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Classical dynamical r -matrices and homogeneous Poisson structures on G/H and K/T

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Abstract

Let G be a finite dimensional simple complex group equipped with the standard Poisson Lie group structure. We show that all G -homogeneous (holomorphic) Poisson structures on G/H , where $H \subset G$ is a Cartan subgroup, come from solutions to the Classical Dynamical Yang-Baxter equations which are classified by Etingof and Varchenko. A similar result holds for the maximal compact subgroup K , and we get a family of K -homogeneous Poisson structures on K/T , where $T = K \cap H$ is a maximal torus of K . This family exhausts all K -homogeneous Poisson structures on K/T up to isomorphisms. We study some Poisson geometrical properties of members of this family such as their symplectic leaves, their modular classes, and the moment maps for the T -action.

Contents

1	Introduction	2
2	The Classical Dynamical Yang-Baxter Equation	4
3	r-matrices and homogeneous Poisson structures on G/H	5
3.1	The main theorem	5
3.2	The Poisson structures $\pi_{r_X(\lambda)}$ on G/H	12
3.3	Comparison with Karolinsky's classification	13
4	r-matrices and homogeneous Poisson structures on K/T	14
5	The Poisson structures $\pi_{X, X_1, \lambda}$ on K/T	17
5.1	Definition	17
5.2	Connections via taking limits in λ	20
5.3	The Lagrangian subalgebras of \mathfrak{g} corresponding to $\pi_{X, X_1, \lambda}$	21

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5.4	Geometrical interpretation of $\pi_{X, X_1, \lambda}$	23
5.5	$\pi_{X, X_1, \lambda}$ as the result of Poisson induction	25
5.6	The symplectic leaves of $\pi_{X, X_1, \lambda}$	29
5.7	The modular vector fields and the leaf-wise moment maps for the T -actions	34

1 Introduction

This paper is motivated by the work of Etingof and Varchenko [E-V] on *classical dynamical r -matrices* for the pair $(\mathfrak{g}, \mathfrak{h})$, where \mathfrak{g} is a complex simple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.

A classical dynamical r -matrix is, by definition, a meromorphic function $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ satisfying the so-called *Classical Dynamical Yang-Baxter Equation* (CDYBE):

$$\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

(See Section 2 for details). One such r -matrix has the form

$$r(\lambda) = \frac{\varepsilon}{2} \Omega + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma} \coth\left(\frac{\varepsilon}{2} \ll \alpha, \lambda \gg\right) E_\alpha \otimes E_{-\alpha},$$

where $\Omega \in (S^2 \mathfrak{g})^\mathfrak{h}$ corresponds to the Killing form \ll, \gg of \mathfrak{g} , Σ is the set of roots of \mathfrak{g} with respect to \mathfrak{h} , the E_α and $E_{-\alpha}$'s are root vectors, and $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function. Other r -matrices can be obtained by performing certain ‘‘gauge transformations’’ to the one above and by taking various limits of it. See Section 2.

We wanted to understand the geometrical meaning of these r -matrices. Etingof and Varchenko show in [E-V] that every classical dynamical r -matrix defines a Poisson groupoid over an open subset of \mathfrak{h}^* . In this paper, we give another geometrical interpretation of the r -matrices by connecting them with Poisson structures on the spaces G/H and K/T , where G is a complex Lie group with Lie algebra \mathfrak{g} , $H \subset G$ its connected subgroup corresponding to \mathfrak{h} , K a compact real form of G , and $T = K \cap H$. We then study some Poisson geometrical properties of these Poisson structures on K/T such as their symplectic leaves, their modular classes, and the moment maps for the T -action. We now explain this in more detail.

A special example of a classical dynamical r -matrix is one that is not ‘‘dynamical’’, i.e., independent of λ . It is given by

$$r_0 = \frac{\varepsilon}{2} \Omega + c + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma_+} E_\alpha \wedge E_{-\alpha}$$

for a choice of positive roots Σ_+ and an element $c \in \mathfrak{h} \wedge \mathfrak{h}$. It defines a (holomorphic) Poisson structure π_G on G by

$$\pi_G(g) = R_g r_0 - L_g r_0,$$

where R_g and L_g are respectively the right and left translations on G by $g \in G$, making (G, π_G) into a Poisson Lie group. This Poisson structure is the semi-classical limit of the quantum group corresponding to G [D1] [D2]. A Poisson structure on G/H is said to be (G, π_G) -homogeneous if the action map $G \times (G/H) \rightarrow G/H$ is a Poisson map [D3].

The first result of this paper, Theorem 3.3, is on the construction of a surjective map from the set of all classical dynamical r -matrices for the pair $(\mathfrak{g}, \mathfrak{h})$ together with their domains to the set of all (holomorphic) (G, π_G) -homogeneous Poisson structures on G/H . More precisely, for any classical dynamical r -matrix r and $\lambda \in \mathfrak{h}^*$ such that $r(\lambda)$ is defined, we show that the bi-vector field $\tilde{\pi}_{r(\lambda)}$ on G defined by

$$\tilde{\pi}_{r(\lambda)} = R_g r_0 - L_g r(\lambda)$$

projects to a holomorphic (G, π_G) -homogeneous Poisson structure on G/H under the projection $G \rightarrow G/H$, and that all (G, π_G) -homogeneous Poisson structures on G/H arise this way. See also [L-X] for another interpretation of classical dynamical r -matrices.

Let $K \subset G$ be a compact real form of G , and let $T = K \cap H$ be the maximal torus of K . Then K also carries a natural Poisson structure π_K such that (K, π_K) is a Poisson Lie group. Theorem 3.3 is then modified to Theorem 4.1 which states that classical dynamical r -matrices give rise to (K, π_K) -homogeneous Poisson structure on K/T and that all (K, π_K) -homogeneous Poisson structures on K/T arise this way.

We point out that a classification of all (G, π_G) or (K, π_K) -homogeneous Poisson structures, not necessarily on G/H or on K/T , has already been obtained by E. Karolinsky [Ka2] [Ka3]. We want to emphasize that what is brought out here is the connection of such Poisson spaces with the CDYBE.

Among all (K, π_K) -homogeneous Poisson structures on K/T , we single out a family denoted by $\pi_{X, X_1, \lambda}$, where X is any subset of the set $S(\Sigma_+)$ of all simple roots, $X_1 \subset X$, and $\lambda \in \mathfrak{h}$ satisfies some regularity condition (Theorem 5.1). This family exhausts all (K, π_K) -homogeneous Poisson structures on K/T up to K -equivariant isomorphisms. Moreover, these Poisson structures are related to each other by taking various limits of the parameter λ (see Section 5.2). We study several Poisson geometrical properties of this family:

The Lagrangian subalgebra of \mathfrak{g} corresponding to each $\pi_{X, X_1, \lambda}$ is described in Section 5.3.

In Section 5.4, we recall the construction in [E-L2] of a Poisson structure Π on the variety \mathcal{L} of all Lagrangian subalgebras in \mathfrak{g} and the fact that each $(K/T, \pi_{X, X_1, \lambda})$ sits inside (\mathcal{L}, Π) as a Poisson submanifold (possibly up to a covering map). The two special cases of $\pi_{X, X_1, \lambda}$ when $X = X_1 = \emptyset$ and when $X = S(\Sigma_+), X_1 = \emptyset$ are considered in more detail here.

In Section 5.5, we show that each $\pi_{X, X_1, \lambda}$ on K/T can be obtained via Poisson induction from a Poisson structure on a smaller manifold.

In Section 5.6, we describe the symplectic leaves of $\pi_{X, X_1, \lambda}$ when X_1 is the empty set. We show that in this case $\pi_{X, X_1, \lambda}$ has a finite number of symplectic leaves. For an arbitrary $\pi_{X, X_1, \lambda}$, we show that it always has at least one open symplectic leaf.

In Section 5.7, we show that with respect to a K -invariant volume form μ_0 on K/T , all the Poisson structures $\pi_{X, X_1, \lambda}$ have the same modular vector field. In the case when X_1 is the empty set, we also describe the moment map for the T -action on each symplectic leaf of $\pi_{X, \emptyset, \lambda}$.

