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A New Condition and Equivalence Results for Robust Stability Analysis of Rationally Time-Varying Uncertain Linear Systems

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Abstract—Uncertain systems is a fundamental area of automatic control. This paper addresses robust stability of uncertain linear systems with rational dependence on unknown time-varying parameters constrained in a polytope. For this problem, a new sufficient condition based on the search for a common homogeneous polynomial Lyapunov function is proposed through a particular representation of parameter-dependent polynomials and LMIs. Relationships with existing conditions based on the same class of Lyapunov functions are hence investigated, showing that the proposed condition is either equivalent to or less conservative than existing ones. As a matter of fact, the proposed condition turns out to be also necessary for a class of systems. Some numerical examples illustrate the use of the proposed condition and its benefits.

I. INTRODUCTION

There have been various contributions in the literature to study robust stability properties of linear systems affected by structured uncertainty. Here the basic problem is to establish whether the origin is a stable equilibrium point over a set of admissible values of the uncertainty. If yes, the system is said to be robustly stable.

The contributions proposed in the literature for addressing this problem are mainly based on linear matrix inequality (LMI) optimizations, and can be classified in various ways. For instance, this classification can be done based on the type of uncertainty, i.e. time-invariant or time-varying, or based on the dependence of the system coefficients on the uncertainty, i.e. linear or rational. See e.g. [1] and references therein.

The type of uncertainty characterizes the type of Lyapunov function that is searched for in order to prove robust stability of the system. Specifically, in the case of time-invariant uncertainty, pioneering methods have looked for a common quadratic Lyapunov function, and more recent ones have proposed the use of parameter-dependent quadratic Lyapunov functions in order to reduce the conservatism. Then, in the case of time-varying uncertainty, we have passed from common quadratic Lyapunov functions to common nonquadratic Lyapunov functions in order to provide results that are as tight as possible. See e.g. [2]–[9].

The dependence of the system coefficients on the uncertainty characterizes the way the Lyapunov function is searched for. Specifically, in the case of linear dependence, existing methods have formulated conditions by exploiting convexity properties of this dependence and of the set of admissible uncertainties (in particular when using common Lyapunov functions) or by studying positivity properties

of polynomial matrix functions (when using parameter-dependent Lyapunov functions). Then, in the case of polynomial or rational dependence, robust stability has been studied also by using the linear fractional representation (LFR). See e.g. [10]–[17].

This paper addresses robust stability of uncertain linear systems with rational dependence on unknown time-varying parameters constrained in a polytope. For this problem, a new sufficient condition based on the search for a common homogeneous polynomial Lyapunov function is proposed in terms of an LMI feasibility test through a particular representation of parameter-dependent polynomials. Hence, the paper investigates relationships with existing conditions based on the same class of Lyapunov functions, specifically our previous conditions [11], [17] which are respectively based on the LFR and on an extended version of Polya's theorem for structured matrices. It is shown that the proposed condition is either equivalent to or less conservative than these previous ones. As a matter of fact, the proposed condition is guaranteed to be also necessary for a class of systems. Some numerical examples illustrate the use of the proposed condition and its benefits.

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries on the representation of polynomials. Section III describes the proposed robust stability condition and its relationships with existing ones. Section IV presents some illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

A. Problem Formulation

The notation used throughout the paper is as follows:

- \mathbb{N}, \mathbb{R} : natural and real number sets;
- 0_n : origin of \mathbb{R}^n ;
- \mathbb{R}_0^n : $\mathbb{R}^n \setminus \{0_n\}$;
- I_n : $n \times n$ identity matrix;
- A' : transpose of A ;
- $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A ;
- $\text{he}(A) = A + A'$;
- $\nabla v(x)$: first derivative row vector of the function $v(x)$;
- $\text{conv}\{a, b, \dots\}$: convex hull of vectors a, b, \dots

