A Model For FMS of Unreliable Machines

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Abstract: - This paper studies Markovian queueing model for flexible manufacturing system. The manufacturing system consists of multiple unreliable machines. Hedging point policy is applied to the system as production control. We model the machine states and inventory levels of the system as a multi-server queueing system. Fast numerical algorithm is presented to solve the steady state probability distribution of the system. Using the probability distribution, the system performance and the effect of machine reliability and maintainability can be evaluated.

Key-Words: Flexible Manufacturing Systems, Queues, Steady State Probability Distribution, Block Gauss-Seidel Method.

1 Introduction

In recent years there has been an increasing role of computer in manufacturing. An important area of the Computer Aided Manufacturing (CAM) is the Flexible Manufacturing System (FMS). The advantage of FMS is that it can reduce work-in-progress and increase machine utilization when suitable production policy is implemented. Moreover, it can also reduce manufacturing lead time and labor. However, the drawbacks of FMS are the cost in setting up, maintenance of machines and management of material, see Buzacott and Shanthikumar [2] for instance. Due to the high capital investment, FMS is considered to operate economically if a high level of system performance is obtained. Mathematical modelling can help with decision required to design and manage an FMS. Queueing theory is an useful tool for many inventory models and manufacturing systems that can assist with long-run decision, see Ching and Zhou [5, 6, 7, 10] for instance. In fact, most analytic models describe FMS as a queueing system, in which the customer are jobs to processed or product in inventory and the servers are simply the reliable machines (workstation) in the system, Buzacott and Yao [3]. However, the assumption that the machines are reliable can greatly affect the performance evaluation of a FMS. Furthermore the effect of machine maintainability is also significant in FMS performance. These are taken into account in our proposed model.

In this paper, we consider FMS of s unreliable machines producing one type of product. When a machine breaks down, it is subject to a repairing process if there is maintenance facility available, otherwise it will queue up and repair in the first come first serve principle. There are $r(\leq s)$ maintenance facilities in the FMS. For simplicity of discussion, we assume the machines are all identical. The up time and down time of each machine are assumed to be exponentially distributed. We assume infinite supply of raw material in the system. Usually, proper positive inventory level is maintained to hedge against the uncertainties in supply, demand or breakdowns of the machine, [1, 12]. Here we consider a order-to-make manufacturing system and Hedging Point Policy (HPP) is applied as the production control. The HPP is characterized by a non-negative number h. The system (machines) keeps on producing products at its maximum rate if the inventory level is less than h. When the inventory level h is reached, all the machines are shut down. In fact, the optimal value of h is the best amount of inventory to store in order to hedge against the uncertainty. It is well known that the hedging point policy is optimal for one-machine manufacturing systems in some simple situations, in the sense that it minimizes the average running cost (or maximize the average profit) of the system, see [1, 12] for instance. When the optimal policy is a zero-inventory policy (i.e. the hedging point is zero), then the policy matches with the just-in-time (JIT) policy. The JIT policies have strongly been favored in real-life production systems for process discipline reasons even when they are not optimal.

In our model, we assume that the inter-arrival time for a demand and processing time for one unit of product are exponentially distributed. The demand is served in a first come first serve principle. Furthermore, we allow a maximum backlog of m in the system. Excessive demand will be rejected and there is a penalty cost associated with the rejection. We are interested in solving the steady state probability distribution for the system because many important performance measures such as the system throughput and machine utilization can be written in terms of the probability distribution. Moreover, we are particularly interested in the average running cost of the system which can also be written down in terms of the steady state probability distribution. The optimal hedging point can be obtained by varying different values of h. Let us give the following notations for our discussion throughout the paper.