Some applications of results in this paper are given in [E-L1], where a Poisson geometrical interpretation of the Kostant harmonic forms on K/T [Ko] is given using the Bruhat Poisson structure $\pi_\infty := \pi_{X, X_1, \lambda}$ for $X = X_1 = \emptyset$. Set $\pi_\lambda = \pi_{X, X_1, \lambda}$ when $X = S(\Sigma_+)$ and $X_1 = \emptyset$. The fact that $\pi_\lambda \rightarrow \pi_\infty$ as $\lambda \rightarrow \infty$ is used in [E-L1] to show that the Kostant harmonic forms are limits of the usual Hodge harmonic forms.

Results in this paper also motivate our work in [E-L2], where, among other things, we show that there is a Poisson manifold (\mathcal{L}_0, Π) such that every $(K/T, \pi_{X, X_1, \lambda})$ is a Poisson submanifold (possibly up to a covering map) of (\mathcal{L}_0, Π) . In fact, \mathcal{L}_0 is an irreducible component of the variety \mathcal{L} of all Lagrangian subalgebras of \mathfrak{g} , and the Poisson structure Π is defined on all of \mathcal{L} . We show in [E-L2] that all the K -orbits in \mathcal{L} with respect to the Adjoint action are (K, π_K) -homogeneous Poisson spaces, and that every (K, π_K) -homogeneous Poisson space maps to (\mathcal{L}, Π) by a Poisson map. Thus, (\mathcal{L}, Π) is a setting for studying all (K, π_K) -homogeneous Poisson spaces.

We point out that many more properties of the Poisson structures $\pi_{X, X_1, \lambda}$ can be studied, among these their Poisson cohomology, their Poisson harmonic forms [E-L1], and their symplectic groupoids. We hope to do this in the future.

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2 The Classical Dynamical Yang-Baxter Equation

Definition 2.1 [F] [E-V] A meromorphic function $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is called a *classical (quasi-triangular) dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$* if it satisfies the following three conditions:

1. *The zero weight condition:* $ad_x r(\lambda) = 0$ for all $x \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$ such that $r(\lambda)$ is defined;
2. *The generalized unitarity condition:* $r^{12} + r^{21} = \varepsilon \Omega$ for some complex number ε and for all $\lambda \in \mathfrak{h}^*$ such that $r(\lambda)$ is defined, where $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is the element corresponding to the

Killing form on \mathfrak{g} ;

3. *The Classical Dynamical Yang-Baxter Equation (CDYBE):*

$$\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

where, for $r = \sum_i u_i \otimes v_i$, we have $r^{12} = \sum_i u_i \otimes v_i \otimes 1$, $r^{13} = \sum_i u_i \otimes 1 \otimes v_i$, $r^{23} = \sum_i 1 \otimes u_i \otimes v_i$,

$$\begin{aligned} \text{CYB}(r) &:= [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \\ &= \sum_{i,j} [u_i, u_j] \otimes v_i \otimes v_j + u_i \otimes [v_i, u_j] \otimes v_j + u_i \otimes u_j \otimes [v_i, v_j], \end{aligned}$$

and $\text{Alt}(dr)(\lambda) \in \wedge^3 \mathfrak{g}$ is the skew-symmetrization of $dr(\lambda) \in \mathfrak{h} \otimes \mathfrak{g} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. The complex number ε is called the *coupling constant* for r .

We now recall the classification of classical dynamical r -matrices for the pair $(\mathfrak{g}, \mathfrak{h})$ as given in [E-V]. Let Σ be the set of all roots for \mathfrak{g} with respect to \mathfrak{h} . For each $\alpha \in \Sigma$, choose root vectors E_α and $E_{-\alpha}$ such that $\ll E_\alpha, E_{-\alpha} \gg = 1$, where \ll, \gg is the Killing form on \mathfrak{g} .

Let ε be a non-zero complex number, let $\mu \in \mathfrak{h}^*$, and let $C = \sum_{i,j} C_{ij} dx_i \wedge dx_j$ be a closed meromorphic 2-form on \mathfrak{h}^* . Let Σ_+ be a choice of positive roots, and let X be a subset of the set $S(\Sigma_+)$ of simple roots in Σ_+ . For each $\alpha \in \Sigma$, define a (scalar-valued) meromorphic function ϕ_α on \mathfrak{h}^* according to the rule: If α is a linear combination of simple roots in X , then

$$\phi_\alpha(\lambda) = \frac{\varepsilon}{2} \coth\left(\frac{\varepsilon}{2} \ll \alpha, \lambda - \mu \gg\right),$$

where $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function; Otherwise, set $\phi_\alpha(\lambda) = \frac{\varepsilon}{2}$ if α is positive and $\phi_\alpha(\lambda) = -\frac{\varepsilon}{2}$ if α is negative.

Theorem 2.2 (Etingof-Varchenko [E-V]) 1. *With the above choices of $\mu, C, \Sigma_+, X \subset S(\Sigma_+)$ and ϕ_α , the meromorphic function $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ defined by*

$$r(\lambda) = \frac{\varepsilon}{2} \Omega + \sum_{i,j} C_{ij}(\lambda) x_i \otimes x_j + \sum_{\alpha \in \Sigma} \phi_\alpha(\lambda) E_\alpha \otimes E_{-\alpha} \quad (1)$$

is a classical dynamical r -matrix with non-zero coupling constant ε ;

2. *Every classical dynamical r -matrix with non-zero coupling constant has this form.*

3 r -matrices and homogeneous Poisson structures on G/H

3.1 The main theorem

Let $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ be any classical dynamical r -matrix as in Definition 2.1. Let

$$A_r(\lambda) = r(\lambda) - \frac{\varepsilon}{2} \Omega$$

be the skew-symmetric part of $r(\lambda)$. Using the fact that Ω is symmetric and ad -invariant, one easily shows that the terms $[\Omega^{ij}, A_r^{kl}]$ in the CDYBE for r all cancel. Moreover, it is well-known that

$$[\Omega^{12}, \Omega^{13}] + [\Omega^{12}, \Omega^{23}] + [\Omega^{13}, \Omega^{23}] = [\Omega^{12}, \Omega^{13}] = [\Omega^{13}, \Omega^{23}] = -[\Omega^{12}, \Omega^{23}] \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}.$$

Therefore, A_r satisfies the following modified CDYBE (see also [E-V]):

$$\text{Alt}(dA_r) + [A_r^{12}, A_r^{13}] + [A_r^{12}, A_r^{23}] + [A_r^{13}, A_r^{23}] = \frac{\varepsilon^2}{4} [\Omega^{12}, \Omega^{23}] \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}. \quad (2)$$

Recall that there is the Schouten bracket $[\ , \]$ on $\wedge \mathfrak{g}$. For $x_1, x_2, \dots, x_k \in \mathfrak{g}$, we use the convention

$$x_1 \wedge x_2 \wedge \dots \wedge x_k = \sum_{\sigma \in S_k} \text{sign}(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \dots \otimes x_{\sigma(k)} \in \mathfrak{g}^{\otimes k}.$$

Then for $X \in \wedge^2 \mathfrak{g}$, the element $\text{CYB}(X)$ and the Schouten bracket $[X, X]$ are related by [D2]

$$\text{CYB}(X) = [X^{12}, X^{13}] + [X^{12}, X^{23}] + [X^{13}, X^{23}] = \frac{1}{2} [X, X].$$

Thus, we can rewrite Equation (2) as

$$[A_r(\lambda), A_r(\lambda)] = \frac{\varepsilon^2}{2} [\Omega^{12}, \Omega^{23}] - 2\text{Alt}(dA_r)(\lambda). \quad (3)$$

It is this form of the CDYBE that we will use to define Poisson structures on G/H .

Recall [D2] that a *classical quasi-triangular r -matrix with coupling constant ε* is an element $r_0 \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\begin{aligned} r_0 + r_0^{21} &= \varepsilon \Omega \\ \text{CYB}(r_0) &= 0. \end{aligned}$$

Remark 3.1 If r_0 has the zero-weight property, i.e., if $r_0 \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{h}}$, then by Theorem 2.2, it must be of the form

$$r_0 = \frac{\varepsilon}{2} \Omega + \sum_{i,j} c_{ij} x_i \wedge x_j + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma_+} E_{\alpha} \wedge E_{-\alpha} \quad (4)$$

for some choice Σ_+ of positive roots and $\sum_{i,j} c_{ij} \in \mathfrak{h} \wedge \mathfrak{h}$. But not every quasi-triangular r_0 has the zero-weight property. For example, for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, we can take $r_0 = \frac{\varepsilon}{2}(\Omega + h \wedge (e + f))$ where h, e and f are the three generators with Lie brackets: $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. See [B-D] for more examples.

