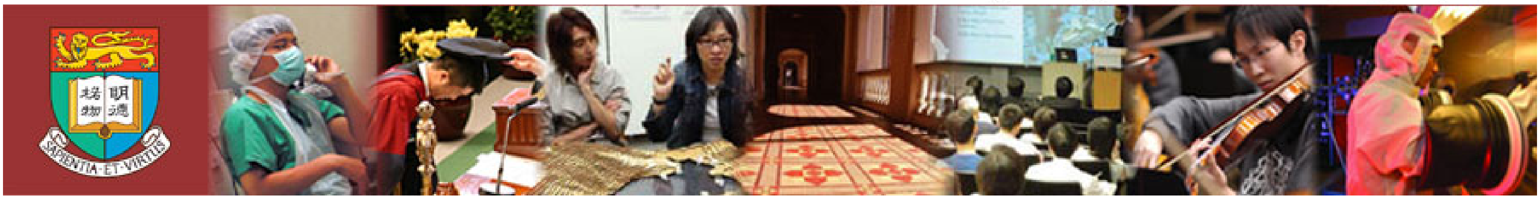

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On a Waring-Goldbach type problem for fourth powers

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Abstract

In this paper, we prove that every sufficiently large positive integer satisfying some necessary congruence conditions can be represented by the sum of a fourth power of integer and twelve fourth powers of prime numbers.

MSC: 11P05, 11P32, 11P55.

Keywords: Waring-Goldbach problem; Circle method

1. Statment of the result

One of the problems of the Waring-Goldbach type is to find the least positive integer s such that every sufficiently large integer satisfying some necessary congruence conditions can be expressed by the sum of s fourth powers of primes. The expected value of s is 5, but this is far from reach by techniques developed so far. The present machinery in the circle method has been able to establish $s = 14$ which is due to Kawada and Wooley [7]. Precisely, they have proved that for all sufficiently large integers $n \equiv 14 (\bmod 240)$, the equation

$$n = p_1^4 + p_2^4 + ... + p_{14}^4$$

is solvable in primes p_j.

On the other hand, concerning the corresponding Waring's problem, Thanigasalam [12] has proved that

$$n = m_1^4 + m_2^4 + ... + m_{13}^4$$

is solvable for every sufficiently large integer n with $n \equiv r (\bmod 16)$ where $1 \le r \le 13$. Here m_j are positive integers. The number of variables 13 has been reduced to 12 by Vaughan [13]. Kawada and Wooley [6] can further reduce 12 to 11 except for $r \equiv 11 (\bmod 16)$. In this paper, we consider the expression

$$n = m^4 + p_1^4 + p_2^4 + ... + p_{12}^4, \tag{1.1}$$

where m is a natural number and p_j are primes. Our result is the following.

Theorem 1. *The equation* (1.1) *is solvable for all sufficiently large integers* n *subject to*

$$n \equiv a (\bmod 240) \quad \text{for any} \quad a \in \mathfrak{A}, \tag{1.2}$$

where

$$\mathfrak{A} = \{12, 13, 28, 93, 108, 157, 172, 237\}.$$

Notation. As usual, $\varphi(n)$ and $\Lambda(n)$ stand for the function of Euler and von Mangoldt respectively, and $d(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. In our context, the letter N stands for a large positive integer, and $L = \log N$. The symbol $r \sim R$ means $R < r \le 2R$. The letters ε and A denote positive constants which are arbitrarily small and arbitrarily large, respectively. We use c_j to denote an absolute positive constant. The letter c denotes an unspecified positive constant which is not neccesarily the same at each occurrence.

2. Outline of the method

Following [7], we introduce some notations. Let $\lambda_0 = 13/16$ and

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \lambda_4 = \lambda_0, \quad \lambda_5 = \lambda_6 = \lambda_0^2, \tag{2.1}$$

$$\lambda_7 = \lambda_8 = 91\lambda_0^2/111, \quad \lambda_9 = ... = \lambda_{12} = 78\lambda_0^2/111, \tag{2.2}$$

$$\mu = (1 + \lambda_1 + \lambda_2 + \dots + \lambda_{12})/4 = 2.22\cdots, \tag{2.3}$$

$$U = N^{1/4}/2, \qquad U_i = U^{\lambda_i}, \quad i = 1, 2, \dots, 12. \tag{2.4}$$

In order to apply the circle method, we set

$$P = U^{2/5}, \qquad Q = NP^{-1}. \tag{2.5}$$

Then by Dirichlet's Lemma on rational approximations for each $\alpha \in [1/Q, 1 + 1/Q]$, there are coprime integers a, q satisfying $1 \le a \le q \le Q$ and

$$\alpha = a/q + \lambda, \qquad |\lambda| \le 1/qQ. \tag{2.6}$$

We denote by $\mathfrak{M}(q, a)$ the set of all α satisfying (2.6). These intervals all lie in $[1/Q, 1+1/Q]$ and for $q \le P$ they are mutually disjoint, since $2P \le Q$. Let the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ be defined as follows:

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a), \qquad \mathfrak{m} = [1/Q, 1 + 1/Q] \setminus \mathfrak{M}.$$

For $W > 0$, we define

$$S(\alpha, W) = \sum_{m \sim W} \Lambda(m) e(m^4 \alpha), \quad \text{and} \quad T(\alpha, W) = \sum_{m \sim W} e(m^4 \alpha), \tag{2.7}$$

where $e(z) = e^{2\pi i z}$. Let

$$R(n) = \sum_{\substack{n = m^4 + m_1^4 + m_2^4 + \dots + m_{12}^4 \\ m \sim U,\, m_i \sim U_i}} \Lambda(m_1) \cdots \Lambda(m_{12}),$$

which is the number of weighted representations of (1.1). Then we have

$$R(n) = \int_{1/Q}^{1+1/Q} \left\{ \prod_{i=1}^{12} S(\alpha, U_i) \right\} T(\alpha, U) e(-n\alpha) d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}. \tag{2.8}$$

To handle the integral on the major arcs, we need the following

Theorem 2. *For all n with $N/2 < n \le N$, we have*

$$\int_{\mathfrak{M}} \left\{ \prod_{i=1}^{12} S(\alpha, U_i) \right\} T(\alpha, U) e(-n\alpha) d\alpha = \mathfrak{S}(n) \mathfrak{J}(n) + O(N^{\mu - 1} L^{-A}). \tag{2.9}$$

