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On Exponential Almost Sure Stability of Random Jump Systems

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Abstract—This paper is concerned with a class of random jump systems represented by transition operators, which includes switched linear systems with strictly stationary switching signals in infinite modes space as its special case. A series of necessary and sufficient conditions are established for almost sure stability of this class of random jump systems under different scenarios. The stability criteria obtained are further extended to Markov jump linear systems with infinite states, and hence a unified approach to describing the almost sure stability of MJLSs is addressed under this context. All the results in the work are developed for both the continuous- and discrete-time systems.

Index Terms—Almost sure stability, Markov processes, random jump systems.

I. INTRODUCTION

IN real-world applications, linear time-invariant models are generally insufficient to describe fully, or even nearly accurately, the behavior of dynamic systems found in industries when the systems are affected by abrupt changes or component failures. Indeed, many systems exhibit random behavior which can be well modeled by certain classes of stochastic switched systems or piecewise deterministic systems. These models consist of a set of subsystems, each associated with a mode, whose operation being governed by a switching signal that specifies the active mode at any time. These switched system models are commonly used to model the robot systems, vehicle systems, and large-scale flexible structures for space stations, for instance.

Stability analysis of switched systems has attracted tremendous attention in recent years. Abundant fundamental results have emerged in this active area [1], [2], [6], [13], [15], [24],

[32], [34]. Among these works, Bolzern *et al.* [6] derive the necessary and sufficient condition for the almost sure stability of continuous-time Markov jump linear systems (MJLSs), Lin and Antsaklis [24] represent a survey of recent results on various stability of switched linear systems, and Xiong *et al.* [34] establish a criterion for testing the robust stability of MJLSs with uncertain switching probabilities in terms of linear matrix inequalities. Moreover, there are various research directions related to switched or jump systems such as the stabilization and filtering [16], [22], [31], [35], [36] problems, model reduction [33] and state estimation [23]. In this paper, we restrict our attention to almost sure stability of random jump systems. This kind of systems encompasses a very important class of switched systems, namely, Markov jump linear systems. It is well known that even if all the subsystems are almost surely exponentially stable, stability may fail for the trajectories of the random jump systems with probability one. Conversely, possessing some unstable subsystems does not mean divergence of the jump systems with positive probability. This interesting phenomenon brings about more significant difficulties and challenges in stability analysis even for linear jump systems. Although random jump systems with finite modes have been extensively studied and have achieved remarkable development in the literature, random jump systems with general mode space are rarely studied. Some works on MJLSs with infinite modes are [7], [10] and [17]. As mentioned in [19], increasing the number of subsystems from finite to infinity may cause totally different properties of the jump systems. For example, mean square stability and stochastic stability are no longer equivalent for the MJLSs with infinite modes, while the two concepts are the same in the case where the modes are finite (see [12], [18]). Indeed, a "good" switching rule which stabilizes the trajectories of random jump systems could "get lost" in a large enough space which consists of infinite modes. Our emphasis in this work is placed on random jump systems with infinite modes.

In this paper, we consider a class of autonomous dynamic systems that jump randomly according to some switching rules. It is worth pointing out that switched linear systems with strictly stationary switching signals belong to the class of dynamic systems treated in this work. The systems under consideration are represented as transition operators in a normed matrix space which may contain uncountably infinitely many elements. A number of necessary and sufficient conditions for the exponentially almost sure stability (EAS-stability) of this class of random jump systems are established under different scenarios. The stability criteria obtained are in fact based on checking the contractivity of the jump systems after a finite number of switches have been

applied. Both continuous- and discrete-time jump systems cases are presented in this paper. It turns out that these results are not only applicable to switched linear systems with stationary switching signals, but also can be extended to MJLSs.

There are plenty of works devoted to the study of almost sure stability of MJLSs. However, most of these earlier work have only considered the case where the Markovian switching signal takes values in a finite state space. For example, the works [11] and [25] offered the sufficient conditions for almost sure stability of MJLSs of finite states. A well-known fact is that all the states of a finite and irreducible Markov process are recurrent and hence admit a stationary probability distribution. However, this fails for most Markov processes defined on infinite state space. Since the existence of stationary probability distribution is a crucial factor in [4] and [6] to establish the stability criteria, we focus our attention on the switching signal which is a positive Harris process on a general metric space. A set of necessary and sufficient conditions for exponentially almost sure stability of both continuous- and discrete-time MJLSs with general metric state space are obtained in the paper. As will be seen in the technical development in this work, several existing results on finite modes switched linear systems become the corollaries of our work. From this point of view, one of the important contributions in this work is to give a unified approach to describing the almost sure stability of MJLSs. Besides, some new scenarios on random jump systems are studied and the stability criteria are established correspondingly. These contractivity criteria for certain cases can be calculated by Monte Carlo algorithms, which are proposed by [4] and [6].

The rest of the paper is organized as follows. Sections II and III discuss the EAS-stability of a class of random jump systems which includes switched linear systems with stationary switching signals as special case. Section IV provides results on the uniform exponentially almost sure stability (UEAS-stability) of MJLSs. All the results are developed on both the continuous- and discrete-time settings. The conclusion of this work is drawn in Section V.

II. EAS-STABILITY OF CONTINUOUS-TIME JUMP SYSTEMS

We will formulate the EAS-stability problems in Section II-A for the continuous-time case. Sections II-B, II-C and II-D will provide some necessary and sufficient conditions for exponentially almost sure stability of random jump systems under different scenarios.

A. Problem Settings

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\phi = \{\phi^t\}_{t \geq 0} : \Omega \to \Omega$ be a measurable semiflow preserving probability P, which is defined as follows:

i) $(\omega, t) \mapsto \phi^t \omega : \Omega \times \mathbb{R}_+ \to \Omega$ is measurable;
ii) ϕ^0 is identity;
iii) $\phi^{s+t} = \phi^s \circ \phi^t$;

where $\mathbb{R}_+ := [0, \infty)$. Also define the nonnegative integer by $\mathbb{Z}_+$. Denote the set of all real $n \times n$ matrices by $\mathcal{M}_n(\mathbb{R})$, and let $(\mathcal{M}_n(\mathbb{R}), d)$ be a Banach space with metric d induced by the norm $\| \cdot \|$, where $\| \cdot \|$ refers to any matrix norm. A map

$(\omega, t) \mapsto T_\omega^t$ from $\Omega \times \mathbb{R}_+$ to $\mathcal{M}_n(\mathbb{R})$ is called cocycle over ϕ^t if

$$T_\omega^{s+t} = T_{\phi^s \omega}^t T_\omega^s \tag{1}$$

with $T_\omega^0 = I$, where I is the unit matrix in $\mathcal{M}_n(\mathbb{R})$. Moreover, let T_ω^t be a differentiable stochastic process taking values in $\mathcal{M}_n(\mathbb{R})$ and let its derivative be a *càdlàg* (right-continuous with finite left-hand limits) process with infinite many jumps. It is well known (cf. [29]) that a *càdlàg* process is measurable and has at most countable discontinuity points with probability one, denote the discontinuity instants of $\partial T_\omega^t / \partial t$ on time t by random sequence $\{\varsigma_i\}$, $i \in \mathbb{Z}_+$ with $\varsigma_0 := 0$ for convention. Assume $\partial T_\omega^t / \partial t = \partial T_\omega^t / \partial t|_{t=\varsigma_i}$ for any $\varsigma_i \leq t < \varsigma_{i+1}, i \geq 0$. Let $\tau_i := \varsigma_{i+1} - \varsigma_i, i \geq 0$ be the holding time, which is the period of time that the process $\partial T_\omega^t / \partial t$ remains at some value after the ith jump. Thus, for each trajectory, $\partial T_\omega^t / \partial t$ remains at the position $\partial T_\omega^t / \partial t|_{t=\varsigma_i}$ for a length of time τ_i and jumps to the value of $\partial T_\omega^t / \partial t|_{t=\varsigma_{i+1}}$ at the instant ς_{i+1}.

Now, consider the autonomous dynamic system:

$$x(t) = T_\omega^t x(0) \tag{2}$$

where the state $x(t) \in \mathbb{R}^n$ and T_ω^t is defined by (1). Then, system (2) is a random jump system. A classical example for system (2) is the linear jump system

$$\dot{x} = A_{\sigma(t)} x \tag{3}$$

with the initial value $x(0)$ and the switching signal $\sigma(t)$ being a strictly stationary process.

We are interested in finding a necessary and sufficient condition for exponentially almost sure stability of system (2) in terms of the jump instants. For this, we naturally assume the expectation of $\log \|T_\omega^{\varsigma_p}\|$ exists for any nonnegative integer p, and let $f^+(\cdot) := \max\{f(\cdot), 0\}$ and $f^-(\cdot) := \max\{-f(\cdot), 0\}$ for function $f(\cdot)$.

Definition 2.1: The random jump system is said to be almost surely exponentially stable (EAS-stable) if there is a random $\rho > 0$ such that for any initial $x(0)$

$$\lim_{t \to \infty} \sup \frac{1}{t} \log \|x(t)\| \leq -\rho, \quad \text{a.s..} \tag{4}$$

B. Nonstationary and Nonergodic Case

Stationarity and ergodicity are two common requirements in studying the kind of random jump systems discussed in this paper. A stochastic process is said to be stationary (or strictly stationary) if its joint probability distribution does not change when shifted in time or space. And ergodicity is used to describe a dynamic system which has the same behavior averaged over time as averaged over space (cf. [20]). However, we would like to consider firstmore general settings without these two requirements under which a criterion can be established. To this end, the following assumptions are made.

