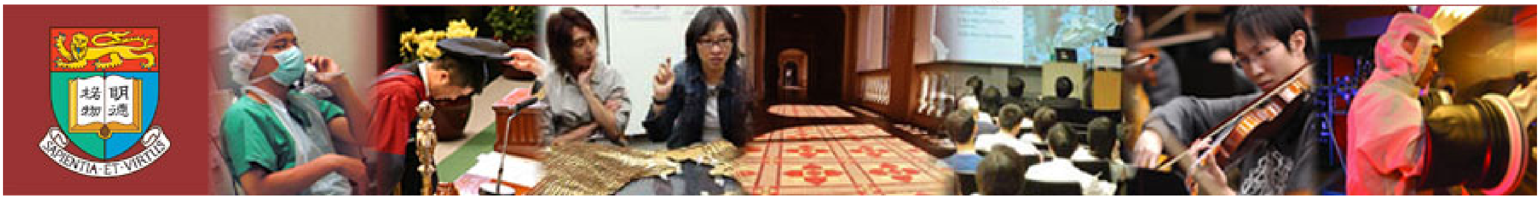

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Multi-objective optimization for a conventional suspension structure

Yinlong Hu[1] and Michael Z. Q. Chen[1,2,*]

***Abstract*— This paper investigates the multi-objective optimization of ride comfort, suspension deflection and tyre grip performance measures for a conventional suspension structure by deriving the analytical solutions for a quarter-car model. The optimization results are compared with two other configurations, one having the same complexity in construction but employing an inerter while the other being the simplest suspension network with one damper and one spring only. The motivation is to investigate the possibility and situations where the inerter can be replaced by some cheaper element such as the spring. The results show that for a low static stiffness and in the situations that ride comfort is less important than suspension deflection and tyre grip (such as race cars), the considered structure would be a reasonable alternative for the one employing an inerter.**

***Index Terms*— passive vehicle suspension; quarter-car model; inerter**

I. INTRODUCTION

Vehicle suspensions play a major role in a vehicle system and decide the overall vehicle performance like ride comfort, vehicle safety and handling. Generally, suspension systems can be divided into three categories, namely, passive suspension systems, semi-active suspension systems and active suspension systems. The advantage of the passive suspension system is its simplicity and low energy consumption.

Inerter is a recently proposed concept and device with the property that the applied force at the terminals is proportional to the relative acceleration between them [6, 12]. The inerter extends the class of mechanical realizations of complex impedances compared to the ones using only springs and dampers and has been applied to various mechanical systems, including vehicle suspensions [2, 13], motorcycle steering systems [7] and building vibration control [15]. It has also rekindled interest in passive network synthesis [1, 3–5].

The advantage of using the inerter for some performance requirements has been well demonstrated. However, since the performance optimization for a vehicle suspension is a compromise among a number of factors such as ride comfort, suspension deflection and tyre grip and for different kinds of vehicles, the requirements for suspension are different, for example, it is reasonable to improve tyre grip and suspension deflection performance at the cost of ride comfort for race cars. The problem that in which situations the inerter is not essential for suspension performance or the inerter can be replaced by other mechanical elements such as springs has not been considered, but such an issue is important since inerter is more expansive and complex to construct.

This paper investigates the multi-objective optimization of ride comfort, suspension deflection and tyre grip performance for a conventional suspension structure, which has been used in [8]. After deriving the analytical solutions for a quarter-car model and comparing it with the simplest suspension structure [10, 13] and one having a similar complexity but employing an ineter [10], the conditions where the considered structure would be an alternative choice compared the one with inerter are obtained and some general guidelines for practice are highlighted.

The rest of the paper is organized as follows: Section II introduces the relevant background on suspension structures and performance measures. Section III derives the optimal performance measures for ride comfort, suspension deflection and tyre grip individually for the considered conventional structure. Section IV investigates the multi-objective performance optimization and compares the three structures considered. Conclusions are drawn in Section V.

II. VEHICLE MODEL, SUSPENSION NETWORKS, AND PERFORMANCE MEASURES

The quarter-car model presented in Fig. 1 is the simplest model for suspension design. It consists of a sprung mass m_s, an unsprung mass m_u and a tyre with spring stiffness k_t [11]. Here, the suspension strut supplying an equal and opposite force on the sprung and unsprung masses is a passive mechanical admittance $Q(s)$. The equations of motion in the Laplace domain are:

$$\begin{aligned} m_s s^2 \hat{z}_s &= \hat{F}_s - sQ(s)(\hat{z}_s - \hat{z}_u), \\ m_u s^2 \hat{z}_u &= sQ(s)(\hat{z}_s - \hat{z}_u) + k_t(\hat{z}_r - \hat{z}_u). \end{aligned}$$

Fig. 2 is the suspension configuration under consideration, which is also a conventional suspension layout employed in [8]. The configuration contains a ‘center spring’ k_1 [13] and a ‘relaxation spring’ k_2 [9]. The static stiffness is

$$K = k + (k_1^{-1} + k_2^{-1})^{-1}. \quad (1)$$

The configurations C_2 and C_3 shown in Fig. 3 are used for comparison. C_2 is the simplest layout and has been discussed in [10, 13]. C_3 is the S_5 layout in [10], which is similar to the considered structure but employing an inerter.