Here $\mathfrak{S}(n)$ is the singular series defined in (4.1) *which satisfies*

$$1 \ll \mathfrak{S}(n) \ll 1 \tag{2.10}$$

for n satisfying (1.2)*; while $\mathfrak{J}(n)$ is defined by* (4.10) *and satisfies*

$$N^{\mu - 1} \ll \mathfrak{J}(n) \ll N^{\mu - 1}. \tag{2.11}$$

Proof of Theorem 1. We first establish the following estimate on the minor arcs.

$$\left| \int_{\mathfrak{m}} \left\{ \prod_{i=1}^{12} S(\alpha, U_i) \right\} T(\alpha, U) e(-n\alpha) d\alpha \right| \ll N^{\mu - 1.01}. \tag{2.12}$$

For $\alpha = a/q + \lambda \in \mathfrak{m}$, we have $P < q \le Q$ and $|\lambda| \le 1/qQ$. If $q > U$, then it follows from Weyl's inequality [14, Lemma 2.4] that $|T(\alpha, U)| \ll U^{7/8+\varepsilon}$. If $P < q \le U$, we apply Lemmas 6.1-6.3 in [14], to get

$$|T(\alpha, U)| \ll \frac{q^{-1/4}U}{1+|\lambda|N} + q^{1/2+\varepsilon} \ll UP^{-1/4} + U^{1/2+\varepsilon} \ll U^{9/10}.$$

Thus we can conclude that

$$\max_{\alpha\in\mathfrak{m}} |T(\alpha, U)| \ll U^{9/10}.$$

On the other hand, a slight modification of Theorem 3 for $j = 1$ in Thanigasalam [12] (or [7, Lemma 4.3]) reveals that

$$\int_0^1 \left| \prod_{i=1}^{12} S(\alpha, U_k) \right| d\alpha \ll (U_1 U_2 \cdots U_{12})^{1/2} U^{\varepsilon}. \tag{2.13}$$

Therefore

$$\begin{aligned} &\left| \int_{\mathfrak{m}} \left\{ \prod_{i=1}^{12} S(\alpha, U_i) \right\} T(\alpha, U) e(-n\alpha) d\alpha \right| \\ &\ll \max_{\mathfrak{m}} |T(\alpha, U)| \int_0^1 \left| \prod_{i=1}^{12} S(\alpha, U_i) \right| d\alpha \\ &\ll U^{9/10+\varepsilon} (U_1 U_2 \cdots U_{12})^{1/2} \ll N^{\mu-1.01}, \end{aligned}$$

by (2.1)-(2.4). This proves (2.12) which in combination with Theorem 2 and (2.8) gives

$$R(n) = \mathfrak{S}(n)\mathfrak{I}(n) + O(N^{\mu-1}L^{-A}).$$

Theorem 1 now follows by summing over dyadic intervals. □

The following sections will be devoted to the proof of Theorem 2.

3. An explicit expression

In this section, we will establish in Lemma 3.1 an explicit expression for the left-hand side of (2.9). For χ mod q, we define

$$C(\chi, a) = \sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{am^4}{q}\right) \quad \text{and} \quad C(q, a) = C(\chi^0, a). \tag{3.1}$$

Then Vinogradov's bound gives [15, Chapter VI, problem 14]

$$|C(\chi, a)| \le 2q^{1/2} d(q)^2. \tag{3.2}$$

For $W > 0$ and $\alpha = a/q + \lambda$ with $(a, q) = 1$, we have

$$S(\alpha, W) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^4}{q}\right) \sum_{\substack{m\sim W \\ m\equiv h (\bmod q)}} \Lambda(m) e(\lambda m^4) + O(L^2).$$

Introducing Dirichlet characters to the above sum over m, one can rewrite $S(\alpha, W)$ as

$$\frac{C(q,a)}{\varphi(q)}\sum_{m\sim W} e(\lambda m^4)+\sum_{\chi \bmod q}\frac{C(\chi,a)}{\varphi(q)}\sum_{m\sim W}(\Lambda(m)\chi(m)-\delta_\chi)e(\lambda m^4)+O(L^2).$$

Here and throughout, $\delta_\chi = 1$ or 0 according as χ is the principal character or not.

By Lemma 4.8 in [11], one has, for $0 < W \le U$ and $\alpha = a/q + \lambda$ subject to (2.6),

$$\sum_{m\sim W} e(\lambda m^4)=\int_W^{2W} e(\lambda u^4)du+O(1).$$

Thus if we denote by $\Phi(\lambda, W)$ the above integral and write

$$\Psi(\chi,\lambda,W)=\sum_{m\sim W}(\Lambda(m)\chi(m)-\delta_\chi)e(\lambda m^4), \tag{3.3}$$

then

$$S(\alpha,W)=S_1(\lambda,W)+S_2(\lambda,W)+O(L^2),$$

where

$$S_1(\lambda,W)=\frac{C(q,a)}{\varphi(q)}\Phi(\lambda,W),\quad S_2(\lambda,W)=\sum_{\chi \bmod q}\frac{C(\chi,a)}{\varphi(q)}\Psi(\chi,\lambda,W). \tag{3.4}$$

For $T(\alpha, U)$, we apply Theorem 4.1 in [14] to get

$$T(\alpha,U)=T(\lambda)+O(q^{1/2+\varepsilon}),$$

where

$$T(\lambda)=\frac{S^*(q,a)}{q}\Phi(\lambda,U)\quad\text{with}\quad S^*(q,a)=\sum_{m=1}^{q} e\left(\frac{am^4}{q}\right). \tag{3.5}$$

Let $\Delta(\lambda)$ be defined by

$$\prod_{i=1}^{12} S(\alpha,U_i)=\prod_{i=1}^{12} S_1(\lambda,U_i)+\Delta(\lambda), \tag{3.6}$$

and let

$$I=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\int_{-1/qQ}^{1/qQ}\left\{\prod_{i=1}^{12} S_1(\lambda,U_i)\right\}T(\lambda)e(-n\lambda)d\lambda, \tag{3.7}$$

$$J=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\int_{-1/qQ}^{1/qQ}\Delta(\lambda)T(\lambda)e(-n\lambda)d\lambda. \tag{3.8}$$

Then we have

$$\int_{\mathfrak{M}}\left\{\prod_{i=1}^{12} S(\alpha,U_i)\right\}T(\alpha,U)e(-n\alpha)d\alpha=I+J+O\left\{P^{1/2+\varepsilon}\int_0^1\left|\prod_{i=1}^{12} S(\alpha,U_i)\right|d\alpha\right\},$$

where, by (2.13) and (2.1)-(2.5), the above O-term is

$$\ll U^{1/5+\varepsilon}(U_1U_2\cdots U_{12})^{1/2}\ll N^{\mu-1}L^{-A}.$$

Therefore we have proved the following.