C1) $\sup_{0 \leq u \leq 1} \log^+ \|T_\omega^u\|$ and $\sup_{0 \leq u \leq 1} \log^+ \|T_{\phi^u \omega}^{1-u}\|$ are in $L^1(\Omega, P)$.

C2) Let the process

$$\{\log \|T_\omega^{\varsigma_i}\|\}_{i\geq 0}$$

(or $\{\log \|T_\omega^{\varsigma_i}\| - \log \|T_\omega^{\varsigma_{i-1}}\|\}_{i\geq 1}$) have (covariance) spectrum with continuity at the origin and possess finite expected values and covariances.

Remark 2.1: Recall that the (covariance) spectrum of a non-stationary process is defined by [27] as a function of some variable in $(-\pi, \pi)$, as a matter of fact, if process $\{X_i\}$ is weakly stationary (the first and second moments do not vary with respect to time, see [20]), it must have a (covariance) spectrum. This is because a sufficient condition for $\{X_i\}$ to possess a (covariance) spectrum is that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i,j=1, i-j=k}^{N} \rho(i,j), \quad k \geq 0$$

exists, where

$$\rho(i,j) = E\left[(X_i - EX_i)\overline{(X_j - EX_j)}\right]. \tag{5}$$

is the covariance. Here, the bar above an expression stands for the complex conjugate.

To facilitate the proof of the main result in this section, we need several lemmas. The first is in fact the continuous-time Multiplicative Ergodic Theorem, which provides a key tool in proving all the main theorems in this section.

Lemma 2.1 [28, Theorem B.3]: Let $\{T_\omega^t\}$ be a measurable cocycle with values in $\mathcal{M}_n(\mathbb{R})$ such that

$$\sup_{0\leq u\leq 1} \log^+ \|T_\omega^u\| \quad \text{and} \quad \sup_{0\leq u\leq 1} \log^+ \left\|T_{\phi^u\omega}^{1-u}\right\|$$

are in $L^1(\Omega, P)$. There is $\Gamma \subseteq \Omega$ with $P(\Gamma) = 1$ such that $\phi^t\Gamma \subseteq \Gamma$ for all $t \geq 0$, and the following properties hold if $\omega \in \Gamma$:
   i) $\lim_{t\to\infty} (T_\omega^{t*}T_\omega^t)^{1/2t} = \Lambda_\omega$ exists.
   ii) Let $\exp\lambda_\omega^1 < \cdots < \exp\lambda_\omega^s$ be the eigenvalues of Λ_ω (where $s = s_\omega$, the $\lambda_\omega^i, 1 \leq i \leq s$ are real and λ_ω^1 may be $-\infty$), and $U_\omega^1, \ldots, U_\omega^s$ the corresponding eigenspaces. Let $d_\omega^i = \dim U_\omega^i, 1 \leq i \leq s, V_\omega^0 = \{0\}$ and $V_\omega^i = U_\omega^1 \oplus \cdots \oplus U_\omega^i$, we have $1 \leq i \leq s$,

$$\lim_{t\to\infty} \frac{1}{t} \log \left\|T_\omega^t x\right\| = \lambda_\omega^i, \quad \text{when} \quad x \in V_\omega^i \setminus V_\omega^{i-1}.$$

Remark 2.2: If the semiflow ϕ is ergodic, s and λ_ω^i are constant almost everywhere.

Based on Multiplicative Ergodic Theorem, the following simple lemma is established.

Lemma 2.2: Under Assumption C1), let $\liminf_{k\to\infty}(\varsigma_k/k) < \infty$, a.s. If there is a subsequence $\{\varsigma_{k_s}\}$ such that

$$\lim_{s\to\infty} \frac{1}{k_s} \log \left\|T_\omega^{\varsigma_{k_s}}\right\| < 0, \quad \text{a.s.} \tag{6}$$

then the system in (2) is EAS-stable. Further, if ϕ is ergodic, the convergence rate ρ in (4) is a constant.

Proof: From the assumption of the lemma, we have

$$\limsup_{s\to\infty} \frac{1}{\varsigma_{k_s}} \log \left\|T_\omega^{\varsigma_{k_s}}\right\|$$
$$= \limsup_{s\to\infty} \frac{k_s}{\varsigma_{k_s}} \cdot \lim_{s\to\infty} \frac{1}{k_s} \log \left\|T_\omega^{\varsigma_{k_s}}\right\|$$
$$= \frac{\lim_{s\to\infty} \frac{1}{k_s} \log \left\|T_\omega^{\varsigma_{k_s}}\right\|}{\liminf_{s\to\infty} \frac{\varsigma_{k_s}}{k_s}} < 0, \quad \text{a.s..} \tag{7}$$

Since the limit of $(1/t) \log \|x(t)\|$ exists as $t \to \infty$ by Assumption C1) and Lemma 2.1, from (7), we have for any $x(0)$

$$\lim_{t\to\infty} \frac{1}{t} \log \|x(t)\| = \lim_{s\to\infty} \frac{1}{\varsigma_{k_s}} \log \|x(\varsigma_{k_s})\|$$
$$\leq \limsup_{t\to\infty} \frac{1}{\varsigma_{k_s}} \left(\log \left\|T_\omega^{\varsigma_{k_s}}\right\| + \log \|x_0\|\right)$$
$$= \limsup_{s\to\infty} \frac{1}{\varsigma_{k_s}} \log \left\|T_\omega^{\varsigma_{k_s}}\right\| < 0, \quad \text{a.s.} \tag{8}$$

which means system (2) is EAS-stable. Further, if ϕ is ergodic, by Remark 2.2 and (8), there are a positive integer s and a set of constants $\lambda^i, 1 \leq i \leq s$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log \|x(t)\| \leq \max_{1\leq i\leq s} \lambda^i < 0, \quad \text{a.s.}$$

which implies the convergence rate is a constant. ∎

We still need a mean ergodic theorem for a class of non-stationary processes and several technical lemmas.

Lemma 2.3 [27, Theorem 2]: For the class of discrete parameter stochastic processes $\{X_k\}$, which have (covariance) spectra, the continuity of the latter at the origin is a sufficient condition for the mean square convergence of

$$\frac{1}{N} \sum_{k=1}^{N} (X_k - m_k)$$

to zero, where $m_k = EX_k$ and $\rho(i,j)$ defined by (5) is finite.

Now, it is ready to represent the sufficient condition for EAS-stability of system (2) under Assumptions C1)–C2).

Lemma 2.4: Let $\liminf_{k\to\infty}(\varsigma_k/k) < \infty$, a.s.. Then, the system in (2) under C1)–C2) is EAS-stable if

$$\liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\| < 0.$$

Proof: With system (2), for any $x(0)$ and $k \geq 1$

$$x(\varsigma_k) = T_\omega^{\varsigma_k} x(0)$$

which immediately gives

$$\log \|x(\varsigma_k)\| \leq \log \|T_\omega^{\varsigma_k}\| + \log \|x(0)\|. \tag{9}$$

Now, since $\{\log \|T_\omega^{\varsigma_i}\|\}_{i\geq 0}$ satisfies Assumption C2), by Lemma 2.3, we have

$$\frac{1}{k} \sum_{i=j}^{k-1+j} \left(\log \|T_\omega^{\varsigma_i}\| - E \log \|T_\omega^{\varsigma_i}\|\right) \to 0 \quad \text{in mean square}$$

where $j = 0, 1$. Hence, for $j = 0, 1$,

$$\frac{1}{k} \sum_{i=j}^{k-1+j} (\log \|T_\omega^{\varsigma_i}\| - E \log \|T_\omega^{\varsigma_i}\|) \to 0 \quad \text{in probability.}$$

$$(10)$$

Since $\limsup_{k\to\infty}(\varsigma_k/k) < \infty$, a.s. and $\liminf_{k\to\infty}(1/k)E\log\|T_\omega^{\varsigma_k}\| < 0$, there is a subsequence $\{k_s\}$ such that

$$\lim_{s\to\infty} \frac{1}{k_s} E \log \|T_\omega^{\varsigma_{k_s}}\| = \liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\|. \quad (11)$$

By (10), we can take a further subsequence $\{k'_s\}$ from $\{k_s\}$ such that (see [3, Th. 20.5(ii)])

$$\lim_{s\to\infty} \frac{1}{k'_s} \sum_{i=1}^{k'_s} (\log \|T_\omega^{\varsigma_i}\| - E \log \|T_\omega^{\varsigma_i}\|) = 0, \quad \text{a.s.}$$

and similarly a further subsequence $\{k''_s\}$ from $\{k'_s\}$ such that

$$\lim_{s\to\infty} \frac{1}{k''_s} \sum_{i=j}^{k''_s-1+j} (\log \|T_\omega^{\varsigma_i}\| - E \log \|T_\omega^{\varsigma_i}\|) = 0, \text{ a.s., } j = 0, 1.$$

Thus, for any $\epsilon > 0$, there is an integer k_ϵ such that if $k''_s > k_\epsilon$

$$\frac{1}{k''_s} \sum_{i=j}^{k''_s-1+j} \log \|T_\omega^{\varsigma_i}\|$$
$$\in \left[\frac{1}{k''_s} \sum_{i=j}^{k''_s-1+j} E \log \|T_\omega^{\varsigma_i}\| - \epsilon, \frac{1}{k''_s} \sum_{i=j}^{k''_s-1+j} E \log \|T_\omega^{\varsigma_i}\| + \epsilon \right]$$
$$\text{a.s.,} \qquad j = 0, 1. \qquad (12)$$