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[1]School of Automation, Nanjing University of Science and Technology, Nanjing. [2]Department of Mechanical Engineering, The University of Hong Kong, Hong Kong. Email: mzqchen@hku.hk. *Corresponding author.

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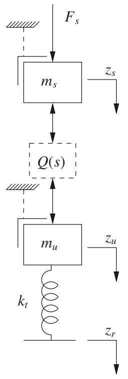

Fig. 1. A quarter-car vehicle model.

The performance measures used in this paper are discussed in detail in [1, 14]. For ride comfort, we use the root-means-square (rms) of body vertical acceleration in response to road disturbances, defined as J_1 as follows

$$J_1 = 2\pi(V\kappa)^{\frac{1}{2}}||sT_{\hat{z}_r \to \hat{z}_s}||_2,$$

where V is the speed of the car, κ is the road roughness parameter. $T_{\hat{z}_r \to \hat{z}_s}$ denotes the transfer function from the road disturbance $\hat{z}_r$ to the displacement of the sprung mass $\hat{z}_s$ and $||\cdot||_2$ is the standard H_2 norm. The rms suspension deflection parameter J_2 is defined as

$$J_2 = 2\pi(V\kappa)^{\frac{1}{2}} \left\| \frac{1}{s} T_{\hat{z}_r \to (\hat{z}_s - \hat{z}_u)} \right\|_2.$$

The rms tyre grip parameter J_3 is defined as

$$J_3 = 2\pi(V\kappa)^{\frac{1}{2}} \left\| \frac{1}{s} T_{\hat{z}_r \to k_t(\hat{z}_u - \hat{z}_r)} \right\|_2.$$

The parameters for the quarter car model and performance measures in this paper are (unless otherwise stated): $m_s = 250$ kg, $k_t = 150$ kNm^{-1}, $\kappa = 5 \times 10^{-7}$ m^{3}cycle^{-1}, $V = 25$ ms^{-1} and $m_u = 35$ kg or 20 kg. Throughout the paper, F_s is equal to 0 since here we are only interested in the responses resulting from the road disturbances.

III. Optimization of J_1, J_2 and J_3 individually for the considered structure

In this section, we derive the analytical solutions of optimal C_1 for J_1, J_2 and J_3 respectively in the approach of [10].

An analytical expression of the H_2-norm of the (stable) transfer function $G(s)$ can be computed from a minimal state-space realization $G(s) = C(sI - A)^{-1}B$ as $||G||_2 = (CLC^T)^{1/2}$, where the matrix L is the unique solution of the Lyapunov equation $AL + LA^T + BB^T = 0$. The performance measures are given by $J_i = 2\pi(V\kappa H)^{1/2}$, where $H = CLC^T$. We evaluate H algebraically as follows.

A. J_1 optimization results

Let m_s, m_u, k_t be fixed and positive. K is the static stiffness shown in (1). Then

$$H_{C_1 J_1}(k_1, k_2, c, K) = c_1 c + c_2 c^{-1}, \qquad (2)$$

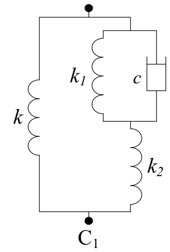

Fig. 2. The conventional configuration.

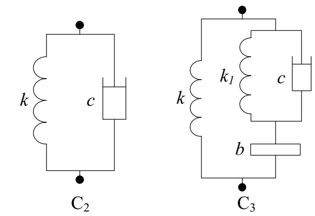

Fig. 3. The comparison configurations.

where $c_1 = d_1 k_2^{-2} + d_2(k_1 + k_2)^{-1} + d_3 k_2^2(k_1 + k_2)$, $c_2 = d_4(k_1 + k_2)^2 k_2^{-2}$, and

$$d_1 = \frac{k_t K^2}{2m_s^2},\ d_2 = \frac{Kk_t}{m_s^2},\ d_3 = \frac{k_t}{2m_s^2},\ d_4 = \frac{(m_u + m_s)K^2}{2m_s^2}.$$

For any $K > 0$, the minimum of $H_{C_1 J_1}$ is achieved with

$$k_2^{-1} = 0,\ \hat{c} = \left(\frac{m_u + m_s}{k_t}\right)^{1/2} K. \qquad (3)$$

The result is consistent with the conclusion of [10] that the relaxation spring itself is not an advantage for ride comfort. When considering the ride comfort performance alone, the optima of C_1 and C_2 coincide, while for mixed performance measures discussed in Section IV, the relaxation spring should not be neglected.