Lemma 3.1. *For all n with $N/2 < n \le N$, we have*

$$\int_{\mathfrak{M}} \left\{ \prod_{i=1}^{12} S(\alpha, U_i) \right\} T(\alpha, U) e(-n\alpha) d\alpha = I + J + O(N^{\mu-1} L^{-A}).$$

In the following sections we will prove that I produces the main term, while J contributes to the error term.

4. Estimation of I

Let $C(q,a)$ and $S^*(q,a)$ be defined by (3.1) and (3.5), respectively. We define

$$B(n,q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) C^{12}(q,a) S^*(q,a),$$

and write

$$A(n,q) = \frac{B(n,q)}{q\varphi^{12}(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q). \tag{4.1}$$

Lemma 4.1. *The singular series $\mathfrak{S}(n)$ is absolutely convergent and satisfies* (2.10).

Proof. By (3.2) and the well known bound $|S^*(q,a)| \ll q^{3/4+\varepsilon}$, one easily obtains

$$|A(n,q)| \ll q^{-21/4+\varepsilon}. \tag{4.2}$$

Therefore the singular series is absolutely convergent and satisfies the second inequality in (2.10). Moreover we have

$$\sum_{q \le P} A(n,q) = \mathfrak{S}(n) + O(P^{-17/4+\varepsilon}). \tag{4.3}$$

To prove the first inequality in (2.10), we first note that $A(n,q)$ is multiplicative with respect to q. We next prove that

$$A(n,p^t) = 0 \quad \text{for} \quad t \ge \alpha, \tag{4.4}$$

where

$$\alpha = \begin{cases} 2, & \text{if } \ p \ge 3, \\ 5, & \text{if } \ p = 2. \end{cases}$$

Actually, when $(a,p)=1$ and $i \ge 2$, we have

$$\begin{aligned} C(p^i,a) &= \sum_{\substack{m=1 \\ (m,p)=1}}^{p^i} e\left(\frac{am^4}{p^i}\right) = \sum_{k=0}^{p-1} \sum_{\substack{m=1 \\ (m,p)=1}}^{p^{i-1}} e\left(\frac{a(kp^{i-1}+m)^4}{p^i}\right) \\ &= \sum_{k=0}^{p-1} \sum_{\substack{m=1 \\ (m,p)=1}}^{p^{i-1}} e\left(\frac{4akm^3p^{i-1}+am^4}{p^i}\right) = \sum_{\substack{m=1 \\ (m,p)=1}}^{p^{i-1}} e\left(\frac{am^4}{p^i}\right) \sum_{k=0}^{p-1} e\left(\frac{4ak}{p}\right). \end{aligned} \tag{4.5}$$

When $p \geq 3$, the inner sum is 0, and hence $A(n, p^i) = 0$. When $p = 2$ and $i \geq 5$, it follows from (4.5) that

$$\begin{aligned} C(2^i, a) &= 2 \sum_{\substack{m=1 \\ (m,2)=1}}^{2^{i-1}} e\left(\frac{am^4}{2^i}\right) = 2 \sum_{k=0}^{2^2-1} \sum_{\substack{m=1 \\ (m,2)=1}}^{2^{i-3}} e\left(\frac{a(k2^{i-3}+m)^4}{2^i}\right) \\ &= 4 \sum_{\substack{m=1 \\ (m,2)=1}}^{2^{i-3}} e\left(\frac{am^4}{2^i}\right) \sum_{k=0}^{1} e\left(\frac{ak}{2}\right) = 0. \end{aligned}$$

Thus $A(n, 2^i) = 0$ for $i \geq 5$. This proves (4.4).

By (4.4) and the multiplicity of $A(n, q)$, we can now write

$$\mathfrak{S}(n) = \big(1 + A(n,2) + A(n,2^2) + A(n,2^3) + A(n,2^4)\big) \prod_{p>2} (1 + A(n,p)). \tag{4.6}$$

Since $S^*(p, a) = C(p, a) + 1$ and $|C(p, a)| \leq 8p^{1/2}$, by (3.2), we have

$$|A(n,p)| \leq \frac{8^{13}p^{11/2} + 8^{12}p^5}{(p-1)^{11}} \leq p^{-2}, \quad \text{when} \quad p \geq c_1,$$

where c_1 is some positive constant. Hence

$$\prod_{p>c_1} (1 + A(n,p)) \geq c_2 > 0. \tag{4.7}$$

On the other hand, we have

$$1 + A(n,2) + A(n,2^2) + A(n,2^3) + A(n,2^4) = \frac{M(2^4, n)}{2^{36}},$$

and for $p > 2$,

$$1 + A(n,p) = \frac{M(p,n)}{(p-1)^{12}}.$$

Here $M(p^j, n)$ is the number of solutions of the congruence

$$m^4 + m_1^4 + m_2^4 + ... + m_{12}^4 = n (\bmod p^j)$$

subject to

$$1 \leq m \leq p^j, \quad 1 \leq m_i < p^j \quad \text{with} \quad p \nmid m_i, \quad i = 1, 2, ..., 12.$$

By Lemma 8.8 in [3], we deduce that $M(p, n) > 0$ for all n and $p \geq 7$, and therefore

$$\prod_{7 \leq p \leq c_1} (1 + A(n,p)) \geq c_3 > 0. \tag{4.8}$$

Moreover, a direct investigation reveals that $M(2^4, n) > 0$ for $n \equiv 12,\ 13 (\bmod 16)$; $M(3, n) > 0$ for $n \equiv 0,\ 1 \pmod 3$ and $M(5, n) > 0$ for $n \equiv \pm 2 \pmod 5$. These estimates together with (4.6)-(4.8) prove that for n satisfying (1.2), $\mathfrak{S}(n) \geq c_4 > 0$. Lemma 4.1 is thus established. □

Lemma 4.2. *Let I be defined by* (3.7). *Then for all $n \in [N/2, N]$ subject to* (1.2), *we have*

$$I = \mathfrak{S}(n)\mathfrak{I}(n) + O(N^{\mu-1}L^{-A}),$$

where $\mathfrak{I}(n)$ *is defined by* (4.10) *and satisfies* (2.11).