Note that $\varsigma_0 = 0$, then

$$\log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - \log \left\| T_\omega^0 \right\| = \sum_{i=1}^{k''_s} \log \|T_\omega^{\varsigma_i}\| - \sum_{i=0}^{k''_s-1} \log \|T_\omega^{\varsigma_i}\|$$

and similarly

$$\frac{1}{k''_s} E \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - E \log \left\| T_\omega^0 \right\|$$
$$= \sum_{i=1}^{k''_s} E \log \|T_\omega^{\varsigma_i}\| - \sum_{i=0}^{k''_s-1} E \log \|T_\omega^{\varsigma_i}\|.$$

Hence, by (12), we immediately obtain that for all $k''_s > k_\epsilon$,

$$\frac{1}{k''_s} \left(E \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - E \log \left\| T_\omega^0 \right\| \right) - 2\epsilon$$
$$\leq \frac{1}{k''_s} \left(\log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - \log \left\| T_\omega^0 \right\| \right)$$
$$\leq \frac{1}{k''_s} \left(E \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - E \log \left\| T_\omega^0 \right\| \right) + 2\epsilon, \quad \text{a.s.}$$

which implies

$$\lim_{s\to\infty} \frac{1}{k''_s} \left(\log \left\| T_\omega^{\varsigma_{k''_s}} \right\| - E \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| \right) = 0, \quad \text{a.s.} \quad (13)$$

since $\log \|T_\omega^0\| = 0$ by the definition of T_ω^t. Consequently,

$$\lim_{s\to\infty} \frac{1}{k''_s} \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| = \lim_{s\to\infty} \frac{1}{k''_s} E \log \left\| T_\omega^{\varsigma_{k''_s}} \right\| < 0 \quad \text{a.s.}$$

$$(14)$$

where the right-hand side (RHS) of (14) exists because of (11). Since Assumption C1) holds, then, by Lemma 2.2, system (2) is EAS-stable.

For $\{\log \|T_\omega^{\varsigma_i}\| - \log \|T_\omega^{\varsigma_{i-1}}\|\}_{i\geq 1}$ satisfying Assumption C2), we can similarly obtain (13) and prove the lemma. ∎

The following lemma gives a necessary condition for EAS-stability of system (2) under Assumptions C1)–C2).

Lemma 2.5: The system in (2) under C1)–C2) is not EAS-stable if

$$\liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\| \geq 0. \qquad (15)$$

Proof: Suppose the condition in (15) is satisfied. In the proof of Lemma 2.4, we know that either $\{\log \|T_\omega^{\varsigma_i}\|\}$ or $\{\log \|T_\omega^{\varsigma_i}\| - \log \|T_\omega^{\varsigma_{i-1}}\|\}$ satisfying Assumption C2) can lead to (14). Thus,

$$\limsup_{k\to\infty} \frac{1}{k} \log \|T_\omega^{\varsigma_k}\| \geq \liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\|, \quad \text{a.s..} \quad (16)$$

Now, let e_i be the ith column of unit matrix I, and let

$$x_i(t) := T_\omega^t e_i. \qquad (17)$$

Denote $X(t)$ as the matrix $(x_1(t), x_2(t), \ldots, x_n(t))$ formed by augmenting the vectors $x_i(t)$, then

$$X(t) = T_\omega^t.$$

Hence, by (16) and the fact that $\liminf_{k\to\infty}(k/\varsigma_k) \geq 0$, we have

$$\limsup_{k\to\infty} \frac{1}{\varsigma_k} \log \|X(\varsigma_k)\|$$
$$\geq \liminf_{k\to\infty} \frac{k}{\varsigma_k} \cdot \limsup_{k\to\infty} \frac{1}{k} \log \|T_\omega^{\varsigma_k}\|$$
$$\geq \liminf_{k\to\infty} \frac{k}{\varsigma_k} \cdot \liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\| \geq 0. \quad (18)$$

As a result, at least one $\|x_i(t)\|$ with initial value $x(0) = e_i$ cannot converge to 0. System (2) is not EAS-stable. ∎

By the above arguments, we immediately obtain the necessary and sufficient condition for EAS-stability of system (2) under Assumptions C1)–C2).

Theorem 2.1: Let $\liminf_{k\to\infty}(\varsigma_k/k) < \infty$, a.s.. Then, the system in (2) under C1)–C2) is EAS-stable if and only if

$$\liminf_{k\to\infty} \frac{1}{k} E \log \|T_\omega^{\varsigma_k}\| < 0. \qquad (19)$$

Proof: It is easy to see that Theorem 2.1 is a straightforward conclusion of Lemmas 2.4 and 2.5. ∎

C. Stationary but Nonergodic Case

An important case is that $\phi^{\varsigma_i} = \phi^{i\tau_0}$ with ϕ^{τ_0} being measure-preserving, which yields the process $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}, i \geq 0\}$ to be a strictly stationary process. In most cases, map ϕ^{τ_0} are also ergodic. Now, we give another assumption instead of ergodicity on this set map to obtain a necessary and sufficient condition of EAS-stability.

C3) Let $\tau_i = f(\phi^{i\tau_0}\omega)$ for some measurable function $f(\cdot)$ and $\phi^{\tau_i} = \phi^{\tau_0}$ be a measure-preserving map for each $i \geq 0$. Also let the inverses of $T^{\tau_i}_{\phi^{\varsigma_i}\omega}$ exist and

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \rho(i) = 0$$

where

$$\rho(i) := \max_{u,v=1,2} E\left[(|\log \|T_0(u)\|| + E|\log \|T_0(u)\||)\right.$$
$$\left. \times (|\log \|T_i(v)\|| + E|\log \|T_i(v)\||)\right]$$

$T_i(1) = T^{\tau_i}_{\phi^{\varsigma_i}\omega}$ and $T_i(2) = (T^{\tau_i}_{\phi^{\varsigma_i}\omega})^{-1}$.

Remark 2.3: As a matter of fact, measure-preserving maps (cf. [20]) $\phi^{\tau_i} = \phi^{\tau_0}$ in Assumption C3) implies the stationarity of $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}, i \geq 0\}$ as we desired. We will see in the subsequent proof that the factor we used directly is the stationarity. This implies that any stationary process $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\}$ is applicable to our case. Also, under Assumption C3), ergodicity in wide sense of $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\}$ (mean-square ergodic in the first moment) is deduced by Lemma 2.7.

Lemma 2.6: If $\tau_i = f(\phi^{i\tau_0}\omega)$ for some measurable function $f(\cdot)$ and $\phi^{\tau_i} = \phi^{\tau_0}$ is a measure-preserving map for each $i \geq 0$, then both $\{\tau_i, i \geq 0\}$ and $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}, i \geq 0\}$ for any positive integer p are strictly stationary processes.

Proof: Let $m \geq 1$ be some integer. For $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m$ and real number $r_i, 1 \leq i \leq m$, define

$$A := \bigcap_{i=1}^{m} \{\tau_{t_i} < r_i\}.$$

Since $\tau_i = f(\phi^{i\tau_0}\omega)$, for any integer $k \geq 0$,

$$\{\tau_{t_i+k} < r_i, 1 \leq i \leq m\}$$
$$= \left\{f(\phi^{(t_i+k)\tau_0}\omega) < r_i, 1 \leq i \leq m\right\}$$
$$= \left\{f\left(\phi^{t_i\tau_0}(\phi^{k\tau_0}\omega)\right) < r_i, 1 \leq i \leq m\right\}. \quad (20)$$

For any $\omega \in \{\tau_{t_i+k} < r_i, 1 \leq i \leq m\}$, let $\omega' = \phi^{k\tau_0}\omega$. Then, by (20), we have $f(\phi^{t_i\tau_0}\omega') < r_i$ for all $1 \leq i \leq m$. Hence,

$$\omega \in \left\{(\phi^{k\tau_0})^{-1}\omega' : (\phi^{t_i\tau_0}\omega') < r_i, 1 \leq i \leq m\right\}. \quad (21)$$

Conversely, if ω satisfies (21), there is a ω' such that $\omega \in \{(\phi^{k\tau_0})^{-1}\omega'\}$ and $f(\phi^{t_i\tau_0}\omega') < r_i$ for all $1 \leq i \leq m$, which obviously gives

$$f\left(\phi^{t_i\tau_0}(\phi^{k\tau_0}\omega)\right) < r_i, \quad 1 \leq i \leq m.$$

Thus, by (20) again, $\omega \in \{\tau_{t_i+k} < r_i, 1 \leq i \leq m\}$. Consequently,

$$\{\omega : \tau_{t_i+k} < r_i, 1 \leq i \leq m\}$$
$$= \left\{(\phi^{k\tau_0})^{-1}\omega : f(\phi^{t_i\tau_0}\omega) < r_i, 1 \leq i \leq m\right\}$$

which implies by the definition of $f(\cdot)$ that

$$\bigcap_{i=1}^{m} \{\tau_{t_i+k} < r_i\} = (\phi^{k\tau_0})^{-1}A = \left[(\phi^{\tau_0})^{-1}\right]^k A.$$

Since ϕ^{τ_0} is a measure-preserving map, we have $P(A) = P([(\phi^{\tau_0})^{-1}]^k A)$. As a result,

$$P\{\tau_{t_i+k} < r_i, 1 \leq i \leq m\} = P\{\tau_{t_i} < r_i, 1 \leq i \leq m\}$$

which means that $\{\tau_i\}$ is strictly stationary.