B. J_2 optimization results

Let m_s, m_u, k_t be fixed and positive. K is the static stiffness shown in (1). Then

$$H_{C_1 J_2}(k_1, k_2, c, K) = e_1 k_2^{-2} c + e_2(k_1 + k_2)^2 k_2^{-2} c^{-1}, \qquad (4)$$

where $e_1 = k_t/2$ and $e_2 = (m_s + m_u)/2$. For $K > 0$, $\min H_{C_1 J_2} = 0$ with $k_2^{-1} = 0$ and $c^{-1} = 0$.

C. J_3 optimization results

Let m_s, m_u, k_t be fixed and positive. K is the static stiffness shown in (1). Then

$$H_{C_1 J_3}(k_1, k_2, c, K) = c_3 c + c_4 c^{-1}, \qquad (5)$$

where $c_3 = a_1 k_2^{-2} + a_2(k_1 + k_2)^{-1} + a_3 k_2^2(k_1 + k_2)^{-2}$, $c_4 = a_4(k_1 + k_2)^2 k_2^{-2}$, and

$$\begin{aligned} a_1 &= \frac{m_u^2 K^2 k_t}{2m_s^2} + \frac{m_u k_t}{2m_s}(2K^2 - 2Kk_t) \\ &\quad + \frac{k_t(K^2 - Kk_t + k_t^2)}{2}, \\ a_2 &= \frac{Kk_t m_u^2}{m_s^2} + \frac{2Kk_t m_u - k_t^2 m_u}{m_s} + Kk_t - \frac{1}{2}k_t^2, \\ a_3 &= \frac{(m_u + m_s)^2 k_t}{2m_s^2}, \\ a_4 &= \frac{(m_u^3 + m_s^3)K^2}{2m_s^2} + \frac{m_u^2(3K - 2k_t)K}{2m_s} \\ &\quad + \frac{m_u(3K^2 - 2Kk_t + k_t^2)}{2}. \end{aligned}$$

Denote $K_0 = \frac{(2m_u + m_s)k_t m_s}{2(m_u + m_s)^2}$. For $K \geq K_0$, $\min H_{C_1 J_3} =$

$2(a_3a_4)^{1/2}$ for $k_1 = 0$, $k_2^{-2} = 0$ and $c = (a_4/a_3)^{1/2}$. The network is effectively reduced to C_2. For $K < K_0$, the unique minimum of $H_{C_1J_3}$ is $\left(\frac{a_4(4a_3a_1-a_2^2)}{a_1}\right)^{1/2}$ and the minimum is achieved with

$$k_2 > -\frac{2a_1}{a_2},\ k_1 = -\left(\frac{a_2}{2a_1}k_2^2 + k_2\right),\ c = \left(\frac{c_2}{c_1}\right)^{1/2}. \quad (6)$$

From the analytical solution of J_3, C_1 performs better than C_2 for $K < K_0$. The result is shown in Fig. 4.

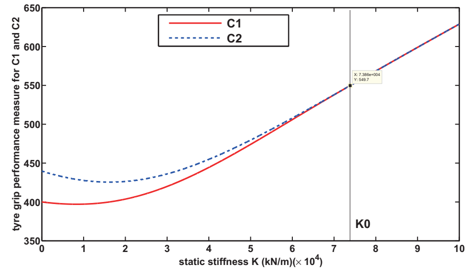

Fig. 4. J_3 performance for C_1 and C_2.

IV. Multi-objective performance optimization

We have obtained the analytical expressions of C_1 for optimal J_1, J_2 and J_3, respectively. The analytical expressions of C_2 and C_3 for optimal J_1 and J_3 are obtained in [10]. J_2 performance measure for C_2 and C_3 can be derived similarly to the C_1 case. The analytical expressions of C_2 and C_3 are shown below.