We consider continuous-time linear systems affected by time-varying uncertainties, in particular described by the model

$$\begin{cases} \dot{x}(t) = A(p(t))x(t) \\ p(t) \in \mathcal{P} \end{cases} \quad (1)$$

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where $x(t) \in \mathbb{R}^n$ is the state vector, $p(t) \in \mathbb{R}^q$ is the time-varying uncertain vector, $A : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$ is a matrix rational function, and $\mathcal{P} \subset \mathbb{R}^q$ is a given bounded convex polytope that we express as

$$\mathcal{P} = \text{conv} \left\{ p^{(1)}, \dots, p^{(r)} \right\} \quad (2)$$

where $p^{(1)}, \dots, p^{(r)} \in \mathbb{R}^q$ are given vectors. The matrix rational function $A(p(t))$ is expressed as

$$A(p) = \frac{B(p)}{b(p)} \quad (3)$$

where $b : \mathbb{R}^q \rightarrow \mathbb{R}$ is a polynomial and $B : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$ is a matrix polynomial, i.e. a matrix whose entries are polynomials. We denote the degree of $B(p)$ and $b(p)$ with δ and δ_{den} , respectively.

Throughout the paper we assume that:

- $p(t)$ ensures the existence of the solution $x(t)$ of the system (1);
- the polynomial $b(p)$ satisfies

$$b(p) > 0 \quad \forall p \in \mathcal{P}. \quad (4)$$

Let us observe that these are mild assumptions. Indeed, the first ensures that $p(t)$ is such that the differential equation in the system (1) admits a solution. Then, the second ensures that $b(p)$ is positive for all admissible values of p , which in turn guarantees that the matrix $A(p)$ is bounded for all admissible values of p . This is reasonable since the entries of $A(p)$ represents physical quantities in a real system, which are bounded. Also, we observe that the second assumption is equivalent to say that the linear fractional representation (LFR) of the system (1) is well-posed [10] (see also Section III-A for further details).

Problem. The problem that we consider in this paper consists of establishing whether the origin is a robustly asymptotically stable equilibrium point for the system (1), i.e.

$$\begin{cases} \forall \varepsilon > 0 \exists \delta > 0 : \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \\ \forall t \geq 0 \forall p(\cdot) \in \mathcal{P} \\ \lim_{t \rightarrow \infty} x(t) = 0_n \quad \forall x(0) \in \mathbb{R}^n \quad \forall p(\cdot) \in \mathcal{P}. \end{cases} \quad (5)$$

B. Positive Polynomials via LMIs

Before proceeding we briefly introduce a key tool that will be exploited in the next sections to derive the proposed conditions. For $x \in \mathbb{R}^n$, let $h(x)$ be a homogeneous polynomial, i.e. a polynomial with all monomials of the same degree, and let $2m$ be the degree of $h(x)$. Let $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ be a vector containing all monomials of degree less equal to m in x , where $\sigma(n, m)$ is the number of such monomials given by

$$\sigma(n, m) = \frac{(n+m-1)!}{(n-1)!m!}. \quad (6)$$

Then, $h(x)$ can be written according to the square matricial representation (SMR) [18] (also known as Gram matrix method [19]) as

$$h(x) = x^{\{m\}'} (H + L(\alpha)) x^{\{m\}} \quad (7)$$

where $H = H' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a symmetric matrix such that $h(x) = x^{\{m\}'} H x^{\{m\}}$, $L(\alpha) = L(\alpha)' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a linear parametrization of the set

$$\mathcal{L} = \{L = L' : x^{\{m\}'} L x^{\{m\}} = 0\}, \quad (8)$$

and $\alpha \in \mathbb{R}^{\omega(n,m)}$ is a vector of free parameters, where $\omega(n, m)$ is the dimension of the linear subspace \mathcal{L} given by

$$\omega(n, m) = \frac{1}{2} \sigma(n, m) (\sigma(n, m) + 1) - \sigma(n, 2m). \quad (9)$$

