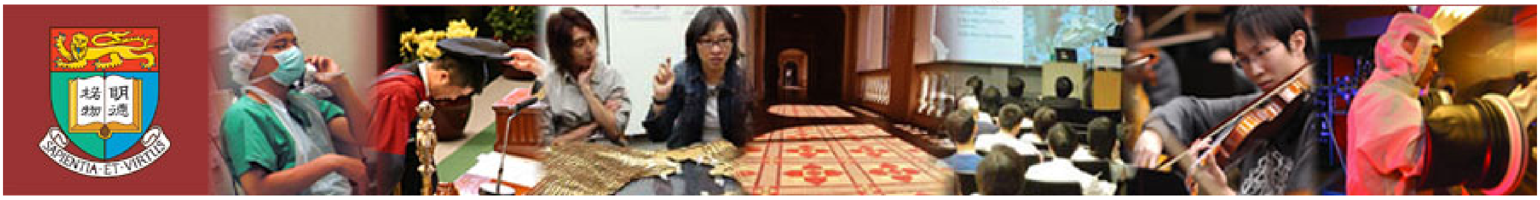

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L_p-mixed intersection bodies and star duality

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Abstract. The paper extends the two notions of the dual mixed volumes and L_p-intersection body to q-dual mixed volumes and L_p-mixed intersection body, respectively. Inequalities for the star dual of L_p-mixed intersection bodies are established.

Keywords. q-dual mixed volumes; L_p-intersection bodies; L_p-mixed intersection bodies; star dual of L_p-mixed intersection bodies.

0. Introduction

The intersection operator and the class of intersection bodies were defined by Lutwak [23]. The closure of the class of intersection bodies was studied by Goody, Lutwak and Weil [13]. The intersection operator and the class of intersection bodies played a critical role in Zhang [34] and Gardner [9] solution of the famous Busemann–Petty problem (see also [12].)

Just as the period from the mid 60's to the mid 80's was a time of great advances in the understanding of the projection operator and the class of projection bodies, during the past 20 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies (see, e.g., [2–8, 10, 11, 13–15, 17–22, 28–34].

As Lutwak [23] shows (and as is further elaborated in Gardner's book [10]), there is a duality between projection and intersection bodies (that at present is not yet understood). Consider the following illustrative example: It is well-known that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the 'dual': When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proven by Fallert, Goodey and Weil [5]. In [26] (see also [24] and [25]), Lutwak introduced mixed projection bodies and derived their fundamental inequalities.

In 1999, Moszyńska [28] introduced the notion of star dual of star body. Recently, Haberl and Ludwig [16] introduced L_p-intersection bodies. Following Haberl, Ludwig and Moszyńska, the paper introduces a new concept of *q-dual mixed volumes* of star bodies which extends the classical dual mixed volumes. Moreover, we extend the notions of L_p-intersection body to L_p-mixed intersection body. Inequalities for star dual of

L_p-mixed intersection bodies are established. Our main results are stated using the following theorems.

Theorem A. *Let K and L be star bodies and* $0 \leq i < n, 0 \leq j < n-1, i, j \in \mathbb{N}$, *and* $p < 1$, *then*

$$\tilde{W}_i(\mathbf{I}_p^\circ(K, L)_j)^{n-1} \leq \tilde{W}_i(\mathbf{I}_p^\circ K)^{n-j-1}\tilde{W}_i(\mathbf{I}_p^\circ L)^j,$$

with equality if and only if K and L are dilates.

$\tilde{W}_i$ is the classical dual quermassintegral. $\mathbf{I}_p K$ denotes the L_p-intersection body of star body K which was defined by Haberl and Ludwig [16]. $\mathbf{I}_p^\circ K$ denotes the star dual of L_p-intersection body which was defined by §2. $\mathbf{I}_p(K, L)_j$ denotes L_p-mixed intersection bodies j-th $\mathbf{I}_p(\underbrace{K, \ldots, K}_{n-j}, \underbrace{L, \ldots, L}_{j})$ and $\mathbf{I}_p^\circ(K, L)_j$ denotes the star dual of the L_p-mixed intersection bodies j-th.