Now, by the definition of ϕ^t, we have

$$\phi^{\varsigma_i}\omega = \phi^{\sum_{j=0}^{i-1}\tau_j}\omega = \phi^{\tau_{i-1}} \circ \cdots \circ \phi^{\tau_1} \circ \phi^{\tau_0}\omega$$

which, under Assumption C3), yields $\phi^{\varsigma_i}\omega = \phi^{i\tau_0}\omega$. Since $\tau_j = f(\phi^{j\tau_0})$, then

$$T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} = T^{\sum_{j=ip}^{(i+1)p-1} f(\phi^{j\tau_0})}_{\phi^{ip\tau_0}\omega}$$
$$= g\left(\phi^{ip\tau_0}\omega, \phi^{(ip+1)\tau_0}\omega, \ldots, \phi^{((i+1)p-1)\tau_0}\omega\right)$$

for some measurable $g(\cdot)$. Similarly, we cab also prove that process $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\}$ is strictly stationary. ∎

Then, we could establish a mean square convergence result under Assumption C3).

Lemma 2.7: Under Assumption C3), for any integer $p \geq 1$, we have

$$\lim_{k\to\infty} E\left|\frac{1}{k}\sum_{i=0}^{k-1}\left(\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\| - E\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\|\right)\right|^2 = 0.$$

Proof: Since $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}, i \geq 0\}$ is strictly stationary by Lemma 2.6, for any $t \geq 0$

$$E\left|\left(\log\left\|T^{\varsigma_{(t+1)p}-\varsigma_{tp}}_{\phi^{\varsigma_{tp}}\omega}\right\| - E\log\left\|T^{\varsigma_{(t+1)p}-\varsigma_{tp}}_{\phi^{\varsigma_{tp}}\omega}\right\|\right)\right.$$
$$\left.\times\left(\log\left\|T^{\varsigma_{(t+i+1)p}-\varsigma_{(t+i)p}}_{\phi^{\varsigma_{(t+i)p}}\omega}\right\| - E\log\left\|T^{\varsigma_{(t+i+1)p}-\varsigma_{(t+i)p}}_{\phi^{\varsigma_{(t+i)p}}\omega}\right\|\right)\right|$$
$$= E\left|(\log\|T^{\varsigma_p}_\omega\| - E\log\|T^{\varsigma_p}_\omega\|)\right.$$
$$\left.\times\left(\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\| - E\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\|\right)\right|.$$

Note that T^t_ω is a real matrix, as a result,

$$E\left|\frac{1}{k}\sum_{i=0}^{k-1}\left(\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\| - E\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\|\right)\right|^2$$
$$\leq \frac{2}{k^2}\sum_{i=0}^{k-1}(k-i)E\left|(\log\|T^{\varsigma_p}_\omega\| - E\log\|T^{\varsigma_p}_\omega\|)\right.$$
$$\times\left(\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\|\right.$$
$$\left.\left. - E\log\left\|T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}\right\|\right)\right|. \quad (22)$$

Denote I_A as the characteristic function, which means $I_A(\omega) = 1$ if $\omega \in A$ and $I_A = 0$ if $\omega \notin A$ for some give set $A \in \Omega$. Now, we estimate the items in the RHS of (22) by (23)

$$
\begin{aligned}
&E \Big| \big(\log \|T_\omega^{\varsigma_p}\| - E \log \|T_\omega^{\varsigma_p}\|\big) \\
&\quad \times \Big(\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| - E \log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\|\Big)\Big| \\
&\leq E\Big|\Big(\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}} - E\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}}\Big) \\
&\quad \times \Big(\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}} \\
&\qquad - E\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}}\Big)\Big| \\
&+ E\Big|\Big(\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}} \\
&\qquad - E\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}}\Big) \\
&\quad \times \Big(\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}} \\
&\qquad - E\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}}\Big)\Big| \\
&+ E\Big|\Big(\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| < 0\}} \\
&\qquad - E\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| < 0\}}\Big) \\
&\quad \times \Big(\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}} \\
&\qquad - E\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}}\Big)\Big| \\
&+ E\Big|\Big(\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| < 0\}} \\
&\qquad - E\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| < 0\}}\Big) \\
&\quad \times \Big(\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}} \\
&\qquad - E\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \\
&\qquad\quad \times I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}}\Big)\Big|.
\end{aligned}
\tag{23}
$$

Note that for $\log\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\| \geq 0$, $i \geq 0$

$$
\Big|\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\|\Big| \leq \sum_{j=0}^{p-1} \log\big\|T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big\|
\tag{24}
$$

and for $\log\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\| < 0$

$$
\Big|\log \big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\|\Big| \leq \log \Big\|\big(T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big)^{-1}\Big\|
\tag{25}
$$

$$
\leq \sum_{j=0}^{p-1} \log \Big\|\big(T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big)^{-1}\Big\|
\tag{26}
$$

where $\big(T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big)^{-1} = \big(T_{\phi^{\varsigma_{ip}}\omega}^{\tau_{ip}}\big)^{-1} \cdots \big(T_{\phi^{\varsigma_{(i+1)p-1}}\omega}^{\tau_{(i+1)p-1}}\big)^{-1}$ exists in (25) by Assumption C3). Then, we immediately obtain by (24) and (26) that

$$
\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}} \leq \sum_{j=0}^{p-1}\Big|\log\big\|T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big\|\Big|
$$

and

$$
-\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}} \leq \sum_{j=0}^{p-1} \log\Big\|\big(T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big)^{-1}\Big\|.
$$

Consequently, for any $i \geq 0$, we have inequalities (27) and (28)

$$
\begin{aligned}
&\Big|\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}} \\
&\quad - E\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}}\Big| \\
&\leq \sum_{j=0}^{p-1}\Big(\Big|\log\big\|T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big\|\Big| + E\Big|\log\big\|T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big\|\Big|\Big)
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\Big|\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}} \\
&\quad - E\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| < 0\right\}}\Big| \\
&\leq \sum_{j=0}^{p-1}\Big(\Big|\log\Big\|\big(T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big)^{-1}\Big\|\Big| \\
&\qquad + E\Big|\log\Big\|\big(T_{\phi^{\varsigma_{ip+j}}\omega}^{\tau_{ip+j}}\big)^{-1}\Big\|\Big|\Big).
\end{aligned}
\tag{28}
$$

Then, by (27), the first term in the RHS of (23) satisfies (29),

$$
\begin{aligned}
&E\Big|\Big(\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}} - E\log\|T_\omega^{\varsigma_p}\| I_{\{\log\|T_\omega^{\varsigma_p}\| \geq 0\}}\Big) \\
&\quad \times \Big(\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}} \\
&\qquad - E\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| I_{\left\{\log\big\|T_{\phi^{\varsigma_{ip}}\omega}^{\varsigma_{(i+1)p} - \varsigma_{ip}}\big\| \geq 0\right\}}\Big)\Big| \\
&\leq E\Bigg[\Bigg(\sum_{j=0}^{p-1}\Big(\Big|\log\big\|T_{\phi^{\varsigma_j}\omega}^{\tau_j}\big\|\Big| + E\Big|\log\big\|T_{\phi^{\varsigma_j}\omega}^{\tau_j}\big\|\Big|\Big)\Bigg)
\end{aligned}
$$

$$\times \left(\sum_{j=0}^{p-1} \left(\left| \log \left\| T^{\tau_{ip+j}}_{\phi^{\varsigma_{ip+j}}\omega} \right\| \right| + E \left| \log \left\| T^{\tau_{ip+j}}_{\phi^{\varsigma_{ip+j}}\omega} \right\| \right| \right) \right) \right]$$

$$\leq p \sum_{j=-p+1}^{p-1} \rho(ip+j) \qquad (29)$$

where the last inequality follows from Assumption C3) and the fact that $\{T^{\tau_i}_{\phi^{\varsigma_i}\omega}\}$ is strictly stationary. For the second term in the RHS of (23), by (27) and (28), it is less than or equal to

$$E \left[\left(\sum_{j=0}^{p-1} \left(\left| \log \left\| T^{\tau_j}_{\phi^{\varsigma_j}\omega} \right\| \right| + E \left| \log \left\| T^{\tau_j}_{\phi^{\varsigma_j}\omega} \right\| \right| \right) \right) \right.$$

$$\times \left(\sum_{j=0}^{p-1} \left(\left| \log \left\| \left(T^{\tau_{ip+j}}_{\phi^{\varsigma_{ip+j}}\omega} \right)^{-1} \right\| \right| \right.$$

$$\left. \left. + E \left| \log \left\| \left(T^{\tau_{ip+j}}_{\phi^{\varsigma_{ip+j}}\omega} \right)^{-1} \right\| \right| \right) \right) \right]$$

$$\leq p \sum_{j=-p+1}^{p-1} \rho(ip+j).$$

Similarly, the remaining two terms in the RHS of (23) are also no more than $p \sum_{j=-p+1}^{p-1} \rho(ip+j)$. Hence, by (23)

$$E \left| \left(\log \|T^{\varsigma_p}_\omega\| - E \log \|T^{\varsigma_p}_\omega\| \right) \right.$$

$$\left. \times \left(\log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| - E \log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| \right) \right|$$

$$\leq 4p \sum_{j=-p+1}^{p-1} \rho(ip+j), \qquad \text{for any } i \geq 0.$$

Then according to (22) and Assumption C3), we have

$$E \left| \frac{1}{k} \sum_{i=0}^{k-1} \left(\log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| - E \log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| \right) \right|^2$$

$$\leq \frac{8p}{k^2} \sum_{i=0}^{k-1} (k-i) \sum_{j=-p+1}^{p-1} \rho(ip+j)$$

$$\leq 16p^2 \left[\frac{1}{kp} \sum_{i=0}^{kp-1} \rho(i) \right] + o(1) \to 0$$

which completes the proof. ∎

The stability criterion of system (2) under Assumptions C1) and C3) is given as follows.