The expression (7) was introduced in [18] in order to investigate positivity of polynomials via LMIs. Indeed, $h(x)$ is non-negative if $s(x)$ is sum of squares of polynomials (SOS), and this latter condition holds if and only if

$$\exists \alpha : H + L(\alpha) \geq 0. \quad (10)$$

The above condition is an LMI feasibility test, which can be solved through a convex optimization since the feasible set of an LMI is convex [2]. See e.g. [1] and references therein for details about SOS polynomials, including algorithms for the construction of the matrices H and $L(\alpha)$. It is worth mentioning that SOS polynomials have been exploited in optimization over polynomials since a long time, see e.g. [20] and the survey [21].

III. ROBUST STABILITY ANALYSIS

A. Equivalent Models

First of all, let us observe that the system (1) can be equivalently represented with other models. A well-known one exploits the LFR, see e.g. [10] and references therein. With the LFR model, the system (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) \\ z(t) = Cx(t) + Dy(t) \\ y(t) = E(p(t))z(t) \\ E(p(t)) = \begin{pmatrix} p_1(t)I_{s_1} & & \\ & \ddots & \\ & & p_q(t)I_{s_q} \end{pmatrix} \\ p(t) \in \mathcal{P} \end{cases} \quad (11)$$

where $x(t)$, $p(t)$ and \mathcal{P} are as in the system (1), $y(t)$, $z(t) \in \mathbb{R}^d$ are auxiliary vectors with $d = s_1 + \dots + s_q$ where s_1, \dots, s_q are nonnegative integers, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$, $C \in \mathbb{R}^{d \times n}$ and $D \in \mathbb{R}^{d \times d}$ are appropriate matrices. Indeed, one has that the matrix $A(p)$ in the system (1) is related to the matrices in the system (11) by

$$A(p) = A + B(I - DE(p))^{-1}E(p)C. \quad (12)$$

Consequently, the LFR is said well-posed if

$$\det(I - DE(p)) = 0 \quad \forall p \in \mathcal{P}. \quad (13)$$

It is useful to observe that, since $b(p)$ in (3) can be selected equal to $\det(I - DE(p))$, assumption (4) coincides with (or is less restrictive than) (13). For the system (11) we define the *LFR degree* as

$$d_{LFR} = \max\{s_1, \dots, s_q\}. \quad (14)$$

We also introduce the polytope of matrices

$$\mathcal{P}_E = \{E(p) : p \in \mathcal{P}\} \quad (15)$$

whose vertices are denoted by

$$E_i = E(p^{(i)}), \quad i = 1, \dots, r. \quad (16)$$

Another model for representing the system (1) consists of adopting a canonical set for the uncertain vectors, in particular the simplex. Indeed, the system (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = D(s(t))x(t) \\ s(t) \in \mathcal{S} \end{cases} \quad (17)$$

where \mathcal{S} is the simplex

$$\mathcal{S} = \left\{ s \in \mathbb{R}^r : \sum_{i=1}^r s_i = 1, s_i \geq 0 \right\}, \quad (18)$$

$s = (s_1, \dots, s_r)' \in \mathbb{R}^r$ is a new time-varying uncertain vector, and $D : \mathbb{R}^r \rightarrow \mathbb{R}^{n \times n}$ is a matrix rational function that we express as

$$D(s) = \frac{C(s)}{c(s)} \quad (19)$$

where $c : \mathbb{R}^r \rightarrow \mathbb{R}$ is a polynomial and $C : \mathbb{R}^r \rightarrow \mathbb{R}^{n \times n}$ is a matrix polynomial. The original uncertain vector p in the system (1) is related to s by $p = \varphi(s)$ where

$$\varphi(s) = \sum_{i=1}^r s_i p^{(i)}. \quad (20)$$

Since p is linear in s and s belongs to the simplex, it follows that the matrix polynomial $C(s)$ and the polynomial $c(s)$ can be chosen homogeneous. Indeed, we have that

$$\begin{aligned} C(s) &= B(p(\varphi(s))) \\ c(s) &= b(p(\varphi(s))). \end{aligned} \quad (21)$$