$$H_{C_2J_1} = d_3c + d_4c^{-1}, \quad (7)$$

$$H_{C_3J_1} = (d_3 + d_5b^{-1} + d_6b^{-2})c + ((d_7 + d_8b^{-1} + d_9b^{-2})k^2 - (d_5 + 2d_6b^{-1})k + d_4)c^{-1}, \quad (8)$$

$$H_{C_2J_2} = e_2c^{-1}, \quad (9)$$

$$H_{C_3J_2} = e_3b^{-2}c + ((e_4b^{-2} + e_5b^{-1})k^2 - 2e_3b^{-1}k + e_2)c^{-1}, \quad (10)$$

$$H_{C_2J_3} = a_3c + a_4c^{-1}, \quad (11)$$

$$H_{C_3J_3} = (a_3 + a_5b^{-1} + a_6b^{-2})c + ((a_7 + a_8b^{-1} + a_9b^{-2})k^2 - (a_5 + 2a_6b^{-1})k + a_4)c^{-1}, \quad (12)$$

where

$$d_5 = -\frac{(m_u+m_s)K}{m_s^2},\ d_6 = \frac{(m_s+m_u)^2K^2 + k_tm_s^2K}{2m_s^2k_t},$$

$$d_7 = \frac{m_u+m_s}{2m_s^2},\ d_8 = -\frac{2(m_s+m_u)^2K + m_s^2k_t}{2m_s^2k_t},$$

$$d_9 = \frac{(m_s+m_u)^3K^2 + 2m_s^2(m_s+m_u)k_tK + m_s^3k_t^2}{2(m_sk_t)^2},$$

$$e_3 = \frac{K(m_u+m_s)^2 + k_tm_s^2}{2Kk_t},\quad e_5 = \frac{m_s^2}{2K^2},$$

$$e_4 = \frac{(m_u+m_s)^3K^2 + 2(m_u+m_s)k_tm_s^2K + k_t^2m_s^3}{2K^2k_t^2},$$

$$a_5 = \frac{-(m_u+m_s)^3K + m_um_s(m_u+m_s)k_t}{m_s^2},$$

$$a_6 = \frac{(m_u+m_s)^4K^2 + (m_u+m_s)^2(m_s-2m_u)m_sk_tK}{2m_s^2k_t} + \frac{m_u^2k_t}{2},\quad a_7 = \frac{(m_s+m_u)^3}{2m_s^2},$$

$$a_8 = \frac{-2(m_u+m_s)^4K + m_s(m_u+m_s)^2(2m_u-m_s)k_t}{2m_s^2k_t},$$

$$a_9 = \frac{m_u+m_s}{2m_s^2k_t^2}((m_u+m_s)^4K^2 + 2m_s(m_u+m_s)^2(m_s - m_u)k_tK + (m_u^2 - m_sm_u + m_s^2)(m_sk_t)^2).$$

A. Mixed performance of J_1 and J_2

We now derive the global optimum for a combined measure $H_{Ci;1,2} = (1-\alpha)H_{C_iJ_1} + \alpha m_s^2 H_{C_iJ_2}$, where $\alpha \in [0,1]$ is a weighting between J_1 and J_2. The scaling factor m_s^2 is inserted to approximately normalize the measures.

1) Mixed performance of J_1 and J_2 for C_1: Let m_s, m_u, k_t be fixed and positive. K is the static stiffness in (1). Consider the mixed performance

$$H_{C_1;1,2}(k_1,k_2,c,K) = (1-\alpha)H_{C_1J_1} + \alpha m_s^2 H_{C_1J_2}, \quad (13)$$

where $H_{C_1J_1}$ and $H_{C_1J_2}$ are given by (2) and (4). For any fixed K and α, $H_{C_1;1,2}$ has a unique minimum with

$$k_2^{-1} = 0,\ c = \left(\frac{(1-\alpha)d_4 + \alpha m_s^2e_2}{(1-\alpha)d_3}\right)^{1/2}. \quad (14)$$

The optimal mixed performance of J_1 and J_2 for C_1 requires $k_2^{-1} = 0$, then C_1 reduces to C_2 and gives no improvement compared with C_2.

2) Mixed performance of J_1 and J_2 for C_2: Let m_s, m_u, k_t be fixed and positive. K is the static stiffness in (1). Consider the mixed performance

$$H_{C_2;1,2}(c,K) = (1-\alpha)H_{C_2J_1} + \alpha m_s^2 H_{C_2J_2}, \quad (15)$$

where $H_{C_2J_1}$ and $H_{C_2J_2}$ are given by (7) and (9). For any fixed K and α, $H_{C_2;1,2}$ has a unique minimum with

$$c = \left(\frac{(1-\alpha)d_4 + \alpha m_s^2e_2}{(1-\alpha)d_3}\right)^{1/2}. \quad (16)$$