Let r_0 be a classical quasi-triangular r -matrix with coupling constant ε (not necessarily of zero weight for \mathfrak{h}). Let $\Lambda = r_0 - \frac{\varepsilon}{2}\Omega \in \mathfrak{g} \wedge \mathfrak{g}$ be the skew-symmetric part of r_0 . Then, as a special case of (3), Λ satisfies the modified Classical Yang-Baxter Equation (CYBE)

$$[\Lambda, \Lambda] = \frac{\varepsilon^2}{2}[\Omega^{12}, \Omega^{23}]. \quad (5)$$

It is well known that the bi-vector field π_G on the group G defined by

$$\pi_G(g) = R_g\Lambda - L_g\Lambda, \quad (6)$$

where for R_g and L_g denote respectively the right and left translations from the identity element to g , defines a Poisson structure on G , and that (G, π_G) is a Poisson Lie group [D2] [STS1].

Remark 3.2 The meaning of the terms $R_g\Lambda$ and $L_g\Lambda$ needs further explanation. Denote by J the complex structure on \mathfrak{g} induced by that on G . Then we can identify (\mathfrak{g}, J) with the holomorphic tangent space $T_e^{1,0}G$ of G at e via $\mathfrak{g} \ni x \mapsto \frac{1}{2}(x - iJ(x))$. For $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$, we regard Λ as an element in $\wedge^2 T_e^{1,0}G$. Then, $L_g\Lambda$ (resp. $R_g\Lambda$), for $g \in G$, is understood to be the image in $\wedge^2 T_g^{1,0}G$ of Λ by the left (resp. right) translation by g . Thus the bi-vector field π_G on G in (6) is holomorphic. All Poisson structures in this section are assumed to be holomorphic.

Recall that an action of the Poisson Lie group (G, π_G) on a Poisson manifold P is said to be Poisson if the action map $G \times P \rightarrow P : (g, p) \mapsto gp$ is a Poisson map, where $G \times P$ is equipped with the product Poisson structure. When the action of G on P is transitive, the Poisson structure on P is said to be (G, π_G) -homogeneous [D3]. The following theorem makes a connection between classical dynamical r -matrices and (G, π_G) -homogeneous Poisson structures on G/H .

Theorem 3.3 *Let $r_0 = \frac{\varepsilon}{2}\Omega + \Lambda$ be any classical quasi-triangular r -matrix (not necessarily of zero-weight) with skew-symmetric part Λ . Let $r(\lambda) = \frac{\varepsilon}{2}\Omega + A_r(\lambda)$ be any classical dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ as in Definition 2.1. For each value λ such that $r(\lambda)$ is defined, define a bi-vector field $\tilde{\pi}_{r(\lambda)}$ on G by*

$$\tilde{\pi}_{r(\lambda)}(g) = R_g\Lambda - L_gA_r(\lambda), \quad g \in G.$$

Let $\pi_{r(\lambda)} = p_\tilde{\pi}_{r(\lambda)}$ be the projection of $\tilde{\pi}_{r(\lambda)}$ to G/H by the map $p : G \rightarrow G/H : g \mapsto gH$. Then*

- 1) $\pi_{r(\lambda)}$ is well-defined and it defines a Poisson structure on G/H ;

2) Equip G with the Poisson structure π_G as defined by (6). Then $\pi_{r(\lambda)}$ is a (G, π_G) -homogeneous Poisson structure on G/H .

3) When r_0 has the zero-weight property, i.e., $r_0 \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{h}}$, every (G, π_G) -homogeneous Poisson structure on G/H arises this way.

The rest of this section is devoted to the proof of this theorem. We first prove the first two parts.

Proof of 1) and 2) in Theorem 3.3. It follows from $A_r(\lambda) \in (\wedge^2 \mathfrak{g})^{\mathfrak{h}}$ that $\pi_{r(\lambda)}$ is well-defined. To show that $\pi_{r(\lambda)}$ defines a Poisson structure on G/H , we calculate the Schouten bracket $[\pi_{r(\lambda)}, \pi_{r(\lambda)}]$ of $\pi_{r(\lambda)}$ with itself. Set $\Lambda^R(g) = R_g \Lambda$ and $A_r(\lambda)^L(g) = L_g A_r(\lambda)$. Then $\tilde{\pi}_{r(\lambda)} = \Lambda^R - A_r(\lambda)^L$. Hence

$$\begin{aligned} [\tilde{\pi}_{r(\lambda)}, \tilde{\pi}_{r(\lambda)}] &= [\Lambda^R, \Lambda^R] - 2[\Lambda^R, A_r(\lambda)^L] + [A_r(\lambda)^L, A_r(\lambda)^L] \\ &= -[\Lambda, \Lambda]^R + [A_r(\lambda), A_r(\lambda)]^L \\ &= -2\text{Alt}(dA_r(\lambda))^L \in (\mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g})^L, \end{aligned}$$

where in the last step, we used Equations (3) and (5). This shows that $\tilde{\pi}_{r(\lambda)}$ is in general not a Poisson bi-vector field on G . However, for $\pi_{r(\lambda)} = p_* \tilde{\pi}_{r(\lambda)}$, we have

$$[\pi_{r(\lambda)}, \pi_{r(\lambda)}] = p_* [\tilde{\pi}_{r(\lambda)}, \tilde{\pi}_{r(\lambda)}] = -2p_* \text{Alt}(dA_r(\lambda))^L = 0.$$

Therefore, $\pi_{r(\lambda)}$ is a Poisson structure on G/H . Now for any g_1 and $g_2 \in G$, we have

$$\begin{aligned} \tilde{\pi}_{r(\lambda)}(g_1 g_2) &= R_{g_1 g_2} \Lambda - L_{g_1 g_2} A_r(\lambda) \\ &= L_{g_1} (R_{g_2} \Lambda - L_{g_2} A_r(\lambda)) + R_{g_2} (R_{g_1} \Lambda - L_{g_1} \Lambda) \\ &= L_{g_1} \tilde{\pi}_{r(\lambda)}(g_2) + R_{g_2} \pi_G(g_1). \end{aligned}$$

Projecting $\tilde{\pi}_{r(\lambda)}$ to $\pi_{r(\lambda)}$, this says that the action map of G on G/H by left translations is a Poisson map. Thus $\pi_{r(\lambda)}$ is a (G, π_G) -homogeneous Poisson structure on G/H . This finishes the proof of 1) and 2) in Theorem 3.3.

We now prove 3) of Theorem 3.3. Assume that $r_0 \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{h}}$. Then by Theorem 2.2, it must be of the form (4) for some choice Σ_+ of positive roots and some $\sum_{i,j} u_{ij} x_i \wedge x_j \in \mathfrak{h} \wedge \mathfrak{h}$.

Let $e = eH$ be the base point of G/H . Recall [D3] that a (G, π_G) -homogeneous Poisson structure π on G/H is determined by its value $\pi(e)$ at e in such a way that

$$\pi(gH) = L_g \pi(e) + p_* \pi_G(g). \quad (7)$$

Moreover, since $\pi_G(g) = 0$ for $g \in H$ (this is why we need the zero weight condition on r_0), we see that $\pi(e)$ is H -invariant, i.e.,

$$\pi(e) \in \wedge^2 T_e(G/H)^H \cong (\wedge^2(\mathfrak{g}/\mathfrak{h}))^H.$$

Let \mathfrak{n}_+ and \mathfrak{n}_- be the nilpotent Lie subalgebras of \mathfrak{g} spanned by the root vectors for the roots in Σ_+ and $-\Sigma_+$ respectively. Identify $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{n}_- + \mathfrak{n}_+$.

