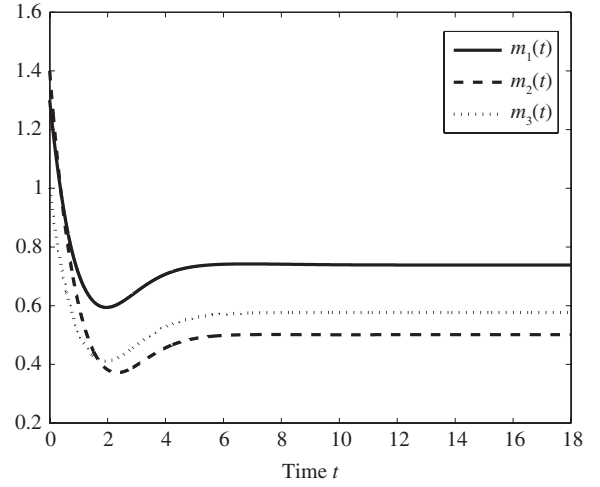

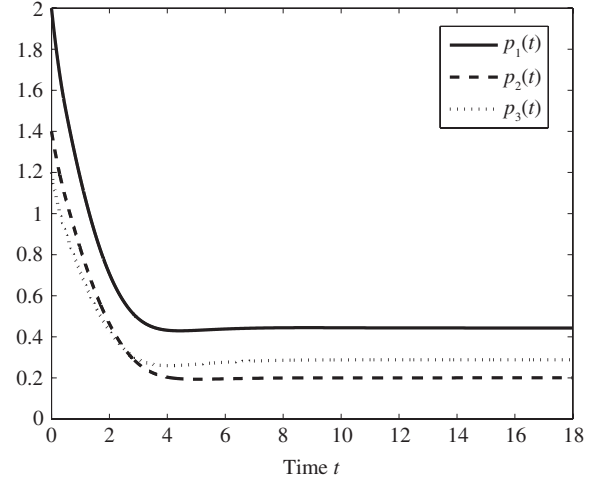Fig. 1. State responses of $m(t)$ of genetic regulatory network (36).

Fig. 2. State responses of $p(t)$ of genetic regulatory network (36).

Example 3: In this example, we will show the application of the proposed result to a biological network, which has been presented as the mathematical model of the repressilator and experimentally studied in *Escherichia coli* [51]. Here, we take the time-varying delay into account and consider the following genetic regulatory network [52], [53]:

$$\begin{cases} \dot{m}(t) = -\hat{A}m(t) + Bf(p(t - d(t))) + \bar{l} \\ \dot{p}(t) = -Cp(t) + Dm(t - d(t)) \end{cases} \quad (36)$$

where $m(t) = [m_1(t) \ m_2(t) \ \cdots \ m_n(t)]^T$, $p(t) = [p_1(t) \ p_2(t) \ \cdots \ p_n(t)]^T$, $\hat{A} = \text{diag}\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\} > 0$, $C = \text{diag}\{c_1, c_2, \ldots, c_n\} > 0$, and $D = \text{diag}\{d_1, d_2, \ldots, d_n\} > 0$, $\bar{l} = [\bar{l}_1 \ \bar{l}_2 \ \cdots \ \bar{l}_n]^T$, $f(p(t)) = [f_1(p_1(t)) f_2(p_2(t)) \ \cdots \ f_n(p_n(t))]^T$, and satisfies

$$0 \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i, \quad s_1 \neq s_2 \in \mathbb{R}. \quad (37)$$