Theorem B. *If* $K_1, \ldots, K_{n-1} \in \varphi^n$, $p < 1$ *and* $1 < r \leq n-1$, $0 \leq i < n, 0 \leq j < n-1, i, j \in \mathbb{N}$, *then*

$$\tilde{W}_i(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{W}_i(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})),$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

$\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1})$ denotes the star dual of L_p-mixed intersection body $\mathbf{I}_p(K_1, \ldots, K_{n-1})$.

The setting for this paper is an n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K, \cdot)$, defined on $\mathbb{R}^n$ by $h(K, \cdot) = \text{Max}\{x \cdot y \colon y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) \colon S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = \text{Max}\{\lambda \geq 0 \colon \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \varphi^n$, then

$$\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty.$$

1. q-dual mixed volume

We first introduce the new notion *q-dual mixed volume*, $\tilde{V}_q(K_1, \ldots, K_n)$, as follows:

DEFINITION 1.1

$$\tilde{V}_q(K_1, \ldots, K_n) = \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^q(K_1, u) \cdots \rho^q(K_n, u) \mathrm{d}S(u) \right)^{1/q}, \quad q \neq 0. \tag{1.1}$$

where $K_1, \ldots, K_n \in \varphi^n$.

By Definition 1.1, $\tilde{V}_q$ is a map

$$\tilde{V}_q \colon \underbrace{\varphi^n \times \cdots \times \varphi^n}_{n} \to \mathbb{R}.$$

We list some of its elementary properties.

(1) *Continuity.* $\tilde{V}_p(K_1, \ldots, K_n)$ is continuous, $K_i \in \varphi^n$ $(i = 1, \ldots, n)$.
(2) *Positivity.* $\tilde{V}_p(K_1, \ldots, K_n) > 0$, for $K_i \in \varphi^n$ $(i = 1, \ldots, n)$.
(3) *Positively homogeneous.* $\tilde{V}_p(\lambda_1 K_1, \ldots, \lambda_n K_n) = \lambda_1 \cdots \lambda_n \tilde{V}_p(K_1, \ldots, K_n)$, $\lambda_i > 0$, $K_i \in \varphi^n (i = 1, \ldots, n)$.

Taking for $q = 1$ in (1.1), we have

$$\tilde{V}_1(K_1, \ldots, K_n) = \tilde{V}(K_1, \ldots, K_n),$$

where $\tilde{V}(K_1, \ldots, K_n)$ is the classical dual mixed volumes which was defined by Lutwak [27].

Moreover, in the paper we will write $\tilde{V}_q(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i})$ as $\tilde{V}_{q,i}(K, L)$, $\tilde{V}_q(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i})$ as $\tilde{V}_{q,i}(K)$, and $\tilde{V}_q(\underbrace{K, \ldots, K}_{n})$ as $\tilde{V}_q(K)$.

For q-dual mixed volume, in view of Definition 1.1 and Hölder inequality for integrals, we obtain a new inequality, *dual Aleksandrov-Fenchel inequality between q-dual mixed volumes* as follows:

Lemma 1.1. *If* $K_1, \ldots, K_n \in \varphi^n$, $1 < r \leq n$, $0 \leq j < n - 1$ *and* $q \neq 0$, *then*

$$\tilde{V}_q(K_1, \ldots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}_q(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n), \tag{1.2}$$

with equality if and only if $K_1, \ldots, K_n$ *are all dilations.*

Taking for $q = 1$ in (1.2), we have

$$\tilde{V}(K_1, \ldots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n), \ 1 < r \leq n,$$

with equality if and only if $K_1, \ldots, K_n$ are all dilations.

This is just the classical Aleksandrov–Fenchel inequality between dual mixed volume which is due to Lutwak [27].

2. L_p-mixed intersection bodies

Recently, Haberl and Ludwig [16] introduced L_p-intersection bodies. For $K \in \mathcal{P}_0^n$, $\mathcal{P}_0^n$ denotes the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. The star body $\mathbf{I}_p^+ K$ is defined for $u \in S^{n-1}$ by

$$\rho(\mathbf{I}_p^+ K, u)^p = \int_{K \cap u^+} |u \cdot x|^{-p} \mathrm{d}x,$$

where $u^+ = \{x \in \mathbb{R}^n \colon u \cdot x \geq 0\}$, and define $\mathbf{I}_p^- K = \mathbf{I}_p^+(-K)$. For $p < 1$, the centrally symmetric star body $\mathbf{I}_p K = \mathbf{I}_p{}^+ K + \mathbf{I}_p{}^- K$ is called the L_p *intersection body* of K. So for $u \in S^{n-1}$,