Lemma 3.4 *Write*

$$\pi(e) = \sum_{\alpha \in \Sigma_+} \left(\frac{\varepsilon}{2} - \phi_\alpha\right) E_\alpha \wedge E_{-\alpha} \in (\wedge^2(\mathfrak{g}/\mathfrak{h}))^H \quad (8)$$

and set $\phi_{-\alpha} = -\phi_\alpha$. Then the bi-vector field π on G/H defined by (7) is Poisson if and only if the function $\phi : \Sigma \rightarrow \mathbb{C}$ satisfies

$$\phi_\alpha \phi_\beta + \phi_\beta \phi_\gamma + \phi_\gamma \phi_\alpha = -\frac{\varepsilon^2}{4}, \text{ whenever } \alpha, \beta, \gamma \in \Sigma \text{ and } \alpha + \beta + \gamma = 0. \quad (9)$$

Proof of Lemma 3.4. For any given $\pi(e)$ in the form of (8), set

$$A = \sum_{\alpha \in \Sigma_+} \phi_\alpha E_\alpha \wedge E_{-\alpha} \in \wedge^2 \mathfrak{g}$$

and introduce the following bi-vector field $\hat{\pi}$ on G :

$$\hat{\pi}(g) = R_g \Lambda - L_g A.$$

Then $\pi = p_* \hat{\pi}$, and hence $[\pi, \pi] = p_* [\hat{\pi}, \hat{\pi}]$. But as in the proof of 1) of Theorem 3.3, we have

$$[\hat{\pi}, \hat{\pi}] = [\Lambda^R, \Lambda^R] - 2[\Lambda^R, A^L] + [A^L, A^L] = -[\Lambda, \Lambda]^R + [A, A]^L.$$

Since Λ satisfies the modified CYBE (5), by writing

$$B = [A, A] - \frac{\varepsilon^2}{2} [\Omega^{12}, \Omega^{23}] \in \wedge^3 \mathfrak{g},$$

we see that $[\hat{\pi}, \hat{\pi}] = B^L$, the left invariant 3-vector field on G with value B at e . Thus $[\pi, \pi] = 0$ if and only if $B \in \mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g}$, or, if and only if

$$[A, A] = \frac{\varepsilon^2}{2} [\Omega^{12}, \Omega^{23}] \text{ mod } \mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g}.$$

A direct calculation shows that

$$\begin{aligned} [A, A] &= \sum_{\alpha \in \Sigma} \phi_\alpha^2 h_\alpha \wedge E_\alpha \wedge E_{-\alpha} \\ &\quad - 2 \sum_{[(\alpha, \beta, \gamma)] \in \tilde{\Sigma}^3} (\phi_\alpha \phi_\beta + \phi_\beta \phi_\gamma + \phi_\gamma \phi_\alpha) N_{\alpha, \beta} E_\alpha \wedge E_\beta \wedge E_\gamma \end{aligned}$$

and

$$[\Omega^{12}, \Omega^{23}] = \frac{1}{2} \sum_{\alpha \in \Sigma} h_\alpha \wedge E_\alpha \wedge E_{-\alpha} + \sum_{[(\alpha, \beta, \gamma)] \in \tilde{\Sigma}^3} N_{\alpha, \beta} E_\alpha \wedge E_\beta \wedge E_\gamma,$$

where $h_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{h}$, $[E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha+\beta}$ when $\alpha, \beta \in \Sigma$ and $\alpha + \beta \in \Sigma$, and the summation over $[(\alpha, \beta, \gamma)] \in \tilde{\Sigma}^3$ means that the summation index runs over all triples $(\alpha, \beta, \gamma) \in \Sigma^3$ such that $\alpha + \beta + \gamma = 0$ but two such triples are considered the same if they only differ by a reordering of the three roots. It then follows immediately that π is a Poisson structure on G/H if and only if Condition (9) is satisfied. This finishes the proof of Lemma 3.4.

It now remains to classify all odd functions ϕ on Σ such that Condition (9) is satisfied. Note that the Weyl group W for $(\mathfrak{g}, \mathfrak{h})$ acts on the set of such functions by $(w \cdot \phi)_\alpha := \phi_{w\alpha}$. We say that two such functions ϕ and ψ are W -related if $\psi = w \cdot \phi$ for some $w \in W$.

Notation 3.5 Let $S(\Sigma_+)$ be the set of simple roots in Σ_+ . For a subset X of $S(\Sigma_+)$, we will use $[X]$ to denote the set of roots in Σ that are in the linear span of X . Also set

$$\mathfrak{h}_X = \text{span}_{\mathbb{C}}\{h_\gamma = [E_\gamma, E_{-\gamma}] : \gamma \in X\}.$$

Lemma 3.6 *For any $X \subset S(\Sigma_+)$ and $h \in \mathfrak{h}_X$ such that $\alpha(h) \notin \pi i\mathbb{Z}$ for any $\alpha \in [X]$, where $\pi = 3.14159\dots$ (we hope that there is no confusion between this notation of $\pi = 3.14159\dots$ and π as a Poisson structure), and \mathbb{Z} is the set of integers, define $\phi : \Sigma \rightarrow \mathbb{C}$ by*

$$\phi_\alpha = \begin{cases} \frac{\varepsilon}{2} \coth \alpha(h), & \alpha \in [X] \\ \frac{\varepsilon}{2}, & \alpha \in \Sigma_+ \setminus [X] \\ -\frac{\varepsilon}{2}, & \alpha \in -(\Sigma_+ \setminus [X]). \end{cases}$$

Then

(1) ϕ satisfies Condition (9);

(2) Any odd function $\psi : \Sigma \rightarrow \mathbb{C}$ satisfying Condition (9) is W -related to one obtained this way.

Proof. (1) can be checked directly. We only show (2). Suppose that $\phi : \Sigma \rightarrow \mathbb{C}$ satisfies Condition (9). Set $Y = \{\alpha \in \Sigma : \phi_\alpha = \frac{\varepsilon}{2}\}$. Then because of (9), Y has two properties:

(A). If $\alpha, \beta \in Y$ and $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in Y$;

(B). If $\alpha \in Y$, then $-\alpha \notin Y$.

It follows [E-V] that there exists a choice of positive roots Σ'_+ such that $Y \subset \Sigma'_+$. Since there exists $w \in W$ such that $w\Sigma'_+ = \Sigma_+$, by considering $w \cdot \phi$ instead of ϕ , we can assume that

$\Sigma'_+ = \Sigma_+$. Set $X = S(\Sigma_+) \cap (\Sigma_+ \setminus Y)$. Since Condition (9) implies that Y has the additional property:

(C) If $\alpha \in Y, \beta \in \Sigma \setminus (-Y)$ are such that $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in Y$,

we claim that $\Sigma_+ = ([X] \cap \Sigma_+) \cup Y$ is a disjoint union. Indeed, suppose that $\alpha \in [X] \cap \Sigma_+$. We first use induction on the height $\text{ht}(\alpha)$ of α with respect to $S(\Sigma_+)$ to show that $\alpha \notin Y$. If $\text{ht}(\alpha) = 1$, then α is simple, so $\alpha \notin Y$ by definition. Suppose that $\text{ht}(\alpha) = k$. We can [Se] write α as $\alpha = \alpha_1 + \cdots + \alpha_k$ such that each α_j is in X and that each $\alpha_1 + \cdots + \alpha_j$ is a root, for $j = 1, \dots, k$. Set $\alpha' = \alpha_1 + \cdots + \alpha_{k-1}$. By induction assumption, $\alpha' \notin Y$. If $\alpha \in Y$, then we know by (C) that $\alpha_k = \alpha - \alpha' \in Y$ which is a contradiction. Thus $\alpha \notin Y$. This shows that $([X] \cap \Sigma_+) \cap Y = \emptyset$. Next, suppose that $\alpha \in \Sigma_+ \setminus Y$. We use induction on $\text{ht}(\alpha)$ again to show that $\alpha \in [X]$. If $\text{ht}(\alpha) = 1$, then $\alpha \in X \subset [X]$ by the definition of X . Suppose that $\text{ht}(\alpha) = k$. Write α as $\alpha = \alpha' + \alpha_k$, where $\alpha' \in \Sigma_+$ and α_k is a simple root. If $\alpha_k \in Y$. Then by (C), we have $-\alpha' = \alpha_k - \alpha \in Y$ which is absurd. Thus $\alpha_k \notin Y$, so $\alpha_k \in X$. If $\alpha' \in Y$, then again by (C), we have $-\alpha_k = \alpha' - \alpha \in Y$ which is also absurd, so $\alpha' \notin Y$. By induction assumption, $\alpha' \in [X]$. Thus $\alpha \in [X]$. Hence we have shown that $\Sigma_+ = ([X] \cap \Sigma_+) \cup Y$ is a disjoint union.