In this example, we consider a three-gene network, that is, $n = 3$. Choose $f_i(s) = s^2/(1 + s^2)$ for any i, which implies
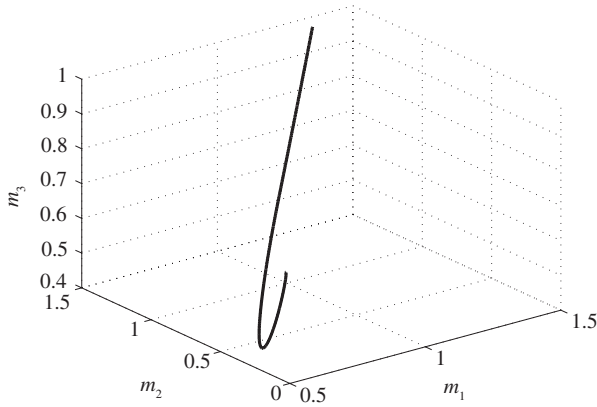
Fig. 3. Phase diagram of $m(t)$ of genetic regulatory network (36).
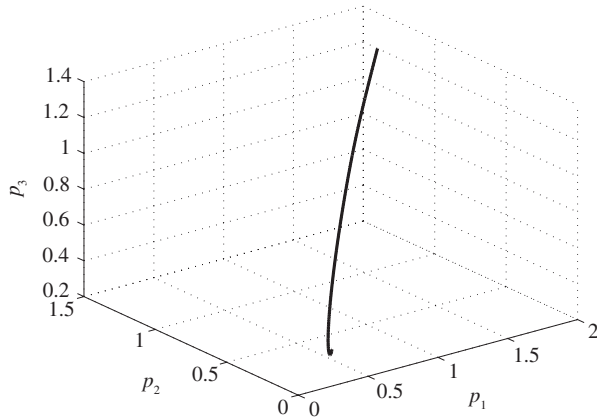

Fig. 4. Phase diagram of $p(t)$ of genetic regulatory network (36).

$l_i = 0.65$ for any i, and

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -0.8 \\ -0.6 & 0 & 0 \\ 0 & -0.6 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \bar{l} = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.6 \end{bmatrix}.$$

Assume (m^*, p^*) be the equilibrium point of (36). By using the transformation $\alpha(t) = m(t) - m^*$, $\beta(t) = p(t) - p^*$, and $h(\beta(t)) = f(\beta(t) + p^*) - f(p^*)$, we can transform genetic regulatory network (36) in the following genetic regulatory network:

$$\begin{cases} \dot{\alpha}(t) = -\hat{A}\alpha(t) + Bh(\beta(t - d(t))) \\ \dot{\beta}(t) = -C\beta(t) + D\alpha(t - d(t)). \end{cases} \tag{38}$$

Letting

$$y(t) = \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}, \quad g(y(t - d(t))) = \begin{bmatrix} \alpha(t - d(t)) \\ h(\beta(t - d(t))) \end{bmatrix}$$

$$A = \begin{bmatrix} \hat{A} & 0 \\ 0 & C \end{bmatrix}, \quad W = \begin{bmatrix} 0 & B \\ D & 0 \end{bmatrix}$$

and $x(t) = W^{-1}y(t)$, we can find that genetic regulatory network (38) can be transformed in static NN (3).

It is assumed that $d(t) = 0.4 + 0.2\sin(t)$. A straightforward calculation gives $d_1 = 0.2$, $d_2 = 0.6$, and $\mu = 0.2$. Applying Theorem 1 with $m = 2$, it can be checked that the NN (3) is globally asymptotically stable, which implies the steady-state ($m^* = \begin{bmatrix} 0.738 & 0.501 & 0.576 \end{bmatrix}^{\mathrm{T}}$, $p^* = \begin{bmatrix} 0.443 & 0.200 & 0.288 \end{bmatrix}^{\mathrm{T}}$) of genetic regulatory network (36) is a stable and unique equilibrium.

Figs. 1 and 2 show the the trajectories of the system states $m(t)$ and $p(t)$, respectively, with the initial states chosen as $m(t) = \begin{bmatrix} 1.3 + 0.5\sin(t) & 1.2 + 0.2\cos(t) & 1 + 0.6\sin(t) \end{bmatrix}^{\mathrm{T}}$ and $p(t) = \begin{bmatrix} 1.5 + 0.5\cos(t) & 1.4 - 0.2\sin(t) & 0.8 + 0.4\cos(t) \end{bmatrix}^{\mathrm{T}}$, $t \in \begin{bmatrix} -0.6 & 0 \end{bmatrix}$, from which we find that the corresponding state responses converge to (m^*, p^*). The corresponding phase diagrams of $m(t)$ and $p(t)$ are given in Figs. 3 and 4, respectively.