Proof. By definition we have

$$I = \sum_{q \le P} A(n,q) \int_{-1/qQ}^{1/qQ} \left\{ \prod_{i=1}^{12} \Phi(\lambda, U_i) \right\} \Phi(\lambda, U) e(-n\lambda) d\lambda. \tag{4.9}$$

Let

$$\mathfrak{I}(n) = \int_{-\infty}^{\infty} \left\{ \prod_{i=1}^{12} \Phi(\lambda, U_i) \right\} \Phi(\lambda, U) e(-n\lambda) d\lambda. \tag{4.10}$$

On using the elementary estimate

$$|\Phi(\lambda, W)| \le \min \left(W, \frac{1}{|\lambda| W^3} \right), \tag{4.11}$$

one easily obtains

$$\left\{ \prod_{k=1}^{12} \Phi(\lambda, U_i) \right\} \Phi(\lambda, U) \ll \frac{U^3 U_3 \cdots U_{12}}{(1+|\lambda| N)^3} \ll \frac{N^\mu}{(1+|\lambda| N)^3}. \tag{4.12}$$

It therefore follows that

$$\int_{1/qQ}^{\infty} \left| \prod_{k=1}^{12} \Phi(\lambda, U_i) \right| |\Phi(\lambda, U)| d\lambda \ll N^\mu \int_{1/qQ}^{\infty} \frac{d\lambda}{(1+|\lambda| N)^3} \ll N^{\mu-1} (qQ/N)^2.$$

Thus

$$\int_{-1/qQ}^{1/qQ} \left\{ \prod_{k=1}^{12} \Phi(\lambda, U_i) \right\} \Phi(\lambda, U) e(-n\lambda) d\lambda = \mathfrak{I}(n) + O(N^{\mu-1} P^{-2} q^2).$$

Putting this in (4.9) and then making use of (4.3) and (4.2), we get

$$\begin{aligned} I &= \mathfrak{I}(n) \sum_{q \le P} A(n,q) + O(N^{\mu-1} P^{-2} \sum_{q \le P} q^2 |A(n,q)|) \\ &= \mathfrak{I}(n) \mathfrak{S}(n) + O(N^{\mu-1} P^{-2}), \end{aligned} \tag{4.13}$$

subject to the validity of (2.11). Now it remains to check (2.11) of which the second inequality is an immediate derivation of (4.12). To prove the first inequality, we apply Fourier's integral formula to get

$$\mathfrak{I}(n) = \frac{1}{4^{13}} \int_D u_1^{-3/4} u_2^{-3/4} \cdots u_{12}^{-3/4} u^{-3/4} du_1 du_2 \cdots du_{12},$$

where $u = n - u_1 - ... - u_{12}$, and D is the set of all vectors $(u_1, u_2, ..., u_{12})$ subject to

$$U_i^4 < u_i \le (2U_i)^4, \quad \text{and} \quad U^4 < u < (2U)^4.$$

Let D^* be the set of those vectors $(u_1, u_2, ..., u_{12})$ such that

$$U_i^4 < u_i \le (3U_i/2)^4 \quad \text{for} \quad i = 1,\ 2, ..., 12.$$

Then it is easy to check that $U^4 < u < (2U)^4$ holds for $(u_1, u_2, ..., u_{12}) \in D^*$. This means that D^* is a nonempty subset of D, and hence

$$\begin{aligned}\mathfrak{I}(n) &\gg \int_{D^*} u_1^{-3/4} u_2^{-3/4} \cdots u_{12}^{-3/4} u^{-3/4} du_1 du_2 \cdots du_{12} \\ &\gg U^{-3} U_1 U_2 \cdots U_{12} \gg N^{\mu-1}.\end{aligned}$$

This proves (2.11), and hence finishes the proof of Lemma 4.2. □

5. Estimation of J

Lemma 5.1. *Let J be as defined in* (3.8). *Then we have*

$$J \ll N^{\mu-1} L^{-A}.$$

To prove Lemma 5.1, we need the following lemma whose proof will be given in the next section.

Lemma 5.2. *Let $W \geq 1$, $R \geq 1$ and $1 < q \leq W^d$ with $d \geq 1$. Then for $k \geq 1$ and $\lambda \in \mathbb{R}$ subject to $|\lambda| W^k \leq R$, we have*

$$\sum_{\chi \bmod q} \left| \sum_{m \sim W} \Lambda(m) \chi(m) e(\lambda m^k) \right| \ll \left\{ \left(R + (WR)^{1/2} \right) q + W^{4/5} q^{1/2} + W \right\} L^c. \tag{5.1}$$

Proof of Lemma 5.1. In view of (3.6) and (3.8), we see that J consists $3^{12} - 1$ terms of the form

$$\sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-1/qQ}^{1/qQ} \left\{ \prod_{i=1}^{12} E(\lambda, U_i) \right\} T(\lambda) e(-n\lambda) d\lambda,$$

where $E(\lambda, U_i) = S_1(\lambda, U_i), S_2(\lambda, U_i)$ or L^2 with the exception that $E(\lambda, U_i) = S_1(\lambda, U_i)$ holds for all $i = 1, 2, ..., 12$. Here $S_1(\lambda, U_i), S_2(\lambda, U_i)$ are defined by (3.4). On using (3.2), we see that

$$E(\lambda, U_i) \ll q^{-1/2+\varepsilon} H(\lambda, U_i),$$

where $H(\lambda, U_i)$ represents any one of the following three expressions

$$|\Phi(\lambda, U_i)|, \qquad \sum_{\chi \bmod q} |\Psi(\chi, \lambda, U_i)|, \qquad q^{1/2} L^2.$$

Using the well known bound $S^*(q, a) \ll q^{3/4+\varepsilon}$ in (3.5), we also see that

$$|T(\lambda)| \ll q^{-1/4+\varepsilon} |\Phi(\lambda, U)|.$$

Therefore we get

$$J \ll \sum_{q \leq P} q^{-21/4+\varepsilon} \max_{|\lambda| \leq 1/qQ} \left\{ \prod_{i=1}^{12} H(\lambda, U_i) \right\} \int_{-1/qQ}^{1/qQ} |\Phi(\lambda, U)| d\lambda,$$

where $H(\lambda, U_i) \neq |\Phi(\lambda, U_i)|$ happens for at least one of $i = 1, 2, ..., 12$. Without loss of generality, we assume $H(\lambda, U_2) \neq |\Phi(\lambda, U_2)|$.