For $\gamma \in X$, since $\phi_\gamma \neq \pm \frac{\varepsilon}{2}$, there exists $\lambda_\gamma \in \mathbb{C}, \lambda_\gamma \notin \pi i\mathbb{Z}$, such that $\phi_\gamma = \frac{\varepsilon}{2} \coth \lambda_\gamma$. Choose $h \in \mathfrak{h}_X$ such that $\gamma(h) = \lambda_\gamma$ for every $\gamma \in X$. We now show that $\alpha(h) \notin \pi i\mathbb{Z}$ and that $\phi_\alpha = \frac{\varepsilon}{2} \coth \alpha(h)$ for all $\alpha \in [X] \cap \Sigma_+$ by using induction on the height $\text{ht}(\alpha)$. This is true when $\text{ht}(\alpha) = 1$. Suppose that $\text{ht}(\alpha) = k$. As before, write $\alpha = \alpha' + \alpha_k$, where $\alpha' \in [X] \cap \Sigma_+, \text{ht}(\alpha') = k - 1$, and $\alpha_k \in X$. Then by induction assumption, $\alpha'(h) \notin \pi i\mathbb{Z}$ and $\phi_{\alpha'} = \frac{\varepsilon}{2} \coth \alpha'(h)$. By Condition (9),

$$-\phi_\alpha(\phi_{\alpha'} + \phi_{\alpha_k}) = -\frac{\varepsilon^2}{4} - \phi_{\alpha'}\phi_{\alpha_k}.$$

If $\phi_{\alpha'} + \phi_{\alpha_k} = 0$, we would have $\phi_{\alpha'}\phi_{\alpha_k} = -\frac{\varepsilon^2}{4}$ and thus $\phi_{\alpha'} = \pm \frac{\varepsilon}{2}$ and $\phi_{\alpha_k} = \mp \frac{\varepsilon}{2}$. This is not possible since $([X] \cap \Sigma_+) \cap Y = \emptyset$. Thus $\phi_{\alpha'} + \phi_{\alpha_k} \neq 0$, so $\alpha(h) = \alpha'(h) + \alpha_k(h) \notin \pi i\mathbb{Z}$, and

$$\phi_\alpha = \frac{\frac{\varepsilon^2}{4} + \phi_{\alpha'}\phi_{\alpha_k}}{\phi_{\alpha'} + \phi_{\alpha_k}} = \frac{\varepsilon}{2} \coth \alpha(h).$$

Q.E.D.

We now continue with the proof of (3) of Theorem 3.3. Let π be a (G, π_G) -homogeneous Poisson structure on G/H . Then by Lemmas 3.4 and 3.6, there exist a choice Σ'_+ of positive roots, a subset X' of the set of simple roots in Σ'_+ , and an element $\lambda_0 \in \mathfrak{h}^*$ such that $\pi = \pi_{r_{X'}(\lambda_0)}$, where

$$r_{X'}(\lambda) = \frac{\varepsilon}{2} \Omega + \frac{\varepsilon}{2} \sum_{\alpha \in [X'] \cap \Sigma_+} \coth \frac{\varepsilon}{2} \ll \alpha, \lambda \gg E_\alpha \wedge E_{-\alpha} + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma'_+ \setminus [X']} E_\alpha \wedge E_{-\alpha} \quad (10)$$

is a classical dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$. This proves part (3) of Theorem 3.3.

Q.E.D.

3.2 The Poisson structures $\pi_{r_X(\lambda)}$ on G/H

In this section, we consider in more detail the case when the Poisson structure on G is defined by a classical quasi-triangular r -matrices r_0 with the zero weight property. In other words, we fix a choice Σ_+ of positive roots, and consider r_0 of the form

$$r_0 = \frac{\varepsilon}{2}\Omega + \sum_{i,j} c_{ij}x_i \wedge x_j + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma_+} E_\alpha \wedge E_{-\alpha}, \quad (11)$$

where $\sum_{i,j} c_{ij}x_i \wedge x_j \in \mathfrak{h} \wedge \mathfrak{h}$. When $\sum_{i,j} c_{ij}x_i \wedge x_j = 0$, the corresponding r_0 is often called the standard r -matrix. The corresponding Poisson structure π_G on G is the semi-classical limit of the quantum group corresponding to G [D2].

For $X \subset S(\Sigma_+)$, set

$$r_X(\lambda) = \frac{\varepsilon}{2}\Omega + \frac{\varepsilon}{2} \sum_{\alpha \in [X] \cap \Sigma_+} \coth \frac{\varepsilon}{2} \langle\langle \alpha, \lambda \rangle\rangle E_\alpha \wedge E_{-\alpha} + \frac{\varepsilon}{2} \sum_{\alpha \in \Sigma_+ \setminus [X]} E_\alpha \wedge E_{-\alpha}. \quad (12)$$

Clearly, the domain $D(r_X)$ of r_X consists of those $\lambda \in \mathfrak{h}^*$ such that $\langle\langle \lambda, \alpha \rangle\rangle \notin \frac{2\pi i\mathbb{Z}}{\varepsilon}$ for all $\alpha \in [X]$. For each such λ , we have the (G, π_G) -homogeneous Poisson structure $\pi_{r_X(\lambda)}$ on G/H : let $p_*\pi_G$ be the projection to G/H of π_G by $p : G \rightarrow G/H : g \mapsto gH$. Then

$$\pi_{r_X(\lambda)} = p_*\pi_G + \left(\sum_{\alpha \in [X] \cap \Sigma_+} \frac{\varepsilon}{1 - e^{\varepsilon \langle\langle \alpha, \lambda \rangle\rangle}} E_\alpha \wedge E_{-\alpha} \right)^L,$$

where where the second term on the right hand side is the G -invariant bi-vector field on G/H whose value at $e = eH$ is the expression given in the parenthesis.

Theorem 3.7 *With the Poisson structure π_G on G defined by r_0 in (11), every holomorphic (G, π_G) -homogeneous Poisson structure on G/H is isomorphic, via a G -equivariant diffeomorphism, to a $\pi_{r_X(\lambda)}$ for some subset $X \subset S(\Sigma_+)$ and $\lambda \in D(r_X)$, where r_X is given in (12).*

Proof. Let π be a (G, π_G) -homogeneous Poisson structure on G/H . By Theorem 3.3, we know that there exists a choice Σ'_+ of positive roots and a subset X' of the set of simple roots in Σ'_+ such that $\pi = \pi_{r_{X'}(\lambda_0)}$ for some $\lambda_0 \in \mathfrak{h}^*$, where $r_{X'}$ is the classical dynamical r -matrix given by (10). Let $\Lambda = r_0 - \frac{\varepsilon}{2}\Omega$ and let $A_{X'}(\lambda_0)$ be the skew-symmetric part of $r_{X'}(\lambda_0)$. Then recall from Section 3 that $\pi = p_*\hat{\pi}'$, where $\hat{\pi}'$ is the bi-vector field on G given by

$$\hat{\pi}'(g) = R_g\Lambda - L_gA_{X'}(\lambda_0), \quad g \in G.$$

Pick $w \in W$ such that $w\Sigma'_+ = \Sigma_+$. Set $X = wX'$. Let \dot{w} be a representative of w in G . We will use $R_{\dot{w}^{-1}}$ to denote the right translation on G by \dot{w}^{-1} as well as the induced diffeomorphism on G/H . Then for any $g \in G$,

$$R_{\dot{w}^{-1}}\hat{\pi}'(g) = R_{\dot{w}^{-1}g}\Lambda - L_gL_{\dot{w}^{-1}}\text{Ad}_{\dot{w}}A_{X'}(\lambda_0) = R_{g\dot{w}^{-1}}\Lambda - L_{g\dot{w}^{-1}}A_X(w\lambda_0),$$

where A_X is the skew-symmetric part of the r -matrix r_X given by (12). It follows from the definition of $\pi_{r_X(w\lambda_0)}$ that $\pi = R_{\dot{w}}\pi_{r_X(w\lambda_0)}$. The map $R_{\dot{w}} : G/H \rightarrow G/H$ is G -equivariant.

Q.E.D.