Theorem 2.2: Let $\log \|T^{\tau_0}_\omega\|$ be integrable and $E\tau_0 < \infty$. The system in (2) under Assumption C1) and C3) is EAS-stable if and only if there is an integer $p > 0$ such that

$$E \log \|T^{\varsigma_p}_\omega\| < 0. \qquad (30)$$

Proof: First, we prove the sufficiency. Let p be the integer such that $E \log \|T^{\varsigma_p}_\omega\| < 0$. Note that Lemma 2.7 implies the existence of a subsequence such that

$$\lim_{k_s \to \infty} \frac{1}{k_s} \sum_{i=0}^{k_s-1} \left(\log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| - \right.$$

$$\left. E \log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| \right) = 0, \quad \text{a.s.}$$

Furthermore, $\{T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega}, i \geq 0\}$ is strictly stationary by Lemma 2.6, then, with probability 1,

$$\lim_{k_s \to \infty} \frac{1}{k_s} \sum_{i=0}^{k_s-1} \log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| = E \log \|T^{\varsigma_p}_\omega\|. \qquad (31)$$

Note that $\{\tau_i\}$ is also strictly stationary, there is a positive random variable ς with $E\varsigma = E\tau_0$ such that

$$\lim_{k \to \infty} \frac{\varsigma_k}{k} = \varsigma, \quad \text{a.s..}$$

Since $E\tau_0 < \infty$ and $\varsigma < \infty$ almost surely,

$$\lim_{k \to \infty} \frac{k}{\varsigma_k} > 0, \quad \text{a.s.}$$

Moreover, for the system in (2),

$$\log \|x(\varsigma_{k_s})\| \leq \sum_{i=0}^{k_s-1} \log \left\| T^{\varsigma_{(i+1)p}-\varsigma_{ip}}_{\phi^{\varsigma_{ip}}\omega} \right\| + \log \|x(0)\|$$

then by using a similar argument as that for Lemma 2.2, we can prove the remaining part of sufficiency.

To prove the necessity, suppose $E \log \|T^{\varsigma_p}_\omega\| \geq 0$ for all integer $p \geq 1$. Hence,

$$\liminf_{k \to \infty} \frac{1}{k} E \log \|T^{\varsigma_k}_\omega\| \geq 0.$$

In fact, from [21, Th. 1], we know that the limit inferior can be replaced by limit, and we rewrite the above inequality as

$$\lim_{k \to \infty} \frac{1}{k} E \log \|T^{\varsigma_k}_\omega\| \geq 0.$$

Since $\log \|T^{\tau_0}_\omega\|$ is integrable, the proof of [21, Th. 2] implies that there is a random variable ψ with

$$E\psi \geq \lim_{k \to \infty} \frac{1}{k} E \log \|T^{\varsigma_k}_\omega\| \geq 0 \qquad (32)$$

such that

$$\liminf_{k \to \infty} \frac{1}{k} \log \left\| T^{\varsigma_{k+1}}_{\phi^{\varsigma_1}\omega} \right\| \geq \psi, \quad \text{a.s..} \qquad (33)$$

Note that, by (32), there is a set $D \subset \Omega$ with $P(D) > 0$ such that $\psi > 0$ on D. This immediately yields by (33) that

$$\liminf_{k \to \infty} \frac{1}{k} \log \left\| T^{\varsigma_{k+1}}_{\phi^{\varsigma_1}\omega} \right\| \geq 0, \quad \text{a.s.}$$

which implies the system in (2) is divergent on D. Thus, if the system in (2) is almost surely stable, there must exist an integer $p \geq 1$ such that (30) holds. ∎

D. Stationary and Ergodic Case

In this subsection, we will provide a necessary and sufficient condition for EAS-stability of system (2) with ergodicity of map ϕ^{τ_0}.

C4) Let $\tau_i = f(\phi^{i\tau_0}\omega)$ for some measurable function $f(\cdot)$ and $\phi^{\tau_i} = \phi^{\tau_0}$ be a measure-preserving and ergodic map for each $i \geq 0$.

To derive the result, we need two technical lemmas. The first lemma relax the condition of Birkhoff–Khinchin Theorem ([20]) to admit the expected value being minus infinite.

Lemma 2.8: Under Assumptions C4), if $E \log^+ \|T_\omega^{\varsigma_p}\| < \infty$, we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \to E \log \|T_\omega^{\varsigma_p}\|, \quad \text{a.s.} \quad (34)$$

Proof: By Assumption C4) and Lemma 2.6, the process

$$\left\{ \log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \right\}$$

is strictly stationary and ergodic; hence, if $E|\log \|T_\omega^{\varsigma_p}\|| < \infty$, the Birkhoff–Khinchin Theorem yields (34). So, we only need to consider the case $E \log \|T_\omega^{\varsigma_p}\| = -\infty$. For any functions f and g, let $f \vee g := \max\{f, g\}$. Since for any real number r

$$E \left| \log \|T_\omega^{\varsigma_p}\| \vee r \right|$$
$$\leq E \left(\log \|T_\omega^{\varsigma_p}\| I_{\{\log \|T_\omega^{\varsigma_p}\| > 0\}} \right)$$
$$+ E \left| \log \|T_\omega^{\varsigma_p}\| I_{\{r < \log \|T_\omega^{\varsigma_p}\| \leq 0\}} \right| + E|r| I_{\{\log \|T_\omega^{\varsigma_p}\| \leq r\}}$$
$$\leq E \log^+ \|T_\omega^{\varsigma_p}\| + |r| < \infty$$

by the Birkhoff–Khinchin Theorem, we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \left(\log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \vee r \right)$$
$$= E(\log \|T_\omega^{\varsigma_p}\| \vee r), \quad \text{a.s.} \quad (35)$$

Since

$$\log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \leq \left(\log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \vee r \right)$$

by (35), we immediately have

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \log \left\| T_{\phi^{\varsigma_{jp}}\omega}^{\sum_{s=0}^{p-1} \tau_{jp+s}} \right\| \leq E \left(\log \|T_\omega^{\varsigma_p}\| \vee r \right), \quad \text{a.s.} \quad (36)$$

Now, let $r \to -\infty$ in the RHS of (36), we obtain (34) again by noting that

$$\lim_{r\to-\infty} E \left(\log \|T_\omega^{\varsigma_p}\| \vee r \right) = -\infty$$

which completes the proof. ∎

Lemma 2.9 [8, Lemma 3.4]: Let $\{M_i\}$ be an ergodic strictly stationary sequence of matrices in $\mathcal{M}_n(\mathbb{R})$. If $E(\log^+ \|M_0\|)$ is finite and that, almost surely,

$$\lim_{i\to\infty} \|M_i M_{i-1} \cdots M_0\| = 0$$

then the top Lyapunov exponent

$$\gamma := \inf \left\{ E \left(\frac{1}{i+1} \log \|M_i M_{i-1} \cdots M_0\| \right), i \in \mathbb{N} \right\}$$

associated with the sequence $\{M_i\}_{i\geq 0}$ is strictly negative.

Now, the stability criterion of system (2) under C1) and C4) is represented by

Theorem 2.3: Let $E \log^+ \|T_\omega^{\tau_0}\| < \infty$ and $E\tau_0 < \infty$. The system in (2) under Assumptions C1) and C4) is EAS-stable if and only if there is an integer $p > 0$ such that (30) holds.

Proof: Since $\phi^{\tau_i} = \phi^{\tau_0}$ is measure-preserving and ergodic, by Lemma 2.6, $\{\tau_i\}$ is ergodic and hence

$$\lim_{k\to\infty} \frac{\varsigma_k}{k} = E\tau_0 < \infty. \quad (37)$$

This together with Lemma 2.8 implies the sufficiency by a similar argument as that of Theorem 2.2.

For the necessity, note that if system (2) is EAS-stable, similar to (18), we have by (37) that

$$\lim_{k\to\infty} \frac{1}{k} \log \|T_\omega^{\varsigma_k}\| < 0, \quad \text{a.s.}$$

which implies $\lim_{k\to\infty} \|T_\omega^{\varsigma_k}\| = 0$. Then, by Lemma 2.9

$$\inf \left\{ E \left(\frac{1}{k} \log \|T_\omega^{\varsigma_k}\| \right) \right\} < 0$$

and hence the lemma is proved ∎

Corollary 2.1: Let the semiflow ϕ be ergodic. Then, the system in (2) under Assumption C1) is EAS-stable with constant convergence rate if and only if there is a finite $t > 0$ such that

$$E \log \|T_\omega^t\| < 0. \quad (38)$$

Proof: Let $t > 0$ satisfy $E \log \|T_\omega^t\| < 0$. Then there is some integer $p \geq 1$ such that

$$\tau_0 := \frac{t}{p} \leq 1. \quad (39)$$

Consequently, by Assumption C1)

$$E \log^+ \|T_\omega^{\tau_0}\| \leq E \sup_{0 \leq u \leq 1} \log^+ \|T_\omega^u\| < \infty. \quad (40)$$

Let $f(\omega) \equiv t/p$ for all $\omega \in \Omega$, then $\tau_i = \tau_0$ for all $i \geq 1$, and hence

$$\phi^{\tau_i} = \phi^{\tau_0} = \phi^{\frac{t}{p}} \quad (41)$$

is measure-preserving and ergodic since ϕ is measure-preserving and ergodic. Obviously, $E\tau_0 = t/p < \infty$, then by Theorem 2.3, system (2) is EAS-stable. Further, by the same

arguments as those for Lemma 2.2, we know the convergence rate ρ is a constant.