3) Mixed performance of J_1 and J_2 for C_3: Let m_s, m_u, k_t be fixed and positive. Consider

$$H_{C_3:1,2} = (1-\alpha)H_{C_3J_1} + \alpha m_s^2 H_{C_3J_2} = f_1c + f_2c^{-1}, \quad (17)$$

where $H_{C_3J_1}$ and $H_{C_3J_2}$ are given by (8) and (10),

$$f_1 = (1-\alpha)(d_3 + d_5b^{-1} + d_6b^{-2}) + \alpha m_s^2e_3b^{-2},$$

$$f_2 = t_2k^2 + t_1k + t_0,$$

where

$$t_2 = (1-\alpha)(d_7 + d_8b^{-1} + d_9b^{-2}) + \alpha m_s^2(e_4b^{-2} + e_5b^{-1}),$$

$$t_1 = -(1-\alpha)(d_5 + 2d_6b^{-1}) - 2\alpha m_s^2e_3b^{-1},$$

$$t_0 = (1-\alpha)d_4 + \alpha m_s^2e_2.$$

For any fixed K and α, $H_{C3:1,2}$ has a unique minimum by $b = \hat{b}$ and

$$k = -\frac{t_1}{2t_2} \quad \text{and} \quad c = \left(\frac{f_2}{f_1}\right)^{1/2}. \qquad (18)$$

Let $\mathcal{Q}$ be the set of real positive solutions of function obtained by substituting (18) into (17) and differentiating with respect to b^{-1}. With $b_0 = \frac{2(d_6(1-\alpha)+m_s^2\alpha e_3)}{(\alpha-1)d_5}$, $\hat{b}$ is given by b_0 or $\mathcal{Q}\bigcap(0, b_0)$.

4) Numerical example: We simulate the cases with static stiffness $K = 15,\ 35,\ 55\ \text{kNm}^{-1}$, respectively. Fig. 6 and Fig. 7 are given by $m_u = 35$ kg.

Fig. 5 shows that the performance difference is decreasing with increasing static stiffness. The use of inerter (C_3) can improve ride comfort performance greatly compared with C_1 and C_2, which is shown in Fig. 6. However, as Fig. 7 shows, C_1 and C_2 requiring less suspension deflection than C_3 does. In some situations that the suspension travel distance is limited or suspension deflection is more essential than ride comfort performance, C_1 and C_2 are better.

B. Mixed performance of J_1 and J_3

In this section, we derive the global optima for a combined measure $H_{Ci;1,3} = (1-\alpha)m_s^2 H_{C_iJ_1} + \alpha H_{C_iJ_3}$, where $\alpha \in [0,1]$ is a weighting between J_1 and J_3. The scaling factor m_s^2 is inserted to approximately normalize the measures.

1) Mixed performance of J_1 and J_3 for C_1: Let m_s, m_u, k_t be fixed and positive. K is the static stiffness in (1). Consider the mixed performance

$$H_{C_1;1,3} = (1-\alpha)m_s^2 H_{C_1J_1} + \alpha H_{C_1J_3} = f_3 c + f_4 c^{-1}, \quad (19)$$

where

$$\begin{aligned} f_3 &= \left((1-\alpha)m_s^2 d_1 + \alpha a_1\right) k_2^{-2} + ((1-\alpha)m_s^2 d_2 + \\ &\quad \alpha a_2)(k_1+k_2)^{-1} + ((1-\alpha)m_s^2 d_3 + \\ &\quad \alpha a_3)k_2^2(k_1+k_2)^{-2}, \\ f_4 &= \left((1-\alpha)m_s^2 d_4 + \alpha a_4\right)(k_1+k_2)^2 k_2^{-2}. \end{aligned}$$

Denote

$$\begin{aligned} K_1 &= \frac{\alpha k_t m_s(2m_u + m_s)}{2(m_s^2 + 2\alpha m_u m_s + \alpha m_u^2)}, \\ K_2 &= -\frac{2\left(m_s^2 d_1(1-\alpha) + \alpha a_1\right)}{m_s^2 d_2(1-\alpha) + \alpha a_2}. \end{aligned}$$

For $K \geq K_1$, $H_{C_1;1,3}$ has a unique minimum given by

$$k_2^{-1} = 0,\ c = \left(\frac{(1-\alpha)m_s^2 d_4 + \alpha a_4}{(1-\alpha)m_s^2 d_3 + \alpha a_3}\right)^{1/2}.$$

For $K < K_1$, $H_{C_1;1,3}$ has a unique minimum given by

$$\begin{aligned} k_2 &> K_2,\ c = \left(\frac{f_4}{f_3}\right)^{1/2}, \\ k_1 &= -\frac{(m_s^2 d_2(1-\alpha) + \alpha a_2)k_2^2}{2(m_s^2 d_1(1-\alpha) + \alpha a_1)} \\ &\quad -\frac{(2m_s^2 d_1(1-\alpha) + 2\alpha a_1)k_2}{2(m_s^2 d_1(1-\alpha) + \alpha a_1)}. \qquad (20) \end{aligned}$$

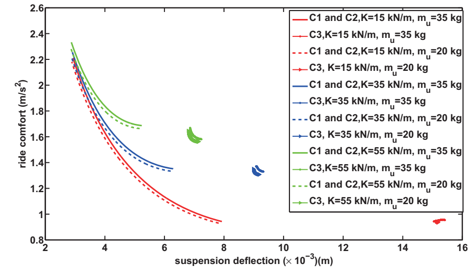

Fig. 5. Mixed J_1 and J_2 performance measure.