VI. Conclusion

The problems of stability and dissipativity analysis were investigated In this paper for static NNs with time-delay. By taking advantage of the delay partitioning technique, some novel Lyapunov–Krasovskii functional candidates were introduced to arrive at the delay-dependent sufficient conditions that warrant the global stability and dissipativity of NNs. The obtained delay-dependent results also rely upon the partitioning size. Finally, three numerical examples were given to demonstrate the reduction of conservatism and effectiveness of the developed approaches.

References

[1] M. M. Gupta, L. Jin, and N. Homma, *Static and Dynamic Neural Networks: From Fundamentals to Advanced Theory*. New York: Wiley, 2003.
[2] Z. Xu, H. Qiao, J. Peng, and B. Zhang, "A comparative study of two modeling approaches in neural networks," *Neural Netw.*, vol. 17, no. 1, pp. 73–85, Jan. 2003.
[3] J. Liang and J. Cao, "A based-on LMI stability criterion for delayed recurrent neural networks," *Chaos Solitons Fractals*, vol. 28, no. 1, pp. 154–160, Apr. 2006.
[4] H. Qiao, J. Peng, Z. Xu, and B. Zhang, "A reference model approach to stability analysis of neural networks," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 33, no. 6, pp. 925–936, Dec. 2003.
[5] B. Du, J. Lam, and Z. Shu, "Stabilization for state/input delay systems via static and integral output feedback," *Automatica*, vol. 46, no. 12, pp. 2000–2007, Dec. 2010.
[6] Y. He, G. Liu, D. Rees, and M. Wu, "Stability analysis for neural networks with time-varying interval delay," *IEEE Trans. Neural Netw.*, vol. 18, no. 6, pp. 1850–1854, Nov. 2007.
[7] Y. He, Q. Wang, C. Lin, and M. Wu, "Delay-range-dependent stability for systems with time-varying delay," *Automatica*, vol. 43, no. 2, pp. 371–376, Feb. 2007.
[8] X. He, Z. Wang, and D. Zhou, "Robust H_∞ filtering for time-delay systems with probabilistic sensor faults," *IEEE Signal Process. Lett.*, vol. 16, no. 5, pp. 442–445, May 2009.
[9] C. Lin, Q. Wang, T. Lee, and B. Chen, "H_∞ filter design for nonlinear systems with time-delay through T-S fuzzy model approach," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 3, pp. 739–746, Jun. 2008.

[10] C. Lin, Z. Wang, and F. Yang, "Observer-based networked control for continuous-time systems with random sensor delays," *Automatica*, vol. 45, no. 2, pp. 578–584, Feb. 2009.

[11] C. Peng and T. Yang, "Improved delay-dependent robust stability criteria for uncertain systems with interval time-varying delay," *IET Control Theory Appl.*, vol. 2, no. 9, pp. 752–761, Sep. 2008.

[12] C. Peng and T. Yang, "Communication-delay-distribution-dependent networked control for a class of T-S fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 2, pp. 326–335, Apr. 2010.

[13] Z. Shu and J. Lam, "Exponential estimates and stabilization of uncertain singular systems with discrete and distributed delays," *Int. J. Control*, vol. 81, no. 6, pp. 865–882, Jun. 2008.

[14] Z. Wang, D. W. C. Ho, Y. Liu, and X. Liu, "Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements," *Automatica*, vol. 45, no. 3, pp. 684–691, Mar. 2009.

[15] Z. Wang, Y. Liu, and X. Liu, "H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities," *Automatica*, vol. 44, no. 5, pp. 1268–1277, 2008.

[16] S. Xu and J. Lam, "On equivalence and efficiency of certain stability criteria for time-delay systems," *IEEE Trans. Autom. Control*, vol. 52, no. 1, pp. 95–101, Jan. 2007.