3.3 Comparison with Karolinsky's classification

When $\sum_{ij} c_{ij}x_i \wedge x_j = 0$ in the definition of r_0 , all (G, π_G) -homogeneous Poisson structures on G/H have been classified by Karolinsky [Ka3] by using Drinfeld's theorem on Poisson homogeneous spaces. We now look at the Poisson structures $\pi_{r_X(\lambda)}$ on G/H in terms of Karolinsky's classification.

Recall that the double Lie algebra associated to the Poisson Lie group (G, π_G) can be identified with the direct sum Lie algebra $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$ equipped with the ad-invariant non-degenerate scalar product given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \frac{1}{\varepsilon}(\langle\langle x_2, y_2 \rangle\rangle - \langle\langle x_1, y_1 \rangle\rangle).$$

The Lie algebra \mathfrak{g} is identified with the diagonal of \mathfrak{d} , and the Lie algebra \mathfrak{g}^* is identified with the subspace

$$\mathfrak{g}^* \cong \{(x_-, x_+) : x_{\pm} \in \mathfrak{b}_{\pm}, (x_-)_{\mathfrak{h}} + (x_+)_{\mathfrak{h}} = 0\}.$$

Here, $\mathfrak{b}_{\pm} = \mathfrak{h} + \mathfrak{n}_{\pm}$ and $(x_{\pm})_{\mathfrak{h}} \in \mathfrak{h}$ is the \mathfrak{h} -component of x_{\pm} . A theorem of Drinfeld [D3] says that (G, π_G) -homogeneous Poisson structures on G/H correspond to Lagrangian (with respect to the scalar product \langle, \rangle) subalgebras \mathfrak{l} of the double $\mathfrak{d} \cong \mathfrak{g} + \mathfrak{g}$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{h}$.

Theorem 3.8 (Karolinsky) [Ka3] *Lagrangian subalgebras \mathfrak{l} of $\mathfrak{g} + \mathfrak{g}$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{h}$ are in 1-1 correspondence with triples $(\mathfrak{p}, \mathfrak{p}', \eta)$, where \mathfrak{p} and \mathfrak{p}' are parabolic subalgebras of \mathfrak{g} such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ is the Levi subalgebra, $\mathfrak{h} \subset \mathfrak{q}$, and η is an interior orthogonal automorphism of \mathfrak{q} with $\mathfrak{q}^{\eta} = \mathfrak{h}$. If $(\mathfrak{p}, \mathfrak{p}', \eta)$ is such a triple, the corresponding subalgebra \mathfrak{l} of $\mathfrak{g} + \mathfrak{g}$ is $\mathfrak{l} = \{(x', x) \in \mathfrak{p}' \times \mathfrak{p} : \eta(x'_q) = x_q\}$, where $x_q \in \mathfrak{q}$ (resp. $x'_q \in \mathfrak{q}'$) is the projection of x (resp. x') to \mathfrak{q} with respect to the Levi decomposition of \mathfrak{p} (resp. \mathfrak{p}').*

For a (G, π_G) -homogeneous Poisson structure π on G/H , the Lagrangian subalgebra $\mathfrak{l}_{\pi(e)}$ of $\mathfrak{g} + \mathfrak{g}$ is by definition [D3]

$$\mathfrak{l}_{\pi(e)} = \{x + \xi : x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, \xi|_{\mathfrak{h}} = 0, \text{ and } \xi \lrcorner \pi(e) = x + \mathfrak{h}\}.$$

For $\pi(e)$ of the form $\pi(e) = \sum_{\alpha \in \Sigma_+} (\frac{\varepsilon}{2} - \phi_\alpha) E_\alpha \wedge E_{-\alpha}$, it is an easy calculation to see that

$$\mathfrak{l}_{\pi(e)} = \mathfrak{h} + \text{span}_{\mathbb{C}}\{\xi_\alpha : \alpha \in \Sigma\},$$

where for $\alpha \in \Sigma$,

$$\xi_\alpha = \left((\phi_\alpha - \frac{\varepsilon}{2}) E_\alpha, (\phi_\alpha + \frac{\varepsilon}{2}) E_\alpha \right) \in \mathfrak{g} + \mathfrak{g}.$$

Thus, for the Poisson structure $\pi_{r_X(\lambda)}$ on G/H , we have

$$\xi_\alpha = \begin{cases} (-\varepsilon E_\alpha, 0) & \text{if } \alpha \in -Y \\ \frac{\varepsilon}{e^{\varepsilon \ll \alpha, \lambda} - 1} (E_\alpha, e^{\varepsilon \ll \alpha, \lambda} E_\alpha) & \text{if } \alpha \in [X] \\ (0, \varepsilon E_\alpha) & \text{if } \alpha \in Y. \end{cases},$$

where $Y = \Sigma_+ \setminus [X]$. Let

$$\mathfrak{p}_X = \mathfrak{h} + \text{span}_{\mathbb{C}}\{E_\alpha : \alpha \in [X] \cup Y\}$$

be the parabolic subalgebra of \mathfrak{g} defined by X , and let

$$\mathfrak{p}'_X = \mathfrak{h} + \text{span}_{\mathbb{C}}\{E_\alpha : \alpha \in \bar{[X]} \cup (-Y)\}$$

be its opposite parabolic subalgebra. Set

$$\mathfrak{m}_X = \mathfrak{h} + \text{span}_{\mathbb{C}}\{E_\alpha : \alpha \in [X]\} \tag{13}$$

so that $\mathfrak{m}_X = \mathfrak{p}_X \cap \mathfrak{p}'_X$. Let η be the interior automorphism of \mathfrak{m}_X given by $\text{Ad}_{e^{\varepsilon h_\lambda}}$, where $h_\lambda \in \mathfrak{h}$ corresponds to $\lambda \in \mathfrak{h}^*$ under the Killing form. Then the triple $(\mathfrak{p}'_X, \mathfrak{p}_X, \eta)$ is the one corresponding to the Poisson structure $\pi_{r_X(\lambda)}$ in the Karolinsky classification.

4 r -matrices and homogeneous Poisson structures on K/T

We pick a compact real form \mathfrak{k} of \mathfrak{g} as follows: For each $\alpha \in \Sigma_+$, set

$$X_\alpha = E_\alpha - E_{-\alpha}, \quad Y_\alpha = i(E_\alpha + E_{-\alpha})$$

and $h_\alpha = [E_\alpha, E_{-\alpha}]$. Then the real subspace

$$\mathfrak{k} = \text{span}_{\mathbb{R}}\{ih_\alpha, X_\alpha, Y_\alpha : \alpha \in \Sigma_+\}$$

is a compact real form of \mathfrak{g} . Set $\mathfrak{t} = \text{span}_{\mathbb{R}}\{ih_\alpha : \alpha \in \Sigma\} \subset \mathfrak{k}$. Let K and $T \subset K$ be respectively the connected compact subgroups of G with Lie algebras \mathfrak{k} and \mathfrak{t} .

It is well-known [So] that every Poisson structure π_K on K such that (K, π_K) is a Poisson Lie group is of the form

$$\pi_K(k) = R_k \Lambda - L_k \Lambda, \quad (14)$$

where

$$\Lambda = u - \frac{i\varepsilon}{2} \sum_{\alpha \in \Sigma_+} \frac{X_\alpha \wedge Y_\alpha}{2} \in \mathfrak{k} \wedge \mathfrak{k} \quad (15)$$

for some $u \in \mathfrak{t} \wedge \mathfrak{t}$, an imaginary complex number ε and a choice Σ_+ of positive roots.

In this section, we will show how (K, π_K) -homogeneous Poisson structures on K/T are related to classical dynamical r -matrices. We remark again that one classification of all (K, π_K) -homogeneous Poisson spaces (by the corresponding Lagrangian Lie subalgebras) has been given by Karolinsky [Ka2].