$$\rho^p(\mathbf{I}_p K, u) = \int_K |u \cdot x|^{-p} \mathrm{d}x. \tag{2.1}$$

Since [15]

$$v(K \cap u^+) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_K |u \cdot x|^{-1+\varepsilon} \mathrm{d}x$$

and

$$\rho(\mathbf{I}K, u) = \lim_{p \to 1_-} \frac{1-p}{2} \rho^p(\mathbf{I}_p K, u),$$

that is, the intersection body of K is obtained as a limit of L_p intersection bodies of K. Also note that a change to polar coordinates in (2.1) shows that up to a normalization factor $\rho^p(\mathbf{I}_p K, u)$ equals the cosine transform of $\rho(K, u)^{n-p}$.

Here, we introduce the L_p-mixed intersection bodies of $K_1, \ldots, K_{n-1}$, $\mathbf{I}_p(K_1, \ldots, K_{n-1})$, whose radial function is defined for $p < 1$ by

$$\rho^p(\mathbf{I}_p(K_1, \ldots, K_{n-1}), u) = \frac{2}{1-p} \tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \tag{2.2}$$

where $\tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ denotes the $q(p)$-mixed volumes of $K_1 \cap E_u, \ldots, K_{n-1} \cap E_u$ in $(n-1)$-dimensional space.

By the above definition, we have

$$\begin{aligned}
&\lim_{p \to 1_-} \frac{1-p}{2} \rho^p(\mathbf{I}_p(K_1, \ldots, K_{n-1}), u) \\
&= \lim_{p \to 1_-} \tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u) \\
&= \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u) \\
&= \rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u).
\end{aligned}$$

The mixed intersection body of $K_1, \ldots, K_{n-1}$ is obtained as a limit of L_p-mixed intersection bodies of $K_1, \ldots, K_{n-1}$.

For the L_p-mixed intersection bodies, $\mathbf{I}_p(K_1, \ldots, K_{n-1})$, if $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}_p(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_p(K, L)_i$. If $L = B$, then $\mathbf{I}_p(K, L)_i$ is written as $\mathbf{I}_p K_i$ and is called the i-th L_p-intersection body of K. For $\mathbf{I}_p K_0$ simply write $\mathbf{I}_p K$, this is just the L_p-intersection bodies of star body K.

On the other hand, Moszyńska [28] introduced the notion of star duality of star body. For the star bodies with 0 in the kernel and positive continuous radial function, such a duality $\circ$ was introduced; it is called the *star duality.*

Let $i\colon \mathbb{R}^n\backslash 0 \longrightarrow \mathbb{R}^n\backslash 0$ be inversion with respect to S^{n-1}:

$$i(x) := \frac{x}{\|x\|^2}.$$

DEFINITION 2.1

For every $K \in \varphi^n$,

$$K^\circ := \mathrm{cl}(\mathbb{R}^n\backslash i(K)).$$

DEFINITION 2.2

For every $K \in \varphi^n$,

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}. \tag{2.3}$$

For the star duality of L_p-mixed intersection bodies, $\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1})$, if $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_p^\circ(K, L)_i$. If $L = B$, then $\mathbf{I}_p^\circ(K, L)_i$ is written as $\mathbf{I}_p^\circ K_i$. For $\mathbf{I}_p^\circ K_0$ simply write $\mathbf{I}_p^\circ K$.

Lemma 2.1. *If* $K, L \in \varphi^n$, $0 \le i < n$, $0 \le j < n-1$, $i, j \in \mathbb{N}$, $q \in (-\infty, 0) \cup (0, +\infty)$ *and* $p < 1$, *then*

$$\tilde{V}_{q,i}(\mathbf{I}_p^\circ K) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_p(K \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{1/q},$$

$$\tilde{V}_{q,i}(\mathbf{I}_p^\circ K_j) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_{p,j}(K \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{1/q},$$

$$\tilde{V}_{q,i}(\mathbf{I}_p^\circ (K, L)_j) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_{p,j}(K \cap E_u, L \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{1/q}.$$

From (1.1), (2.2) and (2.3), Lemma 1.2 easy follows. We shall need the following trivial elementary inequality:

Lemma 2.2 [1]. *If* $a_i \geq 0, b_i > 0$ $(i = 1, 2, \ldots, n)$, *then*

$$\left(\prod_{i=1}^{n}(a_i + b_i)\right)^{1/n} \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n} + \left(\prod_{i=1}^{n} b_i\right)^{1/n}, \tag{2.4}$$

with equality if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

3. Inequalities for star dual of L_p-mixed intersection bodies

3.1 *The Minkowski inequality for star dual of* L_p*-mixed intersection bodies*

The following Minkowski inequality for star dual of L_p-mixed intersection bodies stated in the Introduction will be established: If K and L be star bodies and $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$, and $p < 1$, then

$$\tilde{W}_i(\mathbf{I}_p^{\circ}(K, L)_j)^{n-1} \leq \tilde{W}_i(\mathbf{I}_p^{\circ}K)^{n-j-1}\tilde{W}_i(\mathbf{I}_p^{\circ}L)^{j}, \tag{3.1.1}$$

with equality if and only if K and L are dilates.

This is just a special case of the following result:

Theorem 3.1.1. *Let* $K, L \in \varphi^n$, *and* $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$, $q \neq 0$ *and* $p < 1$, *then*

$$\tilde{V}_{q,i}(\mathbf{I}_p^{\circ}(K, L)_j)^{n-1} \leq \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}K)^{n-j-1}\tilde{V}_{q,i}(\mathbf{I}_p^{\circ}L)^{j}, \tag{3.1.2}$$

with equality if and only if K *and* L *are dilates.*

Proof. Taking for $K_1 = \cdots = K_{n-j} = K$, $K_{n-j+1} = \cdots = K_n = L$ and $r = n$ in (1.2), and in view of taking p for q, we obtain

$$\tilde{V}_{p,i}(K, L) \leq \tilde{V}_p(K)^{n-j}\tilde{V}_p(L)^{j},$$

with equality if and only if K and L are dilates.

Hence, in $(n-1)$-dimensional space, we have

$$\tilde{v}_{p,j}(K \cap E_u, L \cap E_u) \leq \tilde{v}_p(K \cap E_u)^{\frac{n-j-1}{n-1}}\tilde{v}_p(L \cap E_u)^{\frac{j}{n-1}} \tag{3.1.3}$$

with equality if and only if $K \cap E_u$ and $L \cap E_u$ are dilates, it follows if and only if K and L are dilates.

From Lemma 2.1, (3.1.3) and in view of Hölder inequality for integral (see [1]), we have

$$\tilde{V}_{q,i}(\mathbf{I}_p^{\circ}(K, L)_j)$$

$$= \omega_n\left(\frac{1}{n\omega_n}\left(\frac{2}{1-p}\right)^{\frac{-(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_{p,j}(K \cap E_u, L \cap E_u)^{\frac{-(n-i)q}{p}}\mathrm{d}S(u)\right)^{1/q}$$

$$\leq \omega_n\left(\frac{1}{n\omega_n}\left(\frac{2}{1-p}\right)^{\frac{-(n-i)q}{p}}\int_{S^{n-1}}(\tilde{v}_p(K\cap E_u)^{\frac{(n-j-1)}{n-1}}\tilde{v}_p(L\cap E_u)^{\frac{j}{n-1}})^{\frac{-(n-i)q}{p}}\mathrm{d}S(u)\right)^{1/q}$$

$$\leq \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \left(\int_{S^{n-1}} \tilde{v}_p(K \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{n-j-1}{n-1}} \right)^{\frac{1}{q}}$$

$$\times \left(\left(\int_{S^{n-1}} \tilde{v}_p(L \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{j}{n-1}} \right)^{1/q}$$

$$= \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_p(K \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{n-j-1}{q(n-1)}}$$

$$\times \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_p(L \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{j}{q(n-1)}}$$

$$= \tilde{V}_{q,i}(\mathbf{I}_p^{\circ} K)^{\frac{n-j-1}{n-1}} \tilde{V}_{q,i}(\mathbf{I}_p^{\circ} L)^{\frac{j}{n-1}}. \tag{3.1.4}$$

In view of the equality conditions of (3.1.3) and Hölder inequality for integral, it follows that the equality holds if and only if K and L are dilates.

The proof of Theorem 3.1.1 is complete.