$1/\lambda$: the mean arrival time for a demand;
$1/\mu$: the mean processing time for one unit of product;
$1/\sigma$: the mean repair time for a machine;
$1/\gamma$: the mean up time of a machine;
s : number of machines (workstation) in the system;
r : number of maintenance facilities;
m : maximum allowable backlog;
b : the maximum inventory capacity;
$h(\leq b)$: the hedging point;
$c_I > 0$: the unit inventory cost;
$c_B > 0$: the unit backlog cost;
$c_O > 0$: operation cost of a machine per unit time;
$c_R > 0$: repairing cost of a machine per unit time;
$c_D > 0$: penalty cost for rejecting an unit of demand;
$c_P > 0$: profit per unit product.

The remainder of this paper is organized as follow. In §2, we formulate the FMS as a machine-inventory model, and write down the balanced equations of the steady state probability distribution in form of matrix equations. A numerical algorithm is presented in Appendix to solve the system steady state probability distribution. In §3, we illustrate by some numerical examples of our proposed model that the reliability and maintainability of the machines may greatly affect the performance of the FMS. Finally a summary is given in §4 to conclude the paper.

2 The FMS Model

In this section, we establish the mathematical model for the discussed FMS. We construct the balanced equations for the steady state distribution of the machine-inventory system in form of matrix equations.

Under the hedging point policy, the maximum possible inventory level is h. Since the maximum backlog is m, the total number of possible inventory levels is $n = m + h + 1$. In practice the value of n can easily go up to thousands. The number of normal machines can take values in $\{0, 1, \cdots, s\}$. If we let $\alpha(t)$ be the number of normal machines and $x(t)$ be the inventory level at time t then the machine-inventory process

$$\{\alpha(t), x(t)), t \geq 0\} \quad (1)$$

is a continuous time Markov chain taking values in the state space

$$\begin{aligned} S &= \{(\alpha(t), x(t)) : \alpha(t) = 0, \cdots, s, \\ & \quad x(t) = -m, \cdots, h.\}. \end{aligned} \quad (2)$$

The total number of states in S is $n(s + 1)$. Each time when visiting a state, the process stays there for a random period of time that has an Exponential distribution and is independent of the past behavior of the process. If we order the machine states and the inventory levels lexicographically: the machine states are ordered in ascending order of number of normal machines and the inventory levels are ordered in descending order. The steady state probability distribution $\mathbf{p}$ is the solution of the following linear system:

$$A\mathbf{p} = 0 \quad \text{and} \quad \sum_{i=1}^{(s+1)n} p_i = 1. \quad (3)$$

Here the column vector

$$\begin{aligned} \mathbf{p} &= (\mathbf{p}_{0*n+1}, \cdots, \mathbf{p}_{0*n+n}, \mathbf{p}_{1*n+1}, \cdots, \mathbf{p}_{2*n}, \\ & \quad \cdots, \mathbf{p}_{s*n+1}, \cdots, \mathbf{p}_{(s+1)n})^t \end{aligned}$$

is our required steady state probability distribution with $\mathbf{p}_{i*n+j}$ being the steady state probability that the inventory level is $(h-j+1)$ (negative inventory means backlog) and number of normal machine is i. The system *generator* is given by the following tridiagonal block matrix: $A =$

$$\begin{pmatrix} M_0 & -\gamma I & & & 0 \\ -r\sigma I & M_1 & -2\gamma I & & \\ & \ddots & \ddots & \ddots & \\ & & -2\sigma I & M_{(s-1)} & -s\gamma I \\ 0 & & & -\sigma I & M_s \end{pmatrix} \quad (4)$$

and $M_i = \min\{s - i, r\} I_n +$

$$\begin{pmatrix} \lambda & -i\mu & & & 0 \\ -\lambda & \lambda + i\mu & -i\mu & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & \lambda + i\mu & -i\mu \\ 0 & & & -\lambda & i\mu \end{pmatrix} \quad (5)$$

for $i = 0, 1, \cdots, s$. Here I_n is the $n \times n$ identity matrix. The solution of the steady state probability distribution of the system has the following meaning:

$$\begin{aligned} p(\alpha, x) &= \lim_{t \to \infty} \text{Prob}(\alpha(t) = \alpha, x(t) = x) \\ & \quad (\alpha = 0, \cdots, s, x = -m, \cdots, h). \end{aligned}$$

Unfortunately there is no analytic solution for $\mathbf{p}$. A numerical algorithm is proposed in Appendix to solve the problem.