We can write $B(p)$ and $b(p)$ as

$$\begin{aligned} B(p) &= \sum_{i=0}^{\delta} B_i(p) \\ b(p) &= \sum_{i=0}^{\delta_{den}} b_i(p) \end{aligned} \quad (22)$$

where $b_i(p)$ is a homogeneous polynomial and $B_i(p)$ is a matrix homogeneous polynomial, both of degree i . Hence, we obtain

$$\begin{aligned} C(s) &= \sum_{i=0}^{\delta} B_i(\varphi(s)) \\ c(s) &= \sum_{i=0}^{\delta_{den}} b_i(\varphi(s)). \end{aligned} \quad (23)$$

Let us observe that $b_i(\varphi(s))$ and $B_i(\varphi(s))$ are homogenous polynomials in s of degree i since $\varphi(s)$ is a linear function.

Moreover, since $\sum_{i=1}^r s_i = 1$ for $s \in \mathcal{S}$, it follows that $C(s)$ and $c(s)$ can be equivalently rewritten as

$$\begin{aligned} C(s) &= \sum_{i=0}^{\delta} B_i(\varphi(s)) \left(\sum_{i=1}^r s_i \right)^{\delta-i} \\ c(s) &= \sum_{i=0}^{\delta_{den}} b_i(\varphi(s)) \left(\sum_{i=1}^r s_i \right)^{\delta_{den}-i} \end{aligned} \quad (24)$$

which are homogenous polynomials in s . In particular, the degrees of $C(s)$ and $c(s)$ are δ and δ_{den} , respectively.

B. Conditions for Robust Stability

The robust stability of the system (1) and its equivalent models (11) and (11) can be studied by looking for a continuous function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(0) = 0 \\ v(x) > 0 \quad \forall x \in \mathbb{R}_0^n \\ \text{(systems (1) and (11))} & \dot{v}(x, p) < 0 \quad \forall x \in \mathbb{R}_0^n \quad \forall p \in \mathcal{P} \\ \text{(system (17))} & \dot{v}(x, s) < 0 \quad \forall x \in \mathbb{R}_0^n \quad \forall s \in \mathcal{S} \end{cases} \quad (25)$$

where $\dot{v}(x, p)$ and $\dot{v}(x, s)$ are the time derivatives of $v(x)$ along the trajectories of the systems (1) and (11) (for $\dot{v}(x, p)$) and of the systems (17) (for $\dot{v}(x, s)$). If such a function $v(x)$ exists, then $v(x)$ is said a common Lyapunov function for the time-varying uncertain system under investigation.

Let us observe that since the system is linear in the state, the candidate Lyapunov function $v(x)$ can be chosen homogeneous in x . Let us consider the case where such a homogeneous function $v(x)$ is polynomial. We can write the candidate homogeneous polynomial Lyapunov function by using the SMR introduced in Section II-B as

$$v(x) = x^{\{m\}'} V x^{\{m\}} \quad (26)$$

where $m \in \mathbb{N}$ defines the degree of $v(x)$, which is equal to $2m$, and $V = V' \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ is a suitable matrix.

One method for investigating robust stability of the system (1) through homogeneous polynomial Lyapunov functions was proposed in [11] (see also [1] for more details). This method exploits the LFR model (11) and extends the ideas proposed in [10] based on quadratic Lyapunov functions. Specifically, for $A \in \mathbb{R}^{n \times n}$ let us denote with $A^\#$ the matrix satisfying

$$\frac{dx^{\{m\}}}{dx} Ax = A^\# x^{\{m\}}. \quad (27)$$

The matrix $A^\#$ is known as extended matrix of A with respect to the power vector $x^{\{m\}}$, and can be calculated either via the formula