[17] S. Xu, J. Lam, T. Chen, and Y. Zou, "A delay-dependent approach to robust H_∞ filtering for uncertain distributed delay systems," *IEEE Trans. Signal Process.*, vol. 53, no. 10, pp. 3764–3772, Oct. 2005.

[18] S. Xu, J. Lam, and X. Mao, "Delay-dependent H_∞ control and filtering for uncertain Markovian jump systems with time-varying delays," *IEEE Trans. Circuits Syst. I*, vol. 54, no. 9, pp. 2070–2077, Sep. 2007.

[19] L. Zhang, E. K. Boukas, and J. Lam, "Analysis and synthesis of Markov jump linear systems with time-varying delays and partially known transition probabilities," *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2458–2464, Nov. 2008.

[20] J. Cao, K. Yuan, and X. Li, "Global asymptotical stability of generalized recurrent neural networks with multiple discrete delays and distributed delays," *IEEE Trans. Neural Netw.*, vol. 17, no. 6, pp. 1646–1651, Nov. 2007.

[21] H. Li, B. Chen, Q. Zhou, and W. Qian, "Robust stability for uncertain delayed fuzzy Hopfield neural networks with Markovian jumping parameters," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 39, no. 1, pp. 94–102, Feb. 2009.

[22] X. Liao, G. Chen, and E. N. Sanchez, "LMI-based approach for asymptotically stability analysis of delayed neural networks," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 49, no. 7, pp. 1033–1039, Jul. 2002.

[23] X. Liao, G. Chen, and E. N. Sanchez, "Delay-dependent exponential stability analysis of delayed neural networks: An LMI approach," *Neural Netw.*, vol. 15, no. 7, pp. 855–866, Sep. 2002.

[24] S. Mou, H. Gao, J. Lam, and W. Qiang, "A new criterion of delay-dependent asymptotic stability for Hopfield neural networks with time delay," *IEEE Trans. Neural Netw.*, vol. 19, no. 3, pp. 532–535, Mar. 2008.

[25] Z. Wang, Y. Liu, K. Fraser, and X. Liu, "Stochastic stability of uncertain Hopfield neural networks with discrete and distributed delays," *Phys. Lett. A*, vol. 354, no. 4, pp. 288–297, Jun. 2006.

[26] Z. Wang, Y. Liu, L. Yu, and X. Liu, "Exponential stability of delayed recurrent neural networks with Markovian jumping parameters," *Phys. Lett. A*, vol. 356, no. 4, pp. 346–352, May 2006.

[27] L. Wu, Z. Feng, and W. Zheng, "Exponential stability analysis for delayed neural networks with switching parameters: Average dwell time approach," *IEEE Trans. Neural Netw.*, vol. 21, no. 9, pp. 1396–1407, Sep. 2010.

[28] R. Yang, H. Gao, and P. Shi, "Novel robust stability criteria for stochastic Hopfield neural networks with time delays," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 39, no. 2, pp. 467–474, Apr. 2009.

[29] H. Zhang, Z. Liu, and G. Huang, "Novel delay-dependent robust stability analysis for switched neutral-type neural networks with time-varying delays via SC technique," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 40, no. 6, pp. 1480–1491, Dec. 2010.

[30] H. Zhang, Z. Wang, and D. Liu, "Global asymptotic stability of recurrent neural networks with multiple time-varying delays," *IEEE Trans. Neural Netw.*, vol. 19, no. 5, pp. 855–873, May 2008.

[31] B. Zhang, S. Xu, and Y. Zou, "Improved delay-dependent exponential stability criteria for discrete-time recurrent neural networks with time-varying delays," *Neurocomputing*, vol. 72, nos. 1–3, pp. 321–330, Dec. 2008.

[32] B. Zhang, S. Xu, G. Zong, and Y. Zou, "Delay-dependent exponential stability for uncertain stochastic Hopfield neural networks with time-varying delays," *IEEE Trans. Circuits Syst. I*, vol. 56, no. 6, pp. 1241–1247, Jun. 2009.