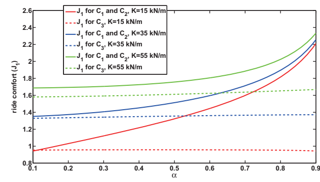

Fig. 6. J_1 performance.

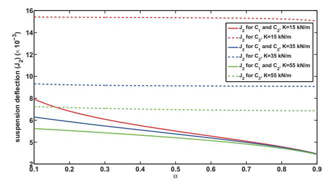

Fig. 7. J_2 performance.

2) Mixed performance of J_1 and J_3 for C_2 ([10, Proposition 13]: Let m_s, m_u, k_t be fixed and positive. Consider

$$H_{C_2;1,3} = (1-\alpha)m_s^2 H_{C_2J_1} + \alpha H_{C_2J_3}, \qquad (21)$$

where $H_{C_2J_1}$ and $H_{C_2J_3}$ are given by (7) and (11). For any fixed K and α, $H_{C_2;1,3}$ has a unique minimum with

$$c = \left(\frac{(m_s^2 d_4 - a_4)\alpha - m_s^2 d_4}{(m_s^2 d_3 - a_3)\alpha - m_s^2 d_3}\right)^{1/2}.$$

3) Mixed performance of J_1 and J_3 for C_3 ([10, Proposition 17]: Let m_s, m_u, k_t be fixed and positive. Consider

$$H_{C_3;1,3} = (1-\alpha)m_s^2 H_{C_3J_1} + \alpha H_{C_3J_3} \qquad (22)$$

Define $\hat{k} = \frac{k_{num}}{k_{den}}$ and $\hat{c} = (c_{num}/c_{den})^{1/2}$, $\gamma = m_s^{-2}$ and

$$\begin{aligned} k_{num} &= 2((\gamma a_6 - d_6)\alpha + d_6)\hat{b}^{-1} + (\gamma a_5 - d_5)\alpha + d_5, \\ k_{den} &= 2(((\gamma a_9 - d_9)\alpha + d_9)\hat{b}^{-2} + ((\gamma a_8 - d_8)\alpha + \\ &\quad d_8)\hat{b}^{-1} + (\gamma a_7 - d_7)\alpha + d_7), \\ c_{num} &= ((\gamma a_9 - d_9)\hat{k}^2\hat{b}^{-2} + (((\gamma a_8 - d_8)\alpha + d_8)\hat{k}^2 \\ &\quad +2((d_6 - \gamma a_6)\alpha - d_6)\hat{k})\hat{b}^{-1} + ((\gamma a_7 - d_7)\alpha \\ &\quad +d_7)\hat{k}^2 + ((d_5 - \gamma a_5)\alpha - d_5)\hat{k} + (a_4\gamma - \\ &\quad d_4)\alpha + d_4, \\ c_{den} &= ((\gamma a_6 - d_6)\gamma + d_6)\hat{b}^{-2} + ((\gamma a_5 - d_5)\alpha + \\ &\quad d_5)\hat{b}^{-1} + (a_3\gamma - d_3)\alpha + d_3. \end{aligned}$$

Let $\mathcal{Q}$ be the set of real, positive solutions of the equation obtained after substituting (8) and (12) into (22) and differentiating with respect to b^{-1}. For any $k \geq 0$, the minimum of $H_{C_3;1,3}$ is achieved with $\hat{k}$, $\hat{c}$ above and $\hat{b} = -2\frac{(\gamma a_6 - d_6)\alpha + d_6}{(\gamma a_5 - d_5)\alpha + d_5} := b_1$ or $\hat{b} \in \mathcal{Q} \cap (0, b_1)$.

4) Numerical results: We simulate cases with static stiffness $K = 20,\ 55,\ 75\ \text{kNm}^{-1}$, respectively. Fig. 9 and Fig. 10 are given by $m_u = 35$ kg.