$$A^\# = (K'K)^{-1} K' \left(\sum_{i=0}^{m-1} I_n^{m-1-i} \otimes A \otimes I_n^i \right) K \quad (28)$$

where $K \in \mathbb{R}^{n^m \times \sigma(n,m)}$ is the matrix satisfying

$$x^{\{m\}} = K x^{\{m\}} \quad (29)$$

and $x^{\{m\}}$ denotes the m -th Kronecker power of x . Define

$$R(V, G, H, E) = R_1 + R_2 \quad (30)$$

where

$$\begin{aligned} R_1 &= \begin{pmatrix} \text{he}(VA^\#) & V(BE)^\# \\ \star & 0 \end{pmatrix} \\ R_2 &= \begin{pmatrix} \text{he}(GC^\#) & G(DE)^\# - G + (HC^\#)' \\ \star & \text{he}(H(DE)^\# - H) \end{pmatrix}. \end{aligned} \quad (31)$$

Define also

$$\mathcal{L}_{LFR} = \{L = L' : f(x, z)'L f(x, z) = 0\} \quad (32)$$

where

$$f(x, z) = \begin{pmatrix} x^{\{m\}} \\ z \otimes x^{\{m-1\}} \end{pmatrix}. \quad (33)$$

Theorem 1 ([11], [1]): Let $m \geq 1$ be an integer, and let $L(\alpha)$ be a linear parametrization of the set \mathcal{L}_{LFR} in (32). The origin of (11) is robustly asymptotically stable if there exist a symmetric matrix V , matrices G and H , and vectors $\alpha^{(1)}, \dots, \alpha^{(r)}$ of suitable size such that the following system of LMIs holds:

$$\begin{cases} V > 0 \\ R(V, G, H, E_i) + L(\alpha^{(i)}) < 0 \quad \forall i = 1, \dots, r. \end{cases} \quad (34)$$

Moreover, if the LFR degree is $d_{LFR} = 1$, a less conservative condition is obtained by requiring that there exist matrices G_i and H_i (in place of G and H) such that

$$\begin{cases} V > 0 \\ R(V, G_i, H_i, E_i) + L(\alpha^{(i)}) < 0 \quad \forall i = 1, \dots, r. \end{cases} \quad (35)$$

Another method for investigating robust stability of the system (1) through homogeneous polynomial Lyapunov functions has been recently proposed in [17] where an extension of Polyá's theorem to the case of structured matrix polynomials has been derived. Specifically, for a nonnegative integer k define

$$G(s) = C(s)^\# \left(\sum_{i=1}^r s_i \right)^k \quad (36)$$

and express $G(s)$ as

$$G(s) = (G_1, \dots, G_l) \left(s^{\{\delta+k\}} \otimes I_{\sigma(n,m)} \right) \quad (37)$$

where $G_1, \dots, G_l \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$ are suitable matrices and $l = \sigma(r, \delta + k)$. The following result holds.

Theorem 2 ([17]): Let $m \geq 1$ and $k \geq 0$ be integers, and let $L(\alpha)$ be a linear parametrization of the set \mathcal{L} in (8). The origin of (1) is robustly asymptotically stable if there exist a symmetric matrix V and vectors $\alpha^{(1)}, \dots, \alpha^{(l)}$ of suitable size such that the following system of LMIs holds:

$$\begin{cases} V > 0 \\ \text{he}(VG_i) + L(\alpha^{(i)}) < 0 \quad \forall i = 1, \dots, l. \end{cases} \quad (38)$$