If we regard $\wedge \mathfrak{g}$ as a real vector space, then

$$\wedge \mathfrak{k} \longrightarrow \wedge \mathfrak{g} : \wedge^l \mathfrak{k} \ni x_1 \wedge \cdots \wedge x_l \longmapsto x_1 \wedge \cdots \wedge x_l \in \wedge^l \mathfrak{g}$$

is an embedding of $\wedge \mathfrak{k}$ into $\wedge \mathfrak{g}$ as a real subspace. This embedding also preserves the Schouten bracket. Thus, for $A \in \wedge^2 \mathfrak{k}$ of the form

$$A = \sum_{\alpha \in \Sigma_+} a_\alpha \frac{X_\alpha \wedge Y_\alpha}{2}, \quad a_\alpha \in \mathbb{R} \text{ for } \alpha \in \Sigma_+,$$

we can calculate $[A, A] \in \wedge^3 \mathfrak{k}$ by first writing $A = \sum_{\alpha \in \Sigma_+} i a_\alpha E_\alpha \wedge E_{-\alpha} \in \wedge^2 \mathfrak{g}$ and calculate $[A, A]$ inside $\wedge \mathfrak{g}$. Indeed, as in the proof of Lemma 3.4, in $\wedge^3 \mathfrak{g}$ we have

$$\begin{aligned} [A, A] &= \frac{1}{2} \sum_{\alpha \in \Sigma_+} a_\alpha^2 (i h_\alpha \wedge X_\alpha \wedge Y_\alpha) \\ &\quad + 2 \sum_{[(\alpha, \beta, \gamma)] \in \tilde{\Sigma}^3} (a_\alpha a_\beta + a_\beta a_\gamma + a_\gamma a_\alpha) N_{\alpha, \beta} E_\alpha \wedge E_\beta \wedge E_\gamma \end{aligned} \quad (16)$$

Clearly, $i h_\alpha \wedge E_\alpha \wedge E_{-\alpha} \in \wedge^3 \mathfrak{k}$ for each $\alpha \in \Sigma_+$. Suppose that $(\alpha, \beta, \gamma) \in \Sigma^3$ are such that $\alpha + \beta + \gamma = 0$. Without loss of generality, we can assume that $\alpha, \beta \in \Sigma_+$ and $\gamma \in -\Sigma_+$. Then

$$N_{\alpha, \beta} E_\alpha \wedge E_\beta \wedge E_\gamma + N_{-\alpha, -\beta} E_{-\alpha} \wedge E_{-\beta} \wedge E_{-\gamma} = N_{\alpha, \beta} (E_\alpha \wedge E_\beta \wedge E_\gamma - E_{-\alpha} \wedge E_{-\beta} \wedge E_{-\gamma}).$$

This element is in $\wedge^3 \mathfrak{k}$ because it is fixed by $\theta \in \text{End}_{\mathbb{R}}(\wedge^3 \mathfrak{g})$ defined by

$$\theta(x_1 \wedge x_2 \wedge x_3) = \theta(x_1) \wedge \theta(x_2) \wedge \theta(x_3), \quad x_1, x_2, x_3 \in \mathfrak{g},$$

where $\theta \in \text{End}_{\mathbb{R}}(\mathfrak{g})$ is the complex conjugation of \mathfrak{g} defined by \mathfrak{k} . The right hand side of (16) is thus the Schouten bracket of A with itself inside $\wedge \mathfrak{k}$.

Now suppose that r is a classical dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ as given in Theorem 2.2. Suppose that $\lambda \in \mathfrak{h}^*$ is in the domain of r such that the skew-symmetric part $A_r(\lambda) = r(\lambda) - \frac{\varepsilon}{2}\Omega$ of $r(\lambda)$ lies in $\wedge^2\mathfrak{k}$. Then

$$[A_r(\lambda), A_r(\lambda)] - [\Lambda, \Lambda] \in (\wedge^3\mathfrak{k}) \cap (\mathfrak{h} \wedge \mathfrak{k} \wedge \mathfrak{k}) = \mathfrak{t} \wedge \mathfrak{k} \wedge \mathfrak{k}.$$

By abuse of notation, we still use $\tilde{\pi}_{r(\lambda)}$ (already used in Theorem 3.3) to denote the bi-vector field on K given by

$$\tilde{\pi}_{r(\lambda)}(k) = R_k\Lambda - L_kA_r(\lambda), \quad k \in K,$$

where R_k and L_k are respectively the right and left translations on K by k . We use $\pi_{r(\lambda)}$ to denote the projection of $\tilde{\pi}_{r(\lambda)}$ to K/T by the map $p : K \rightarrow K/T : k \mapsto kT$.

Theorem 4.1 *Let r be any classical dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ given in Theorem 2.2. Suppose that $\lambda \in \mathfrak{h}^*$ is in the domain of r such that $A_r(\lambda) = r(\lambda) - \frac{\varepsilon}{2}\Omega$ is in $\wedge^2\mathfrak{k}$. Then,*

1) *the bi-vector field $\pi_{r(\lambda)}$ on K/T defines a (K, π_K) -homogeneous Poisson structure on K/T ;*

2) *with the Poisson structure π_K on K given by (14), every (K, π_K) -homogeneous Poisson structure on K/T arises this way.*

Proof. The proof of 1) is similar to that of Theorem 3.3. We prove 2). Assume that π is a (K, π_K) -homogeneous Poisson structure on K/T . Since π is T -invariant, we can write

$$\pi(e) = \sum_{\alpha \in \Sigma_+} \left(-\frac{i\varepsilon}{2} + i\phi_\alpha\right) \frac{X_\alpha \wedge Y_\alpha}{2} \in \wedge^2(\mathfrak{k}/\mathfrak{t}),$$

where $e = eT \in K/T$ and $\phi_\alpha \in i\mathbb{R}$ for each $\alpha \in \Sigma_+$. (Recall that $\varepsilon \in i\mathbb{R}$ is fixed at the beginning.) Set $\phi_{-\alpha} = -\phi_\alpha$ for $\alpha \in \Sigma_+$. Using the same trick for calculating the Schouten bracket in $\wedge\mathfrak{k}$, i.e., by embedding $\wedge\mathfrak{k}$ into $\wedge\mathfrak{g}$, and by using arguments similar to those in the proof of Lemma 3.4, we know that the ϕ_α 's must satisfies Condition (9). Exactly the same as in the proof of the second part of Theorem 3.3, we know that there exist a choice of positive roots Σ'_+ , a choice of a subset X' of the set of simple roots for Σ'_+ , and some (not necessarily unique) $\lambda_0 \in \mathfrak{h}^*$ such that

$$\phi_\alpha = \begin{cases} \frac{\varepsilon}{2} \coth \frac{\varepsilon}{2} \ll \alpha, \lambda_0 \gg & \text{if } \alpha \in [X'] \\ \pm \frac{\varepsilon}{2} & \text{if } \alpha \in \pm(\Sigma'_+ \setminus [X']). \end{cases}$$

Let r be the classical dynamical r -matrix for the pair $(\mathfrak{g}, \mathfrak{h})$ defined by Σ'_+ and X' as in Theorem 2.2 ($\mu = 0$ and $C = 0$), we see that π coincides with the Poisson structure $\pi_{r(\lambda_0)}$ on K/T .

Q.E.D.

5 The Poisson structures $\pi_{X, X_1, \lambda}$ on K/T

5.1 Definition

As in the case for G/H , we will single out a family of (K, π_K) -homogeneous Poisson structures on K/T which exhausts all such Poisson structures on K/T up to K -equivariant isomorphisms.

For a subset $X \subset S(\Sigma_+)$, set

$$\mathfrak{a}_X = \text{span}_{\mathbb{R}}\{h_\gamma = [E_\gamma, E_{-\gamma}] : \gamma \in X\}.$$

Denote by $\{\check{h}_\gamma : \gamma \in S(\Sigma_+)\}$ the set of fundamental co-weights for $S(\Sigma_+)$, i.e., $\check{h}_\gamma \in \mathfrak{a}$ for each $\gamma \in S(\Sigma_+)$ and $\gamma_1(\check{h}_\gamma) = \delta_{\gamma_1, \gamma}$ for all $\gamma_1, \gamma \in S(\Sigma_+)$. For $X_1 \subset S(\Sigma_+)$, set

$$\check{\rho}_{X_1} = \sum_{\gamma \in X_1} \check{h}_\gamma.$$

Define $\check{\rho}_{X_1}$ to be 0 if X_1 is the empty set.