For the FMS, we are interested in the throughput, machine idle time and of course the average profit of the system. Now we let

$$p(i) = \sum_{k=0}^{s} p(k, i), \ i = -m, \cdots, 0, \cdots, h \quad (6)$$

be the marginal steady state probability of the inventory levels of the FMS and

$$q(k) = \sum_{i=-m}^{h} p(k, i), \ k = 0, 1, \cdots, s \quad (7)$$

be the marginal steady state probability of the machine states of the FMS. The throughput of the manufacturing system is then given by

$$T(h,m) = \lambda(1 - p(-m)). \quad (8)$$

The expected inventory management cost can be written as the sum of inventory cost and backlog cost. The sum of inventory cost, backlog cost and rejection cost is given by

$$I(h,m) = c_I \sum_{i=1}^{h} ip(i) - c_B \sum_{i=-m}^{-1} ip(i) - c_D p(-m). \quad (9)$$

The machine operating cost consists of machine repairing cost and machine running cost and is given by

$$\begin{aligned} M(h,m) &= c_O \textstyle\sum_{k=0}^{s} kq(k) \\ &+ c_R \textstyle\sum_{k=0}^{s-r} rq(k) \\ &+ c_R \textstyle\sum_{k=s-r+1}^{s} (s-k)q(k). \end{aligned} \quad (10)$$

Thus the average profit of the FMS is

$$PT(h,m) = c_P T(h,m) - I(h,m) - M(h,m). \quad (11)$$

In the following section, we are going to give some numerical demonstration of the our proposed model.

3 Numerical Examples

In the following numerical examples, for simplicity, we consider FMS with no permitted backlog (i.e. $m = 0$) and zero rejection cost. We let the demand arrival rate λ be 3, the production rate μ of each machine be 1 and the machine repairing rate σ be 1. We also fix the unit inventory cost c_I, operating cost c_O per unit time, repairing cost c_R per unit time, and the unit product profit c_P to be \$5, \$5, \$20 and \$60 respectively.

In the first example, we demonstrate the reliability of machine is an important factor in the performance of a FMS. We consider the system performance by varying the value of γ, the machine breaking down rate. We test two values of γ. The first case is $\gamma = 0.01$ which represents the case that the machine seldom breaks down and is highly reliable. The second case is $\gamma = 1$, the machine is highly unreliable and breaks down very often. In the numerical examples below, we fix the number of machines $s = 4$ and vary the values of r from 1 to 4. Using our proposed model and the BGS algorithm in Appendix, we compute the following tuples (TH, h, IT, PT) for the mentioned values of r and γ. Here TH is the system throughput, h and IT are respectively the optimal hedging point and the percentage of machine idle time and PT is the optimal average profit under the optimal HPP.

r	$\gamma = 0.01$	$\gamma = 1$
1	(2.77,5,30%,131.4)	(0.98,23,0%,31.0)
2	(2.77,5,30%,132.2)	(1.70,16,0%,55.2)
3	(2.77,5,30%,132.2)	(1.95,14,0%,77.2)
4	(2.77,5,30%,132.3)	(1.99,14,0%,93.4)

Table 1

From the numerical results above and many other tested numerical examples, we observe that under optimal HPP, for a given maintenance level r, there are large deviations in average profit and machines idle time for different machine reliability γ. Thus the reliability of machines should be taken in account in the FMS modelling. We also observe that when the machines are highly reliable ($\gamma = 0.01$), the number of maintenance facility can be kept at a minimum level. However, when the machines are highly unreliable ($\gamma = 1$), the number of maintenance facilities available is an important factor for the system performance. Furthermore, we also observed that the more reliable the machines are, the less inventory we need to keep in the system.