In this paper we propose a new method for investigating robust stability of the system (1) through homogeneous polynomial Lyapunov functions. Specifically, let us define the notation

$$\text{sq}(s) = (s_1^2, \dots, s_r^2)' \quad (39)$$

and introduce the function

$$w(x, s) = x^{\{m\}'} \text{he}(VG(\text{sq}(s))) x^{\{m\}}. \quad (40)$$

We have that $w(x, s)$ is a homogeneous polynomial of degree $2m$ in x and $2\delta k$ in s . We can express $w(x, s)$ as

$$w(x, s) = h(x, s)' W h(x, s) \quad (41)$$

where

$$h(x, s) = s^{\{\delta+k\}} \otimes x^{\{m\}} \quad (42)$$

and $H = H' \in \mathbb{R}^{l\sigma(n,m) \times l\sigma(n,m)}$ is a suitable matrix. Define also the linear subspace

$$\mathcal{L}_{PD} = \{L = L' : h(x, s)' L h(x, s) = 0\}. \quad (43)$$

It turns out that the dimension of \mathcal{L}_{PD} is given by

$$\begin{aligned} \nu(r, \delta k, n, m) &= \frac{1}{2} l\sigma(n, m) (l\sigma(n, m) + 1) \\ &\quad - \sigma(r, 2(\delta + k)) \sigma(n, 2m). \end{aligned} \quad (44)$$

Theorem 3: Let $m \geq 1$ and $k \geq 0$ be integers, and let $L(\alpha)$ be a linear parametrization of the set \mathcal{L}_{PD} in (43). The origin of (1) is robustly asymptotically stable if there exist a symmetric matrix V and a vector α of suitable size such that the following system of LMIs holds:

$$\begin{cases} V > 0 \\ W + L(\alpha) < 0. \end{cases} \quad (45)$$

Proof. Suppose that (45) holds. Pre- and post-multiplying the first LMI by $x^{\{m\}'}$ and $x^{\{m\}}$, respectively, we obtain that

$$0 < x^{\{m\}'} V x^{\{m\}} = v(x)$$

which implies that $v(x)$ is positive definite since $x^{\{m\}} \neq 0$ for all $x \neq 0$. Then, pre- and post-multiplying the first LMI by $h(x, s)'$ and $h(x, s)$, respectively, we obtain that

$$0 > h(x, s)' (W + L(\alpha)) h(x, s) = w(x, s)$$

which implies that $w(x, s) < 0$ for all $x, s \neq 0$ since in such a case $h(x, s) \neq 0$. Let us define

$$\hat{w}(x, s) = w(x, \text{sq}(s))$$

where

$$\text{sq}(s) = (\sqrt{s_1}, \dots, \sqrt{s_r})' \quad (46)$$

First, observe that $\hat{w}(x, s)$ is a homogeneous polynomial since $w(x, s)$ is a homogeneous polynomial whose monomials have even degrees in all entries of s . Then, observe that $w(x, s) < 0$ for all $x, s \neq 0$ if and only if

$$\hat{w}(x, s) < 0 \quad \forall x \neq 0 \quad \forall s \in \mathcal{S}.$$

Lastly, observe that

$$\hat{w}(x, s) = c(s) \hat{v}(x, s) \left(\sum_{i=1}^r s_i \right)^k$$

which, concluding, implies that $w(x, s) < 0$ for all $x, s \neq 0$ if and only if

$$\hat{v}(x, s) < 0 \quad \forall x \neq 0 \quad \forall s \in \mathcal{S}.$$

Therefore, the origin of (1) is asymptotically stable, i.e. the theorem holds. \square

Theorem 4: Let $m \geq 1$ be an integer. Then, the condition of Theorem 2 holds for some integer $k = k_1$ if and only if the condition of Theorem 3 holds for some integer $k = k_2$ (with $k_2 \leq k_1$).

Proof. Suppose that the condition of Theorem 2 holds for some integer $k = k_1$. Then, observe that the matrix W can be obtained as

$$W = \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ \star & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & W_l \end{pmatrix}$$

where $W_i = \text{he}(VG_i) + L(\alpha^{(i)})$. Hence, the condition of Theorem 3 holds with $\alpha = 0$ for some integer $k = k_2$, in particular $k_2 = k_1$.