At low static stiffness, C_1 performs better than C_2 as shown in Fig. 8, but tends to coincide with C_2 with increasing static stiffness, which can be explained by the expression of optimal $H_{C_1;1,3}$. There exists a critical point of static stiffness K_1 for the global optimization of $H_{C_1;1,3}$ and for $K > K_1$ C_1 reduces to C_2. From Fig. 9, C_1 has no advantage for ride comfort for both low and high static stiffness compared with C_3. However, for tyre grip performance, C_1 does better than C_2 and C_3 do at low static stiffness, as shown in Fig. 10. In summary, the use of inerter (C_3) has a considerable advantage both in ride comfort and tyre grip performance in general, especially for a high static stiffness. However, if one is more concerned about tyre grip performance (like race cars) for a low static stiffness, C_1 would be a good alternative, since spring is cheaper and easier to construct than inerter.

C. Mixed performance of J_2 and J_3

We now derive the global optima for a combined measure $H_{Ci;2,3} = (1-\alpha)m_s^2 H_{CiJ2} + \alpha H_{CiJ3}$ where $\alpha \in [0,1]$ is a weighting between J_2 and J_3.

1) Mixed performance of J_2 and J_3 for C_1: Let m_s, m_u, k_t be fixed and positive. K is the static stiffness in (1). Consider the mixed performance

$$H_{C_1;2,3} = (1-\alpha)m_s^2 H_{C_1J_2} + \alpha H_{C_1J_3} = f_5 c + f_6 c^{-1}, \quad (23)$$

where

$$\begin{aligned} f_5 &= ((1-\alpha)m_s^2 e_1 + \alpha a_1)k_2^{-2} + \alpha a_2 (k_1+k_2)^{-1} \\ &\quad + \alpha a_3 k_2^2 (k_1+k_2)^{-2}, \\ f_6 &= ((1-\alpha)m_s^2 e_2 + \alpha a_4)(k_1+k_2)^2 k_2^{-2}. \end{aligned}$$

Denote $K_3 = -\frac{2(m_s^2 e_1(1-\alpha)+\alpha a_1)}{\alpha a_2}$. For $K \geq K_0$, $H_{2,3}$ has a unique minimum given by

$$k_2^{-1} = 0 \quad \text{and} \quad c = \left(\frac{(1-\alpha)m_s^2 e_2 + \alpha a_4}{a_3 \alpha}\right)^{1/2}.$$

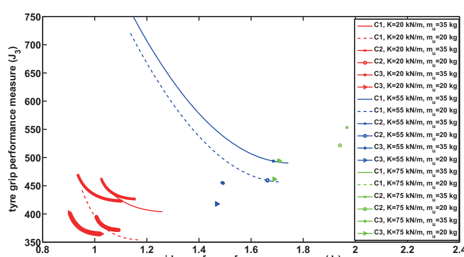

Fig. 8. Mixed J_1 and J_3 performance measure.

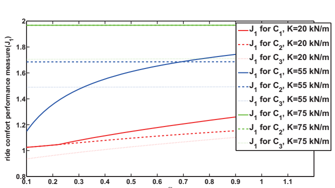

Fig. 9. J_1 performance.

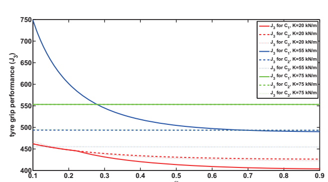

Fig. 10. J_3 performance.

For $K < K_0$, $H_{2,3}$ has a unique minimum given by

$$\begin{aligned} k_2 &> K_3, \; c = \frac{a_2^2 k_2^2}{2}\left(\frac{c_{num}}{c_{den}}\right)^{1/2}, \\ k_1 &= -\frac{k_2(a_2 \alpha k_2 + 2m_s^2 e_1(1-\alpha) + 2a_1\alpha)}{2m_s^2(e_1(1-\alpha)+a_1\alpha)}, \end{aligned}$$

where

$$\begin{aligned} c_{num} &= \alpha^3(m_s^2 e_2(1-\alpha) + \alpha a_4), \\ c_{den} &= (m_s^2 e_1(1-\alpha) + a_1\alpha)^3 (4a_3 e_1 m_s^2(1-\alpha) + \\ &\quad 4a_3 a_1 \alpha - \alpha a_2^2). \end{aligned}$$

2) Mixed performance of J_2 and J_3 for C_2: Let m_s, m_u, k_t be fixed and positive. Consider the mixed performance

$$H_{C_2;2,3} = (1-\alpha)m_s^2 H_{C_2J_2} + \alpha H_{C_2J_3}. \quad (24)$$

For fixed K and α, $H_{C_2:2,3}$ has a unique minimum given by

$$c = \left(\frac{(1-\alpha)m_s^2 e_2 + \alpha a_4}{\alpha a_3}\right)^{1/2}. \quad (25)$$