[33] B. Du and J. Lam, "Stability analysis of static recurrent neural networks using delay-partitioning and projection," *Neural Netw.*, vol. 22, no. 4, pp. 343–347, May 2009.

[34] J. Lam, H. Gao, and C. Wang, "Stability analysis for continuous systems with two additive time-varying delay components," *Syst. Control Lett.*, vol. 56, no. 1, pp. 16–24, Jan. 2007.

[35] D. Peaucelle, D. Arzelier, D. Henrion, and F. Gouaisbaut, "Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation," *Automatica*, vol. 43, no. 5, pp. 795–804, May 2007.

[36] H. Gao, Z. Fei, J. Lam, and B. Du, "Further results on exponential estimates of Markovian jump systems with mode-dependent time-varying delays," *IEEE Trans. Autom. Control*, vol. 56, no. 1, pp. 223–229, Jan. 2011.

[37] Y. Wang, Z. Wang, and J. Liang, "On robust stability of stochastic genetic regulatory networks with time delays: A delay fractioning approach," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 40, no. 3, pp. 729–740, Jun. 2010.

[38] H. Shao, "Delay-dependent stability for recurrent neural networks with time-varying delays," *IEEE Trans. Neural Netw.*, vol. 19, no. 9, pp. 1647–1651, Sep. 2008.

[39] H. Shao, "Less conservative delay-dependent stability criteria for neural networks with time-varying delays," *Neurocomputing*, vol. 73, nos. 7–9, pp. 1528–1532, Mar. 2010.

[40] Z. Zuo, C. Yang, and Y. Wang, "A new method for stability analysis of recurrent neural networks with interval time-varying delay," *IEEE Trans. Neural Netw.*, vol. 21, no. 2, pp. 339–344, Feb. 2010.

[41] M. Wu, Y. He, J. She, and G. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, no. 8, pp. 1435–1439, Aug. 2004.

[42] H. Zeng, Y. He, M. Wu, and C. Zhang, "Complete delay-decomposing approach to asymptotic stability for neural networks with time-varying delays," *IEEE Trans. Neural Netw.*, vol. 22, no. 5, pp. 806–812, May 2011.

[43] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*. London, U.K.: Springer-Verlag, 2007.

[44] H. Zhang, Z. Guan, and G. Feng, "Reliable dissipative control for stochastic impulsive systems," *Automatica*, vol. 44, no. 4, pp. 1004–1010, Apr. 2008.

[45] Z. Feng and J. Lam, "Stability and dissipativity analysis of distributed delay cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 22, no. 6, pp. 976–981, Jun. 2011.

[46] Z. Feng, J. Lam, and H. Gao, "α-dissipativity analysis of singular time-delay systems," *Automatica*, vol. 47, no. 11, pp. 2548–2552, Nov. 2011.

[47] X. Meng, J. Lam, B. Du, and H. Gao, "A delay-partitioning approach to the stability analysis of discrete-time systems," *Automatica*, vol. 46, no. 3, pp. 610–614, Mar. 2010.

[48] K. Gu, V. K. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston, MA: Birkhauser, 2003.

[49] P. Park, J. W. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, Jan. 2011.

[50] S. Xu and J. Lam, "A survey of linear matrix inequality techniques in stability analysis of delay systems," *Int. J. Syst. Sci.*, vol. 39, no. 12, pp. 1095–1113, Dec. 2008.

[51] M. B. Elowitz and S. Leibler, "A synthetic oscillatory network of transcriptional regulators," *Nature*, vol. 403, no. 6767, pp. 335–338, Jan. 2000.

[52] P. Li, J. Lam, and Z. Shu, "On the transient and steady-state estimates of interval genetic regulatory networks," *IEEE Trans. Syst., Man, Cybern., Part B: Cybern.*, vol. 40, no. 2, pp. 336–349, Apr. 2010.