By (4.11) one easily obtains

$$\int_{-1/qQ}^{1/qQ} |\Phi(\lambda, U)| d\lambda \ll U^{-3} L,$$

and hence

$$\begin{aligned} J &\ll U^{-3} L \sum_{q \le P} q^{-21/4+\varepsilon} \max_{|\lambda| \le 1/qQ} \left\{ \prod_{i=1}^{12} H(\lambda, U_i) \right\} \\ &=: U^{-3} L \left(J_1 + J_2 \right). \end{aligned} \tag{5.2}$$

Here J_1 and J_2 represent sums over $q \le L^B$ and $L^B < q \le P$, respectively with $B = 4A$. So to prove Lemma 5.1, we only need to prove that

$$J_1, \ J_2 \ll U_1 U_2 \cdots U_{12} L^{-A}.$$

One notes that for $|\lambda| \le 1/qQ$,

$$|\lambda| U_i^4 \le P/q, \quad \text{for} \quad i = 1, \ 2; \quad \text{and} \quad |\lambda| U_i^4 \le 1, \quad \text{for} \quad i = 3, \ 4, ..., 12.$$

Therefore it follows by trivial estaimates and Lemma 5.2 that for $q \le P = U^{2/5}$,

$$H(\lambda, U_1), \quad H(\lambda, U_2) \ll \left\{ (UPq)^{1/2} + U^{4/5} q^{1/2} + U \right\} L^c \ll U L^c, \tag{5.3}$$

and

$$H(\lambda, U_i) \ll \left\{ q U_i^{1/2} + q^{1/2} U_i^{4/5} + U_i \right\} L^c \quad \text{for} \quad i = 3, \ 4, ..., 12. \tag{5.4}$$

Thus we have

$$J_2 \ll U_1 U_2 \cdots U_{12} L^c \sum_{L^B < q \le P} q^{-21/4+\varepsilon} \prod_{i=3}^{12} (q U_i^{-1/2} + q^{1/2} U_i^{-1/5} + 1).$$

Let

$$\mu_3 = \mu_4 = (1 - \lambda_0)/2, \quad \mu_i = 1 - 5\lambda_i/4 \quad \text{for} \quad i \ge 5,$$

where λ_i are defined by (2.1) and (2.2). Here μ_i are so chosen that for $1 < q \le P$,

$$q^{-\mu_i} (q U_i^{-1/2} + q^{1/2} U_i^{-1/5} + 1) \ll 1, \quad \text{for} \quad i = 3, \ 4, ..., 12.$$

Write

$$\mu^* = 21/4 - (\mu_3 + ... + \mu_{12}) = 2.38 \cdots .$$

Then

$$J_2 \ll U_1 U_2 \cdots U_{12} L^c \sum_{L^B < q \le P} q^{-\mu^* + \varepsilon} \ll U_1 U_2 \cdots U_{12} L^{-A}. \tag{5.5}$$

Now we turn to J_1. For $q \le L^B$, we see from (5.3) and (5.4) that $H(\lambda, U_i) \ll U_i L^c$ for $i = 1, \ 3, \ 4, ..., 12$. Hence

$$J_1 \ll U_1 U_3 \cdots U_{12} L^c \sum_{q \le L^B} q^{-21/4+\varepsilon} \max_{|\lambda| \le 1/qQ} H(\lambda, U_2), \tag{5.6}$$

where $H(\lambda, U_2) = \sum_{\chi \bmod q} |\Psi(\chi, \lambda, U_2)|$ or $q^{1/2}L^2$, by assumption. The desired assertion is obvious if $H(\lambda, U_2) = q^{1/2}L^2$. Otherwise we recall the explicit formula [1, §17, (9)-(10); §19, (4)-(9)]:

$$\sum_{m \le x} \Lambda(m)\chi(m) = \delta_\chi x - \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x(\log qxT)^2}{T}\right),$$

where $2 < T \le x$ is a parameter and $\rho = \beta + i\gamma$ is a typical nontrivial zero of the Dirichlet L-function $L(s, \chi)$. Let $T = PL^B$. Then by integrating by parts,

$$\begin{aligned}
\Psi(\chi, \lambda, U_2) &= \int_{U_2}^{2U_2} e(\lambda u^4) d \sum_{m \le u} (\Lambda(m)\chi(m) - \delta_\chi) \\
&= -\sum_{|\gamma| \le PL^B} \int_{U_2}^{2U_2} u^{\rho-1} e(\lambda u^4) du + O\left(U_2 P^{-1} L^{2-B}(1 + |\lambda| U_2^4)\right) \\
&\ll \sum_{|\gamma| \le PL^B} U_2^\beta + U_2 q^{-1} L^{2-B}.
\end{aligned}$$

Thus we get

$$H(\lambda, U_2) \ll U_2 \sum_{\chi \bmod q} \sum_{|\gamma| \le PL^B} U_2^{\beta-1} + U_2 L^{2-B}.$$

By Satz VIII.6.2 of Prachar [8] and Siegel's theorem [1, §21], there exists a positive constant c_5 such that for $q \le L^B$, $\prod_{\chi \bmod q} L(s, \chi)$ is zero-free in the region

$$\sigma \ge 1 - c_5 / \max\{\log q,\ \log^{4/5} x\}, \qquad |t| \le x.$$

Let $\eta(N) = c_5 \log^{-4/5} N$. By integrating by parts, and then making use of the following well-known zero-density estimates (see for example [4, (1.1)] and [5, Theorem 1])

$$\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (qT)^{(12/5+\varepsilon)(1-\sigma)}, \quad 1/2 \le \sigma \le 1,$$

we have for $q \le L^B$,

$$\begin{aligned}
H(\lambda, U_2) &\ll U_2 \max_{1/2 \le \sigma \le 1-\eta(N)} \left(PL^{2B}\right)^{(12/5+\varepsilon)(1-\sigma)} U_2^{\sigma-1} + U_2 L^{-2A} \\
&\ll U_2 \max_{1/2 \le \sigma \le 1-\eta(N)} U_2^{(\sigma-1)/30} + U_2 L^{-2A} \\
&\ll U_2 \exp(-c_6 L^{1/5}) + U_2 L^{-2A} \ll U_2 L^{-2A}.
\end{aligned}$$