3) Mixed performance of J_2 and J_3 for C_3: Let m_s, m_u, k_t be fixed and positive. Consider

$$H_{C_3;2,3} = (1-\alpha)m_s^2 H_{C_3J_2} + \alpha H_{C_3J_3} = f_7 c + f_8 c^{-1}, \quad (26)$$

where

$$\begin{aligned} f_7 &= (1-\alpha)m_s^2 e_3 b^{-2} + \alpha(a_3 + a_5 b^{-1} + a_6 b^{-2}), \\ f_8 &= ((1-\alpha)m_s^2(e_4 b^{-2} + e_5 b^{-1}) + \alpha(a_7 + a_8 b^{-1} + \\ &\quad a_9 b^{-2}))k^2 - (2(1-\alpha)m_s^2 e_3 b^{-1} + \alpha(a_5 + \\ &\quad 2a_6 b^{-1}))k + (1-\alpha)m_s^2 e_2 + \alpha a_4. \end{aligned}$$

Denote $K_4 = \frac{m_u m_s k_t}{(m_u+m_s)^2}$ and $b_2 = \frac{-2(m_s^2 e_3(1-\alpha)+\alpha a_6)}{\alpha a_5}$. K is the static stiffness. If $K \leq K_4$, the network reduces to S_4 in [10]. For $K > K_4$, the unique minimum is obtained by $b = \hat{b}$,

$$k = \frac{k_{num}}{k_{den}} \text{ and } c = \left(\frac{f_8}{f_7}\right)^{1/2} \quad (27)$$

where

$$\begin{aligned} k_{num} &= (2m_s^2 e_3(1-\alpha) + \alpha a_5 b + 2\alpha a_6)b, \\ k_{den} &= 2(\alpha a_7 b^2 + (m_s^2 e_5(1-\alpha) + \alpha a_8)b + \\ &\quad m_s^2 e_4(1-\alpha) + \alpha a_9). \end{aligned}$$

Let $\mathcal{Q}$ be the set of real, positive solutions b of the equation after substituting (27) into (26) and differentiating with respect to b^{-1}. $\hat{b}$ is equal to b_2 or $\hat{b} \in \mathcal{Q} \bigcap (0, b_2)$.

4) Numerical results: We simulate cases with static stiffness $K = 20, 55, 75$ kNm^{-1}, respectively. Fig. 12 and Fig. 13 are given by $m_u = 35$ kg.

Fig. 11 shows that C_1 perform better than C_2 and C_3 do at low static stiffness and with the increasing of static stiffness, C_1 and C_2 tend to coincide, which is consistent in Fig. 12 and Fig. 13. It is also consistent with the analytical expression of optimal $H_{C_1;2,3}$. There is a critical point of static stiffness K_0 for C_1, beyond which C_1 reduces to C_2. For J_2 performance,

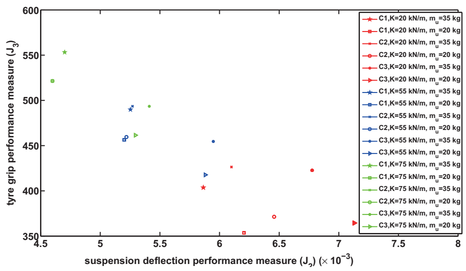

Fig. 11. Mixed J_2 and J_3 performance measure.

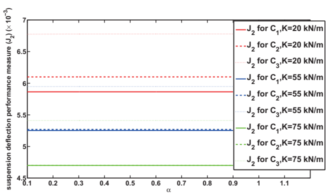

Fig. 12. J_2 performance.

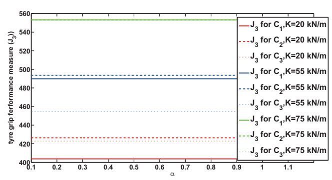

Fig. 13. J_3 performance.

as shown in Fig. 12, C_1 performs better than C_3 for all range K, but for J_3 performance, it only performs better at low static stiffness shown in Fig. 13. In summary, if one is more concerned about suspension deflection and tyre grip performance, for a low static stiffness, C_1 performs better than C_2 and C_3 do, even though an inerter is used in C_3.

V. Conclusion

This paper has presented the analytical solutions for the multi-objective performance optimization including ride comfort, suspension deflection and tyre grip for a quarter-car model. The results show that the considered structure has an advantage in suspension deflection and tyre grip performance at low static stiffness compared with the one with a similar complexity employing the inerter. When considering ride comfort and suspension deflection together the considered structure performs better for suspension deflection performance; for mixed ride comfort and tyre grip performance, it performs better for tyre grip performance at low static stiffness; for mixed suspension deflection and tyre grip performance, it performs better for suspension deflection in all range static stiffness and for tyre grip performance at low static stiffness. In other words, the contributions of inerter for tyre grip performance are mainly for the high static stiffness range. The considered structure may be a good alternative of the one with inerter at low static stiffness where suspension deflection and tyre grip are more important than ride comfort.

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