Conversely, assume that system (2) is EAS-stable. Let $\tau_i = \tau_0$ defined by (39), where $t/p \leq 1$ can be taken as any constant, then from (40), (41) and Theorem 2.3, we know that the necessity is true. ∎

Remark 2.4: By Corollary 2.1, we can immediately deduce [5, Th. 3.2], where the system is linear and switched as in (3).

III. EAS-STABILITY FOR DISCRETE-TIME JUMP SYSTEMS

For the fixed probability space $(\Omega, \mathcal{F}, P)$, let φ be a measurable map preserving P and $T : \Omega \rightarrow \mathcal{M}_n(\mathbb{R})$ a measurable function to the real $n \times n$ matrices. Write

$$T_\omega^k = T(\varphi^{k-1}\omega) \cdots T(\varphi\omega)T(\omega).$$

Consider the dynamic system:

$$x_k = T_\omega^k x_0 \tag{42}$$

where the state $x_k \in \mathbb{R}^n$ and the expected values of $\log \|T_\omega^k\|$ exist for all $k \geq 0$. Now, we present the definition of EAS-stability for discrete-time dynamic systems as follows.

Definition 3.1: The random system in (42) is said to be EAS-stable if there is a random $\nu > 0$ such that for any initial $x(0)$ and initial distribution $\sigma(0)$,

$$\lim_{k \to \infty} \sup \frac{1}{k} \log \|x(k)\| \leq -\nu, \quad \text{a.s.}$$

All the results presented below can be worked out by a similar argument as those of the continuous-time case employing multiplicative ergodic theorem for discrete-time system (cf. [28, Th. 1.6]) and their proofs are omitted for brevity. Now, we list a series of assumptions in the following and then present several necessary and sufficient conditions for almost sure stability under different assumptions.

D1) $\log^+ \|T(\cdot)\|$ is in $L^1(\Omega, P)$.
D2) Let the process $\{\log \|T_\omega^i\|\}_{i \geq 0}$ (or $\{\log \|T_\omega^i\| - \log \|T_\omega^{i-1}\|\}_{i \geq 1}$) have (covariance) spectrum with continuity at the origin and possess finite expected values and covariances.
D3) Let the inverses of $T(\cdot)$ exists and

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \rho(i) = 0$$

where

$$\rho(i) := \max_{u,v=1,2} E\left[(|\log \|T_0(u)\||\right.$$
$$\left. + E |\log \|T_0(u)\||)(|\log \|T_i(v)\|| + E |\log \|T_i(v)\||)\right]$$

$$T_i(1) = T(\varphi^i\omega) \text{ and } T_i(2) = T^{-1}(\varphi^i\omega), i \geq 0.$$

D4) The map φ is ergodic.

Theorem 3.1: The system in (42) under Assumptions D1)–D2) is EAS-stable if and only if

$$\lim_{k \to \infty} \inf \frac{1}{k} E \log \|T_\omega^k\| < 0.$$

Theorem 3.2: Let $\log \|T(\omega)\|$ be integrable. The system in (42) under Assumptions D1) and D3) is EAS-stable if and only if there is an integer $p > 0$ such that

$$E \log \|T_\omega^p\| < 0. \tag{43}$$

Theorem 3.3: Let $E \log^+ \|T(\omega)\| < \infty$. The system in (42) under Assumptions D1) and D4) is EAS-stable with constant convergence rate if and only if there is an integer $p > 0$ such that (43) holds.

IV. UEAS-STABILITY OF MJLSs

We now extend our results for the random jump systems in the previous sections to MJLSs with the switching signal taking values in some general metric spaces. First, we establish a necessary and sufficient condition for uniformly exponentially almost sure stability of a continuous-time MJLSs in Section IV-A. Then, the discrete-time MJLSs will be discussed in Section IV-B.

A. Continuous-Time MJLSs

In this section, we will apply the previous results to the following MJLS:

$$\dot{x} = A_{\sigma(t)} x \tag{44}$$

where $x(t) \in \mathbb{R}^n$, $\sigma(t)$ is a switching signal, which is assumed to be a stationary Markov process with right continuous and finite left-hand limits trajectory, taking values in a general metric state space $(\mathbb{S}, \mathcal{B})$ ($\mathcal{B}$ is the Borel σ-field of $\mathbb{S}$), and $A_\alpha \in \mathcal{M}(\mathbb{R})$, $\alpha \in \mathbb{S}$. As Section II, we can define the jump instants of $\sigma(t)$ by $\{\varsigma_i\}_{i \geq 0}$ with $\varsigma_0 = 0$ and assume $\sigma(t) = \sigma(\varsigma_i)$ for any $t \in [\varsigma_i, \varsigma_{i+1})$. Let $\{\tau_i\}$ be the holding time such that $\tau_i = \varsigma_{i+1} - \varsigma_i$.

Note that the convergence rate ρ in Definition 2.1 is a random variable and the switching rule is fixed, we now give a further definition on uniform exponentially almost sure stability.

Definition 4.1: The random jump system is said to be uniform exponentially almost surely stable (UEAS-stable) if there is a constant $\rho > 0$ such that (4) holds for any initial $x(0)$ and initial distribution $\sigma(0)$.

First, we consider the simple case where $\mathbb{S}$ is a countable state space, say $\mathbb{Z}^+ := \{1, 2, \ldots\}$. Let Π be a transition probability matrix with the property that $\Pi_{ii} = 0$ for all $i \in \mathbb{S}$ and $\mathcal{R} = \{R_i, i \in \mathbb{S}\} \subset [0, \infty)$ a family of rates. Now, construct an $\mathbb{S}$-valued continuous-time Markov process $\{\sigma(t)\}$ with rates $\mathcal{R}$ and transition probability matrix Π by (for example, [30]):
  i) the trajectory of $\sigma(t)$ is piecewise constant and right continuous,
  ii) $P\{\varsigma_i > \varsigma_{i-1} + t, \sigma(\varsigma_i) = j | \sigma(\tau), \tau \in [0, \varsigma_i)\} = e^{-tR_{\sigma(\varsigma_{i-1})}} \Pi_{\sigma(\varsigma_{i-1})j}$ on $\{\varsigma_{i-1} < \infty\}$.

Obviously, $\sigma(t)$ is a stationary Markov process (all the Markov processes mentioned hereafter are also assumed to be time-homogeneous). Let $\sigma_i = \sigma(\varsigma_i), i \geq 0$, then $\{\sigma_i\}$ is an embedded Markov chain of $\{\sigma(t)\}$ with transition probability matrix Π. Now, system (44) turns out to be a Markov jump linear system (MJLS). Let $\Phi(t, s)$ denote the state transition matrix of system (44) over time interval $[s, t]$. Obviously, it is a

random matrix. Also, let E_π be the expected value with respect to distribution π.

To establish the main result in this section, we first provide three lemmas. The first lemma indicates that if the rates of the Markov process $\{\sigma(t)\}$ are bounded from below, the ratio of jump time ς_k and the natural number k cannot be arbitrarily large.

Lemma 4.1: If the rates of Markov process $\sigma(t)$ satisfy $\inf_{R_i \in \mathcal{R}} R_i \geq K_1$ for some constant $K_1 > 0$, then $\liminf_{k\to\infty}(\varsigma_k/k) < \infty$, a.s.

Proof: Since given $\sigma_k \geq 0$, τ_k is exponentially distributed with parameter R_{σ_k}, we have

$$P\{R_{\sigma_k}\tau_k > t | \sigma_k\} = e^{-t} \tag{45}$$

which yields $E\{R_{\sigma_k}\tau_k | \sigma_k\} = 1$. Hence, it can be immediately obtained that for any $k \geq 0$,

$$ER_{\sigma_k}\tau_k = \int E\{R_{\sigma_k}\tau_k | \sigma_k\} \, dP = 1. \tag{46}$$

Furthermore, since $E\{R_{\sigma_k}^2\tau_k^2 | \sigma_k\} = 2$ by (45), we have

$$E(R_{\sigma_k}\tau_k - 1)^2 = ER_{\sigma_k}^2\tau_k^2 - 1 = 1.$$

Consequently,

$$\sum_{k=1}^{\infty} \frac{E(R_{\sigma_k}\tau_k - 1)^2}{k^2} < \infty. \tag{47}$$

Now, note that by the definition of $\sigma(t)$

$$P\{R_{\sigma_k}\tau_k > t | R_{\sigma_s}\tau_s, 0 \leq s \leq k-1\} = e^{-t}$$

which implies that the process $\{R_{\sigma_k}\tau_k - 1\}$ is mutually independent. Note that by (46), $E(R_{\sigma_k}\tau_k - 1) = 0$. Then, from (47) we obtain

$$\frac{\sum_{j=1}^{k}(R_{\sigma_j}\tau_j - 1)}{k} \to 0, \quad \text{a.s.}$$

Moreover, since the rates $\inf_{R_i \in \mathcal{R}} R_i \geq K_1 > 0$; hence,

$$\liminf_{k\to\infty} \frac{\varsigma_k}{k} \leq \lim_{k\to\infty} \frac{\sum_{j=1}^{k} R_{\sigma_j}\tau_j}{K_1 k} = \frac{1}{K_1}$$

which completes the proof. ∎

The following lemma establishes the Strong Law of Large Numbers (SLLN) for the state transition matrix at jump instances.