Then, suppose that the condition of Theorem 3 holds for some integer $k = k_2$. This implies that $-w(x, s)$ is positive definite and SOS. Consequently, one has that

$$\begin{cases} \dot{v}(x, s) = x^{\{m\}'} U(s) x^{\{m\}} \\ U(s) < 0 \quad \forall s \in \mathcal{S}. \end{cases}$$

By proceeding as in the proof of Theorem 2 of [17], this implies that the condition of Theorem 3 holds for some integer $k = k_1$. \square

Corollary 1: Let $m \geq 1$ be an integer, and suppose that at least one of the conditions of Theorem 1 holds. Then, there exists k such that the condition of Theorem 2 and the condition of Theorem 3 hold.

Proof. It follows directly from Theorem 4 of [17]. \square

Corollary 1 states that the condition of Theorem 2 and the condition of Theorem 3 are not more conservative than the conditions of Theorem 1 based on the LFR.

Corollary 2: Suppose that $n = 2$. Then, the origin of (1) is asymptotically stable if and only if there exist m and k such that the conditions of Theorems 2 and 3 hold.

Proof. It follows directly from Theorem 2 of [17]. \square

Although the conditions of Theorems 2 and 3 are equivalent in terms of conservatism according to Theorem 4, the computational burden required by these conditions for establishing robust stability can be significantly different depending on the system under investigation. On this regard, it is useful observing that the computational burden of each LMI problem depends on the number of scalar variables and on the sum of the dimensions of the LMIs. We have that the number of scalar variables in the condition of Theorem 2 is

$$\tau_1 = \frac{1}{2}\sigma(n, m)(\sigma(n, m) + 1) + l\omega(n, m) \quad (47)$$

while in the condition of Theorem 3 is

$$\tau_2 = \frac{1}{2}\sigma(n, m)(\sigma(n, m) + 1) + \nu(r, \delta k, n, m). \quad (48)$$

Then, the sum of the dimensions of the LMIs in both theorems is

$$\xi = (1 + l)\sigma(n, m). \quad (49)$$

| $m \setminus k$ | 0 | 1 | 2 | $m \setminus k$ | 0 | 1 | 2 |
|-----------------|----|----|----|-----------------|-----|-----|-----|
| 1 | 3 | 3 | 3 | 1 | 6 | 6 | 6 |
| 2 | 9 | 10 | 11 | 2 | 39 | 45 | 51 |
| 3 | 19 | 22 | 25 | 3 | 136 | 163 | 190 |

TABLE I

QUANTITY τ_1 IN THE CASE $r = \delta = 2$ FOR $n = 2$ (LEFT) AND $n = 3$ (RIGHT).

| $m \setminus k$ | 0 | 1 | 2 | $m \setminus k$ | 0 | 1 | 2 |
|-----------------|----|----|-----|-----------------|-----|-----|------|
| 1 | 9 | 18 | 31 | 1 | 21 | 42 | 72 |
| 2 | 26 | 49 | 81 | 2 | 117 | 216 | 351 |
| 3 | 53 | 97 | 157 | 3 | 380 | 679 | 1078 |

TABLE II

QUANTITY τ_2 IN THE CASE $r = \delta = 2$ FOR $n = 2$ (LEFT) AND $n = 3$ (RIGHT).

Tables I–III show τ_1 , τ_2 and ξ for some values of n , m and k . Observe that, as explained in Theorem 4, the value of k required to prove stability by Theorem 2 can be different from the one required by Theorem 3.

IV. ILLUSTRATIVE EXAMPLES

A. Example 1

Let us consider the uncertain system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 + p(t) - p(t)^2 & -1 - p(t) \end{pmatrix} x(t)$$

where $p(t)$ is an uncertain time-varying parameter constrained according to $p(t) \in [0, \zeta]$. The problem consists of determining the maximum ζ , denoted by ζ^* , for which the origin is robustly asymptotically stable.