Remark 3.1.1. Taking for $q = 1$ in (3.1.2), (3.1.2) changes to the inequality in Theorem A.

Taking for $i = 0$ in (3.1.2), (3.1.2) changes to the following result.

If $K, L \in \varphi^n$, and $p < 1, q \neq 0$ and $0 < j < n-1, j \in \mathbb{N}$, then

$$\tilde{V}_q(\mathbf{I}_p^{\circ}(K, L)_j)^{n-1} \leq \tilde{V}_q(\mathbf{I}_p^{\circ} K)^{n-j-1} \tilde{V}_q(\mathbf{I}_p^{\circ} L)^j,$$

with equality if and only if K and L are dilates.

3.2 *The Aleksandrov–Fenchel inequality for star dual of L_p-intersection bodies*

In the section, we introduce the concept, *q-dual sum function*, as follows:

If $K, D \in \varphi^n$, then the q-dual sum function of star bodies K and D, $S_{\tilde{v}_{q,i}}(K, D)$, is defined by

$$S_{\tilde{v}_{q,i}}(K, D) = \tilde{V}_{q,i}(K) + \tilde{V}_{q,i}\ (D), 0 \leq i < 0, q \neq 0.$$

The following Aleksandrov–Fenchel inequality for star dual of L_p-mixed intersection bodies stated in the Introduction will be proven:

If $K_1, \ldots, K_{n-1} \in \varphi^n$, $p < 1$ and $1 < r \leq n-1, 0 \leq i < n, 0 \leq j < n-1, i, j \in \mathbb{N}$, then

$$\tilde{W}_i(\mathbf{I}_p^{\circ}(K_1, \ldots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{W}_i(\mathbf{I}_p^{\circ}(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})),$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

This is just a special case of the following result.

Theorem 3.2.1. *If $K_1, \ldots, K_{n-1} \in \varphi^n$, $p < 1$, $q \neq 0$ and $1 < r \leq n-1$, $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$. Let D_i $(i = 1, 2, \ldots, n-1)$ be dilate copies of each other, respectively. Then*

$$S_{\tilde{v}_{q,i}}(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1}), \mathbf{I}_p^\circ(D_1, \ldots, D_{n-1}))^r \leq \prod_{j=1}^{r} S_{\tilde{v}_{q,i}}(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}_p^\circ(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})), \tag{3.2.1}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

Proof. From (1.1) and (2.2), we have that

$$\tilde{V}_{q,i}(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1})) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{1/q}. \tag{3.2.2}$$

By using the inequality in Lemma 1.1, we easily get that

$$\tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u) \leq \left(\prod_{j=1}^{r} \tilde{v}_p(\underbrace{K_j \cap E_u, \ldots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \ldots, K_{n-1} \cap E_u) \right)^{\frac{1}{r}}, \tag{3.2.3}$$

with equality if and only if $K_1 \cap E_u, \ldots, K_{n-1} \cap E_u$ are all dilations of each other, it follows if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

On the other hand, the Hölder's inequality can be stated as

$$\int_{S^{n-1}} \prod_{i=1}^{m} f_i(u) \mathrm{d}S(u) \leq \prod_{i=1}^{m} \left(\int_{S^{n-1}} (f_i(u))^m \mathrm{d}S(u) \right)^{\frac{1}{m}}, \tag{3.2.4}$$

with equality if and only if all f_i are proportional.

From (3.2.2), (3.2.3) and (3.2.4), we obtain that

$$\tilde{V}_{q,i}(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1})) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_p(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{1}{q}}$$

$$\leq \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \right.$$

$$\left. \times \int_{S^{n-1}} \prod_{j=1}^{r} \tilde{v}_p(\underbrace{K_j \cap E_u, \ldots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \ldots, K_{n-1} \cap E_u)^{\frac{-(n-i)q}{rp}} \mathrm{d}S(u) \right)^{\frac{1}{q}}$$

$$\leq \omega_n \left(\prod_{j=1}^{r} \frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{-(n-i)q}{p}} \right.$$

$$\left. \times \int_{S^{n-1}} \tilde{v}_p(\underbrace{K_j \cap E_u, \ldots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \ldots, K_{n-1} \cap E_u)^{\frac{-(n-i)q}{p}} \mathrm{d}S(u) \right)^{\frac{1}{rq}}$$

$$= \left(\prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \right)^{\frac{1}{r}}.$$