Lemma 4.2: If $\{\sigma(t)\}$ is a Q-irreducible and Q-recurrent Markov process with transition probability matrix P and rates $\mathcal{R} = \{R_i, i \in \mathbb{S}\} \subset [K_1, K_2]$ for some $0 < K_1 < K_2 < \infty$, then for any initial distribution μ, there is a distribution $\pi = \pi\Pi$ such that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \log\left\|\Phi\left(\varsigma_{ip}, \varsigma_{(i-1)p}\right)\right\| = E_\pi \log\|\Phi(\varsigma_p, 0)\|, \quad \text{a.s.} \tag{48}$$

Proof: Since $\sigma(t)$ is Q-irreducible and Q-recurrent, we know that the embedded Markov chain $\{\sigma_i\}$ is irreducible and recurrent. Hence, there is a unique invariant measure π for $\{\sigma_i\}$. Moreover, by Q-positive recurrence, $\sigma(t)$ possesses a unique stationary probability $\tilde{\pi}$ with $\tilde{\pi}_j > 0$ for all $j \in \mathbb{S}$, where $\tilde{\pi}_j$ is the jth component of $\tilde{\pi}$.

Note that $0 < K_1 < R_i < K_2 < \infty, \forall i \in \mathbb{S}$, it can be concluded that the invariant measure π of $\{\sigma_i\}$ satisfies (see [9])

$$\pi_i = \frac{\tilde{\pi}_i R_i}{\sum_{j\in\mathbb{S}} \tilde{\pi}_j R_j} > 0. \tag{49}$$

Hence, the embedded Markov chain $\{\sigma_i\}$ is an irreducible and positive recurrent chain taking values on a countable state space, which means that π is a probability distribution defined by the lemma.

Now, since $\sup_{i\in\mathbb{S}} \|A_i\| < \infty$

$$E_\pi \log^+ \|\Phi(\varsigma_p, 0)\| = E_\pi \log^+ \left\|e^{\sum_{i=1}^{p} A_{\sigma(\varsigma_i)}\tau_{i-1}}\right\|$$

$$\leq \left(\sup_{i\in\mathbb{S}} \|A_i\|\right)\left(\sum_{i=1}^{p} E\tau_{i-1}\right)$$

$$\leq \frac{p}{K_1}\left(\sup_{i\in\mathbb{S}} \|A_i\|\right) < \infty$$

where the first inequality follows from

$$E\tau_i = \sum_{j\in\mathbb{S}} \frac{1}{R_j}P\{\sigma_i = j\} \leq \frac{1}{K_1}.$$

Thus, SLLN holds for $\{\log\|\Phi(\varsigma_{ip}, \varsigma_{(i-1)p})\|\}$ (see [26, Th. 17.0.1]) and the case

$$E_\pi \log^- \|\Phi(\varsigma_p, 0)\| = \infty$$

can be treated similarly as Lemma 2.8 and yields (48). ∎

Note that the stability criteria in Section II are all derived for some given initial distributions, to apply the previous results in the current case where the initial distribution is arbitrary, we need the following lemma:

Lemma 4.3: Let $\{\sigma(t)\}$ be a Q-irreducible and Q-recurrent Markov process with state space $(\mathbb{S}, \mathcal{B})$. If, for some initial distribution μ_0 of $\sigma(t)$,

$$\limsup_{t\to\infty} \frac{1}{t}\log\|x(t)\| < 0, \quad \text{a.s.} \quad \forall x(0) \tag{50}$$

then, for all initial distribution μ, (50) still holds.

Proof: First, note that when the initial distribution of $\sigma(t)$ is μ_0

$$\limsup_{t\to\infty} \frac{1}{t}\log\|\Phi(t, 0)\| < 0, \quad \text{a.s.} \tag{51}$$

because of (50). Now, for any $x \in \mathbb{S}$, let

$$g(x) = P_x\left\{\limsup_{t\to\infty} \frac{1}{t}\log\|\Phi(t, 0)\| < 0\right\}$$

where P_x denotes the probability of events conditional on the chain beginning with $\sigma(0) = x$. Hence, by (51)

$$\int g(x)d\mu_0 = 1. \tag{52}$$

Define its transition function for the time-homogeneous $\{\sigma(t)\}$ by

$$P_s(\sigma(t), B) := P\{\sigma(t+s) \in B | \mathcal{F}_t^\sigma\}, \quad \forall s \geq t \geq 0$$

where $B \in \mathcal{B}$ and $\mathcal{F}_t^\sigma$ is the σ-field generated by $\{\sigma(u), u \leq t\}$. Then, for any $s \geq 0$, from the Chapman–Kolmogorov property, we have

$$\int P_s(x, dy)g(y)$$
$$= \int P_s(x, dy)P_y\left\{\limsup_{t \to \infty} \frac{1}{t} \log \|\Phi(t, 0)\| < 0\right\}$$
$$= \int P_s(x, dy)$$
$$\quad \times P\left\{\limsup_{t \to \infty} \frac{1}{t} \log \|\Phi(t+s, s)\| < 0 | \sigma(s) = y\right\}$$
$$= P_x\left\{\limsup_{t \to \infty} \frac{1}{t} \log \|\Phi(t+s, s)\| < 0\right\}$$
$$= P_x\left\{\limsup_{t \to \infty} \frac{1}{t+s} \log \|\Phi(t+s, 0)\| < 0\right\}$$

where the last equality follows from the fact that the inverse of $\Phi(s, 0)$ exists and

$$\log \|\Phi(t+s, s)\| - \log \|\Phi^{-1}(s, 0)\|$$
$$\leq \log \|\Phi(t+s, 0)\|$$
$$\leq \log \|\Phi(t+s, s)\| + \log \|\Phi(s, 0)\|.$$

Consequently,

$$\int P_s(x, dy)g(y) = g(x). \tag{53}$$

Thus, for any $t_1 \geq t_2 \geq 0$ and any initial distribution μ, by the Markov property and (53), we have

$$E\left[g(\sigma(t_1)) | \mathcal{F}_{t_2}^\sigma\right] = \int P_{t_1-t_2}(\sigma(t_2), dy) g(y)$$
$$= g(\sigma(t_2)), \quad \text{a.s.}$$

which means $\{g(A_{\sigma(t)}), \mathcal{F}_t^A\}$ is a martingale process. Furthermore, note that $0 \leq g(\sigma(t)) \leq 1$, by the martingale convergence theorem, for any initial distribution μ, $G := \lim_{t \to \infty} g(\sigma(t))$ exists almost surely. And hence by Lebesgue's dominated convergence theorem, given $x \in \mathbb{S}$

$$E_x[G] = \lim_{t \to \infty} E_x g(\sigma(t)) = g(x) \tag{54}$$

where E_x denotes the expected value with respect to P_x.

Now, since $\{\sigma(t)\}$ is Q-irreducible and Q-recurrent, if there is a real constant $c \in [0, 1]$ such that $g(x_0) = c$ for some $x_0 \in \mathbb{S}$, then we have

$$P_x\{g(\sigma(t)) = c, \text{i.o.}\} = 1$$

where "i.o." means "infinitely often." Hence, for the initial distribution μ,

$$P\{g(\sigma(t)) = c, \text{i.o.}\} = \int P_x\{g(\sigma(t)) = c, \text{i.o.}\}\, d\mu = 1.$$

Therefore, there is a random sequence $\{t_n\}$ such that

$$G = \lim_{n \to \infty} g(\sigma(t_n)) = c, \quad \text{a.s.}$$

Consequently, by (54), for all $x \in \mathbb{S}$ we have

$$g(x) = E_x[G] = c.$$

Hence, by (52) we know that $c = 1$ and, for the initial distribution μ

$$\int g(x)\mu = 1.$$

This shows that for any initial distribution μ, (51) still holds and hence leads to (50). ∎

Remark 4.1: The conclusion of Lemma 4.3 also holds for the case where $\mathbb{S}$ is a general metric state space. The analysis before (54) works well and what remains is just to show that $g(x) \geq c$ for some constant c if $\{x : g(x) \geq c\} \in \mathcal{B}^+$ and $g(x) \leq c$ if $\{x : g(x) \leq c\} \in \mathcal{B}^+$, respectively, where $\mathcal{B}^+ := \{B \in \mathcal{B} : \psi(B) > 0\}$ (For the definition of ψ-measure, see [26]). The arguments of the remainder are similar to those in the above proof.

Theorem 4.1: Let $\mathbb{S}$ be a countable state space and $\sup_{i \in \mathbb{S}} \|A_i\| < \infty$. If $\sigma(t)$ is a Q-irreducible and Q-positive recurrent Markov process with transition probability matrix Π and bounded rates $\mathcal{R} = \{R_i, i \in \mathbb{S}\} \subset [K_1, K_2]$ for some $0 < K_1 < K_2 < \infty$, the MJLS (44) is UEAS-stable if and only if there is an integer $p > 0$ such that

$$E_\pi \log \|\Phi(\varsigma_p, 0)\| < 0 \tag{55}$$

where π is the distribution satisfying $\pi = \pi\Pi$.