Let us use the condition proposed in this paper. We have that $n = 2$, $r = 2$, and $\delta = 2$. The system (17) turns out to have a matrix homogeneous polynomial $D(s)$ as in (19) with

$$\begin{aligned} C(s) &= \begin{pmatrix} 0 \\ (-2 + \zeta - \zeta^2)s_1^2 + (-4 + \zeta)s_1s_2 - 2s_2^2 \\ s_1^2 + 2s_1s_2 + s_2^2 \\ (-1 + \zeta)s_1^2 + (-2 + \zeta)s_1s_2 - s_2^2 \end{pmatrix} \\ c(s) &= s_1^2 + 2s_1s_2 + s_2^2. \end{aligned}$$

Theorem 3 provides the lower bounds of ζ^* shown in Table IVa. These lower bounds are found through a line search over ζ performed via a bisection algorithm. Observe that, according to Corollary 2, these lower bounds are guaranteed to approximate arbitrarily well ζ^* .

| $m \setminus k$ | 0 | 1 | 2 | $m \setminus k$ | 0 | 1 | 2 |
|-----------------|----|----|----|-----------------|----|----|----|
| 1 | 8 | 10 | 12 | 1 | 12 | 15 | 18 |
| 2 | 12 | 15 | 18 | 2 | 24 | 30 | 36 |
| 3 | 16 | 20 | 24 | 3 | 40 | 50 | 60 |

TABLE III

QUANTITY ξ IN THE CASE $r = \delta = 2$ FOR $n = 2$ (LEFT) AND $n = 3$ (RIGHT).

| $m \setminus k$ | 0 |
|-----------------|-------|
| 1 | 3.871 |
| 2 | 5.081 |
| 3 | 5.329 |

(a)

| $m \setminus k$ | 0 | 1 | ... | 9 |
|-----------------|-------|-------|-----|-------|
| 1 | 3.812 | 3.871 | ... | 3.871 |
| 2 | 3.907 | 4.528 | ... | 4.999 |
| 3 | 4.000 | 4.866 | ... | 4.999 |

(b)

TABLE IV

EXAMPLE 1: LOWER BOUNDS OF ζ^* PROVIDED BY THEOREM 3 (A) AND BY THEOREM 2 (B).

| $m \setminus k$ | 0 |
|-----------------|-------|
| 1 | 0.891 |
| 2 | 1.478 |
| 3 | 1.605 |

TABLE V

EXAMPLE 2: LOWER BOUNDS OF ζ^* PROVIDED BY THEOREM 3.

For comparison, we report in Table IVb the lower bounds provided by [17], i.e. Theorem 2. As we can see, even by using much larger values of k (i.e., 9 versus 0), these lower bounds are more conservative than those provided by Theorem 3 proposed in this paper. Also, it should be remarked that the computational time required for computing the lower bounds in Table IVb for $k = 9$ is larger than that required for computing the lower bounds in Table IVa.

B. Example 2

Here we consider the uncertain system

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 & 1+p(t) \\ 0 & -1 & -p(t) \\ \frac{-3-p(t)^2}{1+p(t)} & 0 & -2 \end{pmatrix} x(t)$$

where $p(t)$ is an uncertain time-varying parameter constrained according to $p(t) \in [0, \zeta]$. The problem consists of determining the maximum ζ , denoted by ζ^* , for which the origin is robustly asymptotically stable.

In this case we have $n = 3$, $r = 2$, and $\delta = 2$. Theorem 3 provides the lower bounds of ζ^* shown in Table V.

V. CONCLUSION

This paper has addressed robust stability of uncertain linear systems with rational dependence on unknown time-varying parameters constrained in a polytope. A new sufficient condition has been proposed in terms of an LMI feasibility test based on homogeneous polynomial Lyapunov functions and a particular representation of parameter-dependent polynomials. It has been shown that the proposed condition is either equivalent to or less conservative than existing ones that exploit this class of Lyapunov functions and are based on other techniques for their search. The

proposed condition turns out to be also necessary for a class of systems.

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