This together with (5.6) prove

$$J_1 \ll U_1 U_2 \cdots U_{12} L^{-A}.$$

With this, Lemma 5.1 follows from (5.2) and (5.5). □

6. Proof of Lemma 5.2.

Let $M \geq 1$ be a real number. For $j = 1, ..., 10$, let M_j be positive integers such that

$$2^{-10}M \leq M_1 \cdots M_{10} \leq 2M, \quad \text{and} \quad 2M_6, ..., 2M_{10} \leq (2M)^{1/5}. \tag{6.1}$$

For a positive integer m, let

$$a_j(m) = \begin{cases} \log m, & \text{if} \quad j = 1, \\ 1, & \text{if} \quad j = 2, ..., 5, \\ \mu(m), & \text{if} \quad j = 6, ..., 10. \end{cases} \tag{6.2}$$

We define the following functions of a complex variable s:

$$f_j(s, \chi) = \sum_{m \sim M_j} \frac{a_j(m)\chi(m)}{m^s}, \qquad F(s, \chi) = f_1(s, \chi) \cdots f_{10}(s, \chi). \tag{6.3}$$

To prove Lemma 5.2, we need the following mean value estimate for $F(1/2 + it, \chi)$.

Lemma 6.1. *Let $d \geq 1$ and $g \geq 1$. For $2 \leq T \leq M^g$ and $1 < q \leq M^d$, we have*

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi^0}} \int_{-T}^{T} |F(1/2 + it, \chi)| \, dt \ll \{qT + (qT)^{1/2} M^{3/10} + M^{1/2}\} L^c. \tag{6.4}$$

To prove Lemma 6.1, we quote the following two well known results (see for example [10, Theorems 2.5 and 3.17]).

Lemma 6.2. *Let $q \geq 1$, $T \geq 1$, $M_0 \geq 1$ and $D \geq 1$. Let a_m be complex numbers. Then we have*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| \sum_{m=M_0}^{M_0+D} \frac{a_m \chi(m)}{m^{it}} \right|^2 dt \ll \sum_{m=M_0}^{M_0+D} (qT + m) |a_m|^2.$$

Lemma 6.3. *Let $a_0 = 5$ and $a_1 = 9$. Then for $q > 2$, $T \geq 2$ and $\nu = 0, 1$, we have*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| L^{(\nu)}(1/2 + it, \chi) \right|^4 dt \ll qT \log^{a_\nu}(qT).$$

Proposition 6.4. *If there exist M_i and M_j with $1 \leq i < j \leq 5$ such that $M_i M_j > M^{2/5}$, then* (6.4) *is true.*

Proof. Without loss of generality, we may suppose that $i = 1$ and $j = 2$. Using Perron's summation formula [11, Lemma 3.12] and then shifting the path of integration to the left, we get for $\chi \neq \chi_0$

$$\begin{aligned} f_1(1/2 + it, \chi) &= -\frac{1}{2\pi i} \int_{1/2+1/L-iT_0}^{1/2+1/L+iT_0} L'(1/2 + it + w, \chi) \frac{(2M_1)^w - M_1^w}{w} dw + O(L^2) \\ &= -\frac{1}{2\pi i} \left\{ \int_{1/2+1/L-iT_0}^{-iT_0} + \int_{-iT_0}^{iT_0} + \int_{iT_0}^{1/2+1/L+iT_0} \right\} + O(L^2), \end{aligned}$$

where $T_0 = M^{d+g}$. One notes that the function $\frac{(2M_1)^w - M_1^w}{w}$ has a removable singularity at $w = 0$. Thus, on the above vertical segment from $-iT_0$ to iT_0, we have

$$\frac{(2M_1)^{iv} - M_1^{iv}}{iv} \ll \frac{1}{1+|v|}.$$

On using the well-known bounds (see for example [9, pp.271, Exercise 6 and pp.269, (13)]): For $q \geq 1$, $\chi \neq \chi^0$ and $\nu \geq 0$,

$$L^{(\nu)}(\sigma + it, \chi) \ll \log^{(\nu+1)}(q(|t|+2)) \max\left\{1, q^{(1-\sigma)/2}|t|^{1-\sigma}\right\}, \quad \sigma > 0,$$

we see that the contribution from the two horizontal segments is

$$\ll L^2 \max_{0 \leq u \leq 1/2+1/L} q^{(1-(1/2+u))/2}(T_0 + |t|)^{1-(1/2+u)} M_1^u / T_0 \ll L^2 q^{1/4} T_0^{-1/2} \ll 1,$$

since $q \leq M^d \leq T_0$. Therefore we get

$$f_1(1/2 + it, \chi) \ll \int_{-T_0}^{T_0} \left|L'(1/2 + it + iv, \chi)\right| \frac{dv}{1+|v|} + L^2,$$

and by Hölder's inequality,

$$\begin{aligned} &\sum_{\substack{\chi \bmod q \\ \chi \neq \chi^0}} \int_{-T}^{T} |f_1(1/2 + it, \chi)|^4 \, dt \\ &\ll L^3 \sum_{\chi \bmod q} \int_{-T}^{T} \int_{-T_0}^{T_0} \left|L'(1/2 + it + iv, \chi)\right|^4 \frac{dvdt}{1+|v|} + O(qTL^8). \end{aligned} \tag{6.5}$$

Write $\int_{-T_0}^{T_0} = \int_{|v| \leq 2T} + \int_{2T < |v| \leq T_0}$. Then the first term in (6.5) splits into two quantities, which we denote by $\varSigma_1$ and $\varSigma_2$, respectively. By Lemma 6.3, we have

$$\begin{aligned} \varSigma_1 &\ll L^3 \sum_{\chi \bmod q} \int_{-2T}^{2T} \frac{dv}{1+|v|} \int_{-T+v}^{T+v} \left|L'(1/2 + iw, \chi)\right|^4 dw \\ &\ll L^4 \sum_{\chi \bmod q} \int_{-3T}^{3T} \left|L'(1/2 + iw, \chi)\right|^4 dw \ll qTL^{13}. \end{aligned}$$