Proof: First, we prove the sufficiency. By the conditions of the theorem, we know that $\{\sigma(t)\}$ possesses a unique stationary probability distribution $\tilde{\pi}$, which is assumed to be the initial distribution at this point. Hence, $\{\sigma(t)\}$ turns out to be a strictly stationary process. To apply the results in Section II, let $T_\omega^t = \Phi(t, 0)$, which is a cocycle as we discussed before. Thus, the system (44) coincides with (2). By Lemmas 4.1 and 4.2, $\liminf_{k \to \infty}(\varsigma_k/k) < \infty$, a.s. and for all $k \geq 1$

$$\lim_{k \to \infty} \frac{1}{kp} \log \|\Phi(\varsigma_{kp}, 0)\|$$
$$\leq \lim_{k \to \infty} \frac{1}{kp} \sum_{i=1}^{k} \log \|\Phi(\varsigma_{ip}, \varsigma_{(i-1)p})\|$$
$$= \frac{1}{p} E_\pi \log \|\Phi(\varsigma_p, 0)\| < 0, \quad \text{a.s.}$$

From Lemma 2.2, we know that (4) holds when the initial distribution of $\sigma(t)$ is $\tilde{\pi}$, and hence holds for all initial distributions by Lemma 4.3. The UEAS-stability is thus proved.

The necessity part is obvious by Theorem 3.3 as $\{\Phi(\varsigma_i, \varsigma_{(i-1)})\}$ is strictly stationary and ergodic if the initial distribution is chosen as π. ∎

Remark 4.2:
i) The condition $K_2 < \infty$ guarantees $\varsigma_i \to \infty$ as $i \to \infty$ almost surely.
ii) For irreducible Markov process $\sigma(t)$ with finite state space $\mathbb{S}$, all the conditions in Theorem 4.1 are satisfied automatically. Hence, system (44) with switching signal $\{\sigma(t)\}$ being finite and irreducible Markov process is EAS-stable if and only if there is an integer $p > 0$ such that (55) holds, which is the result of [6, Th. 3].

For the general case, where $\sigma(t)$ takes values in a metric state space $(\mathbb{S}, \mathcal{B})$, we consider the simplest but a very important class of Markov jump processes with a bounded generator (see [14] for detailed construction). Intuitively speaking, the process starts from a point $\sigma(0) = \alpha_0 \in \mathbb{S}$ and remains there for an exponentially distributed holding time τ_0 with parameter $R(\alpha_0)$, at which time it jumps to a new position $\sigma(1) = \alpha_1$ according to the Markov transition function $P(\alpha_0, \cdot)$. It then remains at α_1 for another length of time τ_1, which is exponentially distributed with parameter $R(\alpha_1)$. The holding time τ_1 is independent of τ_0 given α_1. Then, it jumps to α_2 according to $P(\alpha_1, \cdot)$, and so on. Obviously, $\sigma(t)$ is generated by the embedded Markov chain $\{\sigma_i\}$ with the transition function $P(\alpha, \cdot)$, where $\sigma(\varsigma_i) = \sigma_i$ as defined before. The stationary Markov processes taking values on a countable state space that we discussed above belong to this class.

We can obtain a similar result as that of Theorem 4.1 for the class of Markov jump processes $\{\sigma(t)\}$ with bounded generator. The proof idea is the same as that of Theorem 4.1, and the details are omitted. In fact, recall that $\{\sigma(t)\}$ is Harris if and only if there exists a nonzero σ-finite measure μ on its state space $(\mathbb{S}, \mathcal{B})$ such that for all $B \in \mathcal{B}$,

$$\mu(B) > 0 \Rightarrow P_\alpha \left\{ \int_0^\infty 1_B(\sigma(t))\, dt = \infty \right\} = 1, \quad \forall \alpha \in \mathbb{S}.$$

Now, let $\{\sigma(t)\}$ be positive Harris recurrent, by the construction of $\{\sigma(t)\}$, we have

$$\mu(B) > 0 \Rightarrow P_\alpha\{\sigma_i \in B \text{ for some } i \geq 1\} = 1, \quad \forall \alpha \in \mathbb{S}$$

where μ is defined above. Hence, $\{\sigma_i\}$ is a Harris chain. Moreover, assume $0 < \inf_{\alpha \in \mathbb{S}} R(\alpha) < \sup_{\alpha \in \mathbb{S}} R(\alpha) < \infty$, it is easy to see from [14] that the invariant measure π of $\{\sigma_i\}$ satisfies

$$\pi(A) = \frac{\int_A R(\alpha)\, d\tilde{\pi}}{\int R(\alpha)\, d\tilde{\pi}} \in (0, \infty), \qquad \forall A \in \mathcal{B} \qquad (56)$$

where $\tilde{\pi}$ is the invariant probability distribution of $\{\sigma(t)\}$. Then, $\{\sigma_i\}$ is a positive Harris chain. As a result, SLLN holds for $\{\sigma_i\}$. Moreover, note that the holding times are constructed almost the same as those of Markov processes taking values on a countable state space, except that the set of parameters are uncountable, the proof of Lemma 4.1 also works for the current case. Consequently, $\liminf_{k \to \infty}(\varsigma_k/k) < \infty$, a.s.. By Remark 4.1, we immediately deduce the following result.

Theorem 4.2: Let $\sup_{\alpha \in \mathbb{S}} \|A_\alpha\| < \infty$ and the switching signal $\{\sigma(t)\}$ be a Markov jump process with a bounded generator. If $\{\sigma(t)\}$ is positive Harris recurrent and the parameters of the holding times $\mathcal{R} = \{R(\alpha), \alpha \in \mathbb{S}\} \subset [K_1, K_2]$, where $0 < K_1 < K_2 < \infty$ are constants. Then, system (44) is UEAS-stable if and only if there is an integer $p > 0$ such that

$$E_\pi \log \|\Phi(\varsigma_p, 0)\| < 0 \qquad (57)$$

where π is defined by (56).

Remark 4.3: From the previous definitions of Markov processes in this section, it is clear that the rates of $\{\sigma(t), t \geq 0\}$ depend on the past states. That is, for the ith jump, rate $R_{\sigma(\varsigma_{i-1})}$ is determined by state $\sigma(\varsigma_{i-1})$, where process $\sigma(t)$ remains during the time interval $[\varsigma_{i-1}, \varsigma_i)$. Thus, the continuous-time MJLSs concerned with both Theorems 4.1 and 4.2 have the rates depending on the states (it is also the case for the discrete-time MJLSs discussed in Section IV-B). Note that $\{\sigma(t)\}$ is a first-order Markov process, for the linear systems with random piecewise constant parameters following a high-order Markov process rule, the stability properties can be dealt with as well in principle by a similar approach.

B. Discrete-Time MJLSs

Finally, we study the discrete-time MJLS:

$$x_{k+1} = A_{\sigma_k} x_k \qquad (58)$$

where $x_k \in \mathbb{R}^n$, σ_k is a discrete-time switching signal taking values in a general metric state space $(\mathbb{S}, \mathcal{B})$ and $A_\alpha \in \mathcal{M}(\mathbb{R})$, $\alpha \in \mathbb{S}$. Let $\Phi(k, l)$ denote the state transition matrix of system (58) over time interval $[k, l]$, where $k, l \geq 0$ are integers. Now, assume $\{\sigma_k\}$ is a positive Harris chain, which means it possesses a stationary probability distribution π. Coinciding with the continuous-time case, we state the following result without proof since the proof idea is the same while the techniques are much easier.

Theorem 4.3: Let $\sup_{\alpha \in \mathbb{S}} \|A_\alpha\| < \infty$ and the switching signal $\{\sigma_k\}$ be a positive Harris chain. Then, system (58) is UEAS-stable if and only if there is an integer $p > 0$ such that

$$E_\pi \log \|\Phi(p, 0)\| < 0 \qquad (59)$$

where π is the stationary probability distribution of $\{\sigma_i\}$.

Since an irreducible and positive recurrent chain taking values on a countable state space is a positive Harris chain, we obtain the following corollary from Theorem 4.3 directly.

Corollary 4.1: Let $\sup_{i \in \mathbb{S}} \|A_i\| < \infty$ with $\mathbb{S}$ being a countable state space and let the switching signal $\{\sigma_k\}$ be an irreducible and positive recurrent chain. Then, system (58) is UEAS-stable if and only if there is an integer $p > 0$ such that (59) holds.

Remark 4.4: Every finite irreducible Markov chain is positive recurrent, thus we can conclude that system (58) with the switching signal $\{\sigma_k\}$ being a finite and irreducible Markov chain is UEAS-stable if and only if there is an integer $p > 0$ such that (59) holds, which reduces to the result of [4, Proposition 3.4].

V. CONCLUSION

This paper has studied the exponentially almost sure stability of random jump systems arising in modeling certain classes of stochastic switched systems. The problem requires the modes of the random jump systems to take values from a normed matrix space. The solution was obtained by checking the contractivity of the jump systems after a finite number of switches being applied. A series of necessary and sufficient conditions for exponentially almost sure stability of this class of random jump systems were obtained for both the continuous- and discrete-

time cases. These results were further extended to the Markov jump linear systems with general metric state space and obtained a series of necessary and sufficient conditions for exponentially almost sure stability of both continuous- and discrete-time MJLSs, establishing a unified approach to describing the almost sure stability of MJLSs.

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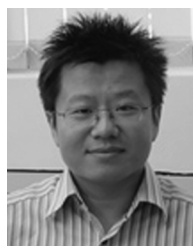
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