[53] Z. Wang, H. Gao, J. Cao, and X. Liu, "On delayed genetic regulatory networks with polytopic uncertainties: Robust stability analysis," *IEEE Trans. Nanobiosci.*, vol. 7, no. 2, pp. 154–163, Jun. 2008.

Zheng-Guang Wu was born in 1982. He received the M.S. degree from Zhejiang Normal University, Jinhua, China, in 2007, and the Ph.D. degree from Zhejiang University, Hangzhou, China, in March 2011.

He was a Research Assistant with the Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, from February 2010 to June 2010, and a Research Associate the Department of Mechanical Engineering, University of Hong Kong, from December 2010 to February 2011. He is currently with the Institute of Cyber-Systems and Control, Zhejiang University. His current research interests include robust controls and systems theory.

International Federation of Automatic Control Technical Committee on Control Design. He has served on the Engineering Panel of the Research Grants Council, Hong Kong Special Administrative Region. He is a co-recipient of the International Journal of Systems Science Prize Paper Award, and a recipient of the A. H. Gibson Prize awarded by the Ashbury Scholarship, the H. Wright Baker Prize for academic performance, the Outstanding Researcher Award of the University of Hong Kong, and a Distinguished Visiting Fellowship of the Royal Academy of Engineering. His doctoral and post-doctoral research projects were supported by the Croucher Foundation Scholarship and Fellowship.

James Lam (SM'99) received the B.Sc. (first class) degree in mechanical engineering from the University of Manchester, Manchester, U.K., and the M.Phil. and Ph.D. degrees from the University of Cambridge, Cambridge, U.K.

He held lectureships at the City University of Hong Kong, Kowloon, Hong Kong, and the University of Melbourne, Melbourne, Australia, prior to joining the University of Hong Kong, Hong Kong in 1993. He has held guest professorships in many universities in China. His current research interests include model reduction, robust controls and filtering, delay, singular systems, Markovian jump systems, multidimensional systems, networked control systems, vibration controls, and biological networks.

Prof. Lam is a Chartered Mathematician, a Chartered Scientist, a fellow of the Institute of Mathematics and its Applications, and a fellow of the Institution of Engineering and Technology. Apart from serving as Subject Editor of the *Journal of Sound and Vibration*, he is Associate Editor of an *Asian Journal of Control*, the *International Journal of Systems Science*, the *International Journal of Applied Mathematics and Computer Science*, and the *Journal of the Franklin Institute, Automatica, and Multidimensional Systems and Signal Processing*, and is on the Editorial Board of *IET Control Theory and Applications, Dynamics of Continuous, Discrete and Impulsive Systems: Series B (Applications & Algorithms)*, and *Proc. IMechE Part I: Journal of Systems and Control Engineering*. He was an Editor-in-Chief of the *IEE Proceedings: Control Theory and Applications*, an Associate Member of the IEEE TRANSACTIONS ON SIGNAL PROCESSING, and a member of the

Hongye Su was born in 1969. He received the B.S. degree in industrial automation from the Nanjing University of Chemical Technology, Jiangsu, China, in 1990, and the M.S. and Ph.D. degrees from Zhejiang University, Hangzhou, China, in 1993 and 1995, respectively.

He was a Lecturer with the Department of Chemical Engineering, Zhejiang University, from 1995 to 1997. From 1998 to 2000, he was an Associate Professor with the Institute of Advanced Process Control, Zhejiang University. Currently, he is a Professor with the Institute of Cyber-Systems and Control, Zhejiang University. His current research interests include robust controls, time-delay systems, and advanced process control theory and applications.

Jian Chu was born in 1963. He received the B.S., M.S., and Ph.D. degrees from Zhejiang University, Hangzhou, China, in 1982, 1984, and 1989, respectively.

He joined the Institute of Advanced Process Control, Zhejiang University, as a Post-Doctoral Researcher. In 1991, he was made an Associate Professor and in 1993 a Full Professor. In 1994, he was approved as a Doctorial Advisor. His current research interests include control theory and applications, research and development of computer control systems, and advanced process control software.