As regards $\varSigma_2$, let $v = w - t$. Note that $2T \leq |w - t| \leq T_0$ and $|t| \leq T$ imply $|w - t| \geq |w|/2$ and $T \leq |w| \leq 2T_0$. Therefore

$$\begin{aligned} \varSigma_2 &\ll TL^3 \sum_{\chi \bmod q} \int_{T}^{2T_0} \left|L'(1/2 + iw, \chi)\right|^4 \frac{dw}{1+|w|} \\ &\ll TL^4 \max_{T \leq X \leq T_0} \frac{1}{X} \sum_{\chi \bmod q} \int_{X}^{2X} \left|L'(1/2 + iw, \chi)\right|^4 dw \ll qTL^{13}, \end{aligned}$$

by Lemma 6.3. Collecting these estimates, we get

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi^0}} \int_{-T}^{T} |f_1(1/2 + it, \chi)|^4 \, dt \ll qTL^{13}. \tag{6.6}$$

A similar argument also leads to

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \int_{-T}^{T} |f_2\left(1/2+it,\chi\right)|^4\, dt \ll qTL^9. \tag{6.7}$$

On the other hand, we have

$$\prod_{j=3}^{10} f_j\left(1/2+it,\chi\right) = \sum_{M_3\cdots M_{10}<m\leq 2^8 M_3\cdots M_{10}} \frac{b(m)\chi(m)}{m^{1/2+it}},$$

where $|b(m)| \leq d_8(m)$. Thus by Lemma 6.2,

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \int_{-T}^{T} \left|\prod_{j=3}^{10} f_j\left(1/2+it,\chi\right)\right|^2 dt$$

$$\ll \sum_{M_3\cdots M_{10}<m\leq 2^8 M_3\cdots M_{10}} (qT+m)\,\frac{d_8^2(m)}{m} \ll \left(qT+M^{3/5}\right)L^c, \tag{6.8}$$

since $M_3\cdots M_{10} \ll M/(M_1M_2) \ll M^{3/5}$. Writing

$$F\left(1/2+it,\chi\right) = f_1\left(1/2+it,\chi\right) f_2\left(1/2+it,\chi\right) \prod_{j=3}^{10} f_j\left(1/2+it,\chi\right),$$

then by Hölder's inequality and (6.6)-(6.8), we get

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \int_{T}^{2T} |F\left(1/2+it,\chi\right)|\, dt$$

$$\ll \left\{\prod_{j=1}^{2}\left(\sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \int_{T}^{2T} |f_j\left(1/2+it,\chi\right)|^4\, dt\right)^{1/4}\right\}\left\{\sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \int_{T}^{2T} \left|\prod_{j=3}^{10} f_j\left(1/2+it,\chi\right)\right|^2 dt\right\}^{1/2}$$

$$\ll (qT)^{1/2}\left(qT+M^{3/5}\right)^{1/2} L^c \ll \left(qT+(qT)^{1/2}M^{3/10}\right)L^c.$$

This proves Proposition 6.4. □

Proposition 6.5. *If there is a partition* $\{J_1, J_2\}$ *of the set* $\{1, ..., 10\}$ *such that*

$$\prod_{j\in J_1} M_j + \prod_{j\in J_2} M_j \ll M^{3/5},$$

then (6.4) *is true.*

Proof. For $\nu = 1, 2$, define

$$F_\nu(s,\chi) := \prod_{j\in J_\nu} f_j(s,\chi) = \sum_{n\leq N_\nu} \frac{b_\nu(n)\chi(n)}{n^s},$$

where $N_\nu = \prod_{j\in J_\nu}(2M_j)$ and $b_\nu(n) \ll Ld_{10}(n)$. By Lemma 6.2, we have

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}} \int_T^{2T} |F_1(1/2+it,\chi)|^2 \ll \sum_{n\le N_1} (qT+n)\frac{|b_1(n)|^2}{n} \ll (qT+N_1)L^c,$$

and similarly

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}} \int_T^{2T} |F_2(1/2+it,\chi)|^2 \ll (qT+N_2)L^c.$$

Write $F(s,\chi) = F_1(s,\chi)F_2(s,\chi)$. Then by Cauchy's inequality we get

$$\begin{aligned}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}} \int_T^{2T} |F(1/2+it,\chi)|\, dt &\ll (qT+N_1)^{1/2}(qT+N_2)^{1/2} L^c\\ &\ll \left(qT + (qT)^{1/2}M^{3/10} + M^{1/2}\right) L^c,\end{aligned}$$

since $N_1+N_2 \ll M^{3/5}$, and $N_1N_2 \ll M$. This proves Proposition 6.5. □

Proof of Lemma 6.1. In view of Proposition 6.4, we may assume that $M_iM_j \le M^{2/5}$ for all i,j with $1\le i<j\le 5$. It follows that there is at most one M_j with $1\le j\le 5$ such that $M_j > M^{1/5}$. Without loss of generality, we can suppose this exceptional M_j is M_1, so we have $M_j \le M^{1/5}$ for $j=2,...,5$, and also for $j=6,...,10$, by assumption. Let l be the integer with $2\le l<8$ such that

$$M_1\cdots M_l \le M^{2/5}, \quad \text{but} \quad M_1\cdots M_{l+1} > M^{2/5}.$$

Let $J_1 = \{1,2,...,l+1\}$ and $J_2 = \{l+2,...,10\}$. And write $N_1 = M_1\cdots M_{l+1}$ and $N_2 = M_{l+2}\cdots M_{10}$. Then we have

$$M^{2/5} \ll N_1 \ll M^{2/5}M_{l+1} \ll M^{2/5}M^{1/5} \ll M^{3/5}, \quad \text{and} \quad N_2 \ll M/N_1 \ll M^{3/5}.$$

This proves $N_1+N_2 \ll M^{3/5}$, i.e. the assumption of Proposition 6.5 is satisfied. Lemma 6.1 thus follows. □