The second example is related to the design of the FMS. Suppose that in the FMS, there can be only one maintenance facility, i.e. $r = 1$. Moreover, due to limited capital, at most four machines can be implemented in the system and each machine has a failure rate γ of 1. Assuming the other systems parameters are kept the same as in Example 1. What is the optimal number of machines to be placed in the system? Again use our proposed model and the BGS algorithm in Appendix, we compute the following results.

$s = 1$	$s = 2$
(0.8,26,0%,26.2)	(0.5,27,0%,17.4)
$s = 3$	$s = 4$
(0.94,24,0%,29.9)	(0.98,23,0%,31.0)

Table 2

In this case, the optimal number of machines to be placed in the system is 4.

In the third example, we consider the problem of minimum maintenance facility. Suppose there are eight ($s = 8$) moderate reliable ($\gamma = 0.1$) in the FMS. Assuming the other systems parameters are kept the same as in Example 1. what is the optimal number of maintenance facility should be placed in the system?

Maintenance Facilities	Performance Measure
$r = 1$	(2.8,4,54%,112.2)
$r = 2$	(2.8,4,59%,117.9)
$r = 3$	(2.8,3,60%,123.1)
$r = 4$	(2.8,3,60%,124.6)
$r = 5$	(2.8,3,60%,124.9)
$r = 6$	(2.8,3,60%,124.9)
$r = 7$	(2.8,3,60%,124.9)
$r = 8$	(2.8,3,60%,124.9)

Table 3

In this case, the minimum number of maintenance facility to be placed in the system is 5. Further increase in maintenance facility does not improve the performance of the FMS.

4 Summary

A Markovian multi-server queueing model is proposed for flexible manufacturing system of multiple unreliable machines under HPP policy. A numerical algorithm, the BGS method is presented to solve the steady state probability distribution. Advanced numerical methods based on preconditioned conjugate gradient methods for solving the probability vector can be found in [4, 10]. Numerical examples are given to demonstrate that the machine reliability and maintainability have important effects on the performance of the system. Other applications of the model are also illustrated.

Our proposed model can still cope with the case when machines are not identical. It is interesting to extend our model to non Markovian repairing and production processes. For example, our model can be extended to handle the case when the repairing process is a sequence of exponential distributed repairing steps.

5 Appendix

Block Gauss-Seidel (BGS) Algorithm

A numerical algorithm namely Block Gauss-Seidel (BGS) is proposed here to solve the steady state probability distribution in (3). Let $\mathbf{p} = (\mathbf{p_0}, \mathbf{p_1}, \cdots, \mathbf{p_s})^t$ be the steady state probability distribution. Each vector $\mathbf{p_i}$ is a $n \times 1$ vector representing the steady state inventory level when the number of normal machines in the system is i. The BGS algorithm reads (see Golub and van Loan [11] for more detail about the algorithm):

Initialize $\mathbf{p} = (0, 0, \cdots, 0, 1)^t$;
Initialize error $= 1$;
While error $> 10^{-10}$ do the following: (where 10^{-10} is the error tolerance)
for $i = 1$ to s do
$\mathbf{p_{i-1}} = \gamma i \mathbf{p_i}$;

end;
$\mathbf{p_s} = \mathbf{0}$; (where $\mathbf{0}$ is the zero vector)
$\mathbf{p_0} = M_0^{-1}\mathbf{p_0}$;
for $i = 1$ to s do
$\mathbf{p_i} = M_i^{-1}(\mathbf{p_i} + \sigma \min(s-i+1, r)\mathbf{p_{i-1}})$;
end;
$\mathbf{p} = \mathbf{p}/(\mathbf{1}^t\mathbf{p})$; (where $\mathbf{1}$ is the column vector with all entries being one)
error $= \sqrt{(A\mathbf{p})^t * (A\mathbf{p})}$;
end;

All the computations are done in a NEC Celeron 300Hz notebook with MATLAB. We remark that the BGS algorithm converges [11] for our problem and the total computational cost is around $O(n^2(s+1)^2)$, see [10] for instance.

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