In view of the equality conditions of (3.2.3) and (3.2.4), it follows that the equality holds if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

In view of D_i $(i = 1, 2, \ldots, n-1)$ being homothetic copies of each other, we have

$$\tilde{V}_{q,i}(\mathbf{I}_p^{\circ}(D_1, \ldots, D_{n-1}))^r = \prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})).$$

Hence

$$\begin{aligned} &S_{v_{q,i}}(\mathbf{I}_p^{\circ}(K_1, \ldots, K_{n-1}),\ \mathbf{I}_p^{\circ}\ (D_1, \ldots, D_{n-1})) \\ &\quad = \left(\prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \right)^{\frac{1}{r}} \\ &\qquad + \left(\prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})) \right)^{\frac{1}{r}}, \end{aligned} \tag{3.2.5}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

By using the inequality in Lemma 2.2 in the right side of inequality (3.2.5), we obtain

$$\begin{aligned} &S_{v_{q,i}}(\mathbf{I}_p^{\circ}(K_1, \ldots, K_{n-1}), \mathbf{I}_p^{\circ}(D_1, \ldots, D_{n-1})) \\ &\quad \leq \prod_{j=1}^{r} (\tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \\ &\qquad + \tilde{V}_{q,i}(\mathbf{I}_p^{\circ}\underbrace{(D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})))^{\frac{1}{r}} \end{aligned}$$

$$= \left(\prod_{j=1}^{r} S_{v_{q,i}}(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}_p^\circ(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})) \right)^{\frac{1}{r}}.$$

In view of the equality conditions of inequality (2.4), it follows that the equality holds if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

The proof is complete.

COROLLARY 3.2.1

If $K_1, \ldots, K_{n-1} \in \varphi^n$, $0 \le i < n$, $0 \le j < n-1$, $i, j \in \mathbb{N}$, $0 < r < n-1$, *and* $p < 1$, *then*

$$S_{\tilde{w}_i}(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1}),\ \mathbf{I}_p^\circ(D_1, \ldots, D_{n-1}))^r \le \prod_{j=1}^{r} S_{\tilde{w}_i}(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}_p^\circ(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})), \tag{3.2.6}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ *are all dilations of each other.*

Let D_i $(i = 1, 2, \ldots, n-1)$ be a single point in (3.2.6). Then (3.2.6) changes to the inequality in Theorem B stated in the Introduction.

COROLLARY 3.2.2

If $K_1, \ldots, K_{n-1} \in \varphi^n$, $0 \le j < n-1$, $j \in \mathbb{N}$, $0 < r < n-1$, $p < 1$ *and* $q \ne 0$, *then*

$$S_{\tilde{v}_q}(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1}),\ \mathbf{I}_p^\circ(D_1, \ldots, D_{n-1}))^r \le \prod_{j=1}^{r} S_{\tilde{v}_q}(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}_p^\circ(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})), \tag{3.2.7}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ *are all dilations of each other.*

Let D_i $(i = 1, 2, \ldots, n-1)$ be a single point in (3.2.7). Then (3.2.7) reduces to

$$\tilde{V}_q(\mathbf{I}_p^\circ(K_1, \ldots, K_{n-1}))^r \le \prod_{j=1}^{r} \tilde{V}_q(\mathbf{I}_p^\circ(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})),$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

COROLLARY 3.2.3

Let $K, L, D, D' \in \varphi^n$ *and* D' *be a dilate copy of* D, *and* $0 \le i < n$, $0 \le j < n-1$, i, $j \in \mathbb{N}$, $q \ne 0$ *and* $p < 1$. *Then*

$$S_{\tilde{v}_{q,i}}(\mathbf{I}_p^\circ(K,L)_j,\mathbf{I}_p^\circ(D,D')_j)^{n-1} \le S_{\tilde{v}_{q,i}}(\mathbf{I}_p^\circ K,\mathbf{I}_p^\circ D)^{n-j-1} S_{\tilde{v}_{q,i}}(\mathbf{I}_p^\circ L,\mathbf{I}_p^\circ D')^j, \tag{3.2.8}$$

with equality if and only if K and L are dilates.

Let D and D' be single point in (3.2.8). Then (3.2.8) changes to inequality (3.1.2).

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