Proof of Lemma 5.2. For $W>0$, one has

$$\begin{aligned}&\sum_{\chi \bmod q}\left|\sum_{m\sim W}\Lambda(m)\chi(m)e(\lambda m^k)\right|\\ &= \left|\sum_{\substack{m\sim W\\ (m,q)=1}}\Lambda(m)e(\lambda m^k)\right| + \sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}}\left|\sum_{m\sim W}\Lambda(m)\chi(m)e(\lambda m^k)\right|. \qquad (6.9)\end{aligned}$$

Obviously the first term is bounded by W. By integrating by parts, we have

$$\sum_{m\sim W}\Lambda(m)\chi(m)e(\lambda m^k) = \int_W^{2W} e(\lambda u^k)d\sum_{W<m\le u}\Lambda(m)\chi(m).$$

Now we apply Heath-Brown's identity [2, Lemma 1] for $k=5$ which states that for $m \le 2W$,

$$\Lambda(m) = \sum_{j=1}^{5} \binom{5}{j} (-1)^{j-1} \sum_{\substack{m_1 \cdots m_{2j} = m \\ m_{j+1}, \dots, m_{2j} \le (2W)^{1/5}}} (\log m_1) \mu(m_{j+1}) \cdots \mu(m_{2j}).$$

With this, the sum $\sum_{W<m\le u} \Lambda(m)\chi(m)$ decomposes into a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\Sigma(u;\mathbf{M}) = \sum_{\substack{\mathrm{m}_1 \sim \mathrm{M}_1 \\ W<m_1\cdots m_{10}\le u}} \cdots \sum_{\mathrm{m}_{10} \sim \mathrm{M}_{10}} a_1(m_1)\chi(m_1) \cdots a_{10}(m_{10})\chi(m_{10}),$$

where $a_i(m)$ are given by (6.2), and M_j are positive integers satisfying (6.1) with $M = W$. Here $\mathbf{M}$ denotes the vector $(M_1, M_2, ..., M_{10})$. We notice that for $j = 1, 2, ..., 10$, the function $f_j(s,\chi)$ in (6.3) is a finite sum and has no poles for $\sigma \ge 1/2$. So by applying Perron's summation formula and then shifting the contour to the left, the above $\Sigma(u;\mathbf{M})$ becomes

$$\begin{aligned} &\frac{1}{2\pi i} \int_{1+1/L-iT_1}^{1+1/L+iT_1} F(s,\chi) \frac{u^s - W^s}{s} ds + O\left(L^2\right) \\ &= \frac{1}{2\pi i} \left\{ \int_{1+1/L-iT_1}^{1/2-iT_1} + \int_{1/2-iT_1}^{1/2+iT_1} + \int_{1/2+iT_1}^{1+1/L+iT_1} \right\} + O\left(L^2\right), \end{aligned}$$

where $T_1 = 4k\pi(R+W)$. The integral on the two horizontal segments above is bounded by

$$\max_{1/2\le\sigma\le 1+1/L} |F(\sigma \pm iT_1,\chi)| \frac{u^\sigma}{T_1} \ll L,$$

since $W < u \le 2W$ and

$$|F(\sigma \pm iT_1,\chi)| \ll \prod_{j=1}^{10} |f_j(\sigma \pm iT_1,\chi)| \ll L \prod_{j=1}^{10} M_j^{1-\sigma} \ll W^{1-\sigma} L.$$

Thus we get

$$\Sigma(u;\mathbf{M}) = \frac{1}{2\pi} \int_{-T_1}^{T_1} F\left(1/2+it,\chi\right) \frac{u^{1/2+it} - W^{1/2+it}}{1/2+it} dt + O(L^2).$$

And therefore

$$\begin{aligned} &\int_W^{2W} e(\lambda u^k) d\Sigma(u;\mathbf{M}) \\ &= \frac{1}{2\pi} \int_{-T_1}^{T_1} F\left(1/2+it,\chi\right) \int_W^{2W} u^{-1/2+it} e(\lambda u^k) du dt + (1+|\lambda| W^k) L^2. \end{aligned}$$

Here by making use of Lemmas 4.3 and 4.5 in [11], the inner integral is

$$\ll \quad W^{1/2}\min\left\{\frac{1}{\min\limits_{W^k<v\le(2W)^k}|t+2k\pi\lambda v|},\frac{1}{\sqrt{1+|t|}}\right\}$$

$$\ll \quad W^{1/2}\begin{cases}\frac{1}{\sqrt{1+|t|}}, & \text{if} \quad |t|\le 4k\pi R,\\ \frac{1}{|t|}, & \text{if} \quad |t|>4k\pi R,\end{cases}$$

since $|\lambda|W^k \le R$. Therefore we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}|\Psi(\chi,\lambda,W)|\\ &\ll W^{1/2}\sum_{\mathbf{M}}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\int_{|t|\le 4k\pi R}|F(1/2+it,\chi)|\frac{dt}{\sqrt{1+|t|}}\\ &\quad+W^{1/2}\sum_{\mathbf{M}}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\int_{4k\pi R<|t|\le T_1}|F(1/2+it,\chi)|\frac{dt}{|t|}+qRL^{12}.\end{aligned}$$

By Lemma 6.1, we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\int_{|t|\le 4k\pi R}|F(1/2+it,\chi)|\frac{dt}{\sqrt{1+|t|}}\\ &\ll L\max_{1\le T\le 2k\pi R}\frac{1}{\sqrt{1+T}}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\int_{T<|t|\le 2T}|F(1/2+it,\chi)|\,dt\\ &\ll \max_{1\le T\le 2k\pi R}\frac{1}{\sqrt{1+T}}\left(qT+(qT)^{1/2}W^{3/10}+W^{1/2}\right)L^c\\ &\ll \left(qR^{1/2}+q^{1/2}W^{3/10}+W^{1/2}\right)L^c.\end{aligned}$$

Similarly we have

$$\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\int_{4k\pi R<|t|\le T_1}|F(1/2+it,\chi)|\frac{dt}{|t|}\ll\left(q+q^{1/2}W^{3/10}R^{-1/2}+W^{1/2}R^{-1}\right)L^c.$$

These estimates together with (6.9) show that

$$\sum_{\chi \bmod q}\left|\sum_{m\sim W}\Lambda(m)\chi(m)e(\lambda m^k)\right|\ll\left\{\left(R+(WR)^{1/2}\right)q+W^{4/5}q^{1/2}+W\right\}L^c.$$

This finishes the proof of Lemma 5.2. □

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