Theorem 5.1 For $X \in S(\Sigma_+)$, $X_1 \subset X$ and $\lambda = \lambda_1 + \frac{i\pi}{2}\check{\rho}_{X_1} \in \mathfrak{a}_X + \frac{i\pi}{2}\check{\rho}_{X_1}$ such that $\alpha(\lambda_1) \neq 0$ for all $\alpha \in [X]$ with $\alpha(\check{\rho}_{X_1})$ even, let $\pi_{X, X_1, \lambda}$ be the bi-vector field on K/T given by

$$\pi_{X, X_1, \lambda} = p_*\pi_K - \frac{i\varepsilon}{2} \left(\sum_{\alpha \in [X] \cap \Sigma_+} \frac{1}{1 - e^{2\alpha(\lambda)}} X_\alpha \wedge Y_\alpha \right)^L,$$

where the second term on the right hand side is the K -invariant bi-vector field on K/T whose value at $e = eT$ is the expression given in the parenthesis. Then

- 1) $\pi_{X, X_1, \lambda}$ is a (K, π_K) -homogeneous Poisson structure on K/T , and
- 2) every (K, π_K) -homogeneous Poisson structure on K/T is K -equivariantly isomorphic to some $\pi_{X, X_1, \lambda}$.

Remark 5.2 Note that the condition on $\lambda_1 \in \mathfrak{a}_X$ is equivalent to $\alpha(\lambda) \notin \pi i\mathbb{Z}$ for all $\alpha \in [X]$, so that $e^{2\alpha(\lambda)} \neq 1$ for all $\alpha \in [X]$.

Proof. 1). The number $e^{2\alpha(\lambda)}$ is real for each $\alpha \in [X]$. Thus $\pi_{X, X_1, \lambda}$ is a (K, π_K) -homogeneous Poisson structure coming from a classical dynamical r -matrix.

2) Assume that π is a (K, π_K) -homogeneous Poisson structure on K/T . By Theorem 4.1 and by a proof similar to that of Theorem 3.7, there exist $X \subset S(\Sigma_+)$ and some $\lambda_0 \in \mathfrak{h}^*$ such that π is isomorphic, via a K -equivariant diffeomorphism of K/T , to the Poisson structure π' given by

$$\pi' = p_*\pi_K - \frac{i\varepsilon}{2} \left(\sum_{\alpha \in [X] \cap \Sigma_+} k_\alpha X_\alpha \wedge Y_\alpha \right)^L,$$

where

$$k_\alpha = \frac{1}{2}(1 - \coth(\frac{\varepsilon}{2} \ll \alpha, \lambda_0 \gg)) = \frac{1}{1 - e^{\varepsilon \ll \alpha, \lambda_0 \gg}} \in \mathbb{R}.$$

Let $h_{\lambda_0} \in \mathfrak{h}$ be the element in \mathfrak{h} corresponding to λ_0 under the Killing form, so that $\ll \alpha, \lambda_0 \gg = \alpha(h_{\lambda_0})$ for all $\alpha \in \Sigma$. It remains to show that $\frac{\varepsilon}{2}h_{\lambda_0}$ can be replaced by some $\lambda \in \mathfrak{a}_X + \frac{i\pi}{2}\check{\rho}_{X_1}$. To this end, consider the function $f(z) = 1/(1 - e^z)$ for $z \in \mathbb{C}$. It takes values in all of \mathbb{C} except for 0 and 1. Moreover, $f(\mathbb{R} \setminus \{0\}) = (-\infty, 0) \cup (1, \infty)$ and $f(\mathbb{R} + i\pi) \in (0, 1)$. Set

$$X_1 = \{\gamma \in X : k_\gamma \in (0, 1)\}.$$

Then for each $\gamma \in X$, there exists $\mu_\gamma \in \mathbb{R}$ such that

$$\begin{cases} k_\gamma = f(\mu_\gamma + i\pi) & \text{if } \gamma \in X_1 \\ k_\gamma = f(\mu_\gamma) & \text{if } \gamma \in X \setminus X_1. \end{cases}$$

Let $\lambda_1 \in \mathfrak{a}_X$ be such that $2\gamma(\lambda_1) = \mu_\gamma$ for each $\gamma \in X$ (such a λ_1 exists), and let $\lambda = \lambda_1 + \frac{\pi i}{2}\check{\rho}_{X_1}$. Then $k_\gamma = f(2\gamma(\lambda))$ for all $\gamma \in X$. Consequently, by writing $\alpha \in [X] \cap \Sigma_+$ as a linear combination of elements in X , we see that $k_\alpha = f(2\alpha(\lambda))$ for all $\alpha \in [X]$.

Q.E.D.

Notation 5.3 For reasons given in Section 5.2, we will use π_∞ to denote the Poisson structure $p_*\pi_K$ on K/T . It is called the *Bruhat Poisson structure* [Lu-We] because its symplectic leaves are Bruhat cells in K/T . See Section 5.6 for more details.

Example 5.4 Consider

$$K = SU(2) = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} : u, v \in \mathbb{C}, |u|^2 + |v|^2 = 1 \right\},$$

$T = \{\text{diag}(e^{ix}, e^{-ix}) : x \in \mathbb{R}\} \cong S^1$ and the root $\alpha(x, -x) = 2x$ is taken to be the positive root. Then

$$X_\alpha = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y_\alpha = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

With

$$\Lambda = -\frac{i\varepsilon}{2} \frac{X_\alpha \wedge Y_\alpha}{2} \in \mathfrak{su}(2) \wedge \mathfrak{su}(2)$$

and the Poisson structure π_K on $K = SU(2)$ defined by

$$\pi_K = \Lambda^R - \Lambda^L,$$

the Poisson brackets among the coordinate functions u, v, \bar{u} and \bar{v} on $SU(2)$ are given by

$$\{u, \bar{u}\} = -\frac{\varepsilon}{4}|v|^2, \quad \{u, v\} = \frac{\varepsilon}{8}uv, \quad \{u, \bar{v}\} = \frac{\varepsilon}{8}u\bar{v}, \quad \{v, \bar{v}\} = 0.$$

Let π_0 be the $SU(2)$ -invariant bivector field on $SU(2)/S^1$ whose value at the point $e = eS^1$ is $X_\alpha \wedge Y_\alpha$. It is symplectic.

Case 1: $X = X_1 = \emptyset$. Then $\pi_{X, X_1, \lambda} = \pi_\infty$;

Case 2: $X = \{\alpha\}$, $X_1 = \emptyset$. Then $\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}$ with $\lambda_1 \neq 0$, and

$$\pi_{X, X_1, \lambda} = \pi_\infty - \frac{i\varepsilon}{2} \frac{1}{1 - e^{4\lambda_1}} \pi_0.$$

Case 3: $X = X_1 = \{\alpha\}$. Then

$$\lambda = \begin{pmatrix} \lambda_1 + \frac{\pi i}{4} & 0 \\ 0 & -\lambda_1 - \frac{\pi i}{4} \end{pmatrix}$$

with $\lambda_1 \in \mathbb{R}$ arbitrary, and

$$\pi_{X, X_1, \lambda} = \pi_\infty - \frac{i\varepsilon}{2} \frac{1}{1 + e^{4\lambda_1}} \pi_0.$$

Note that the range of the function $\frac{1}{1 - e^{4\lambda_1}}$ for $\lambda_1 \in \mathbb{R} \setminus \{0\}$ is $(-\infty, 0) \cup (1, +\infty)$, and the range of $\frac{1}{1 + e^{4\lambda_1}}$ for $\lambda_1 \in \mathbb{R}$ is $(0, 1)$. Thus, for all possible choices of X, X_1 and λ , we get all the Poisson structures of the form

$$\pi^a = \pi_\infty - \frac{i\varepsilon}{2} a \pi_0$$

for $a \in \mathbb{R}$ except for $a = 1$. But the Poisson structure π^a when $a = 1$ is easily seen to be isomorphic to π_∞ (corresponding to $a = 0$) by the $SU(2)$ -equivariant diffeomorphism on $SU(2)/S^1$ defined by the right translation by the non-trivial Weyl group element. The fact that every $(SU(2), \pi_K)$ -homogeneous Poisson structures on S^2 is of the form π^a for some $a \in \mathbb{R}$ is very easy to check directly [Sh].

Identify the Lie algebra $\mathfrak{su}(2)$ with \mathbb{R}^3 by

$$\begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \mapsto (x, y, z)$$

so the Adjoint orbit through $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ can be identified with the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Consequently, we have the identification

$$SU(2)/S^1 \rightarrow S^2 : kS^1 \mapsto \text{Ad}_k \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

or

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} S^1 \longmapsto (|u|^2 - |v|^2, -i(uv - \bar{u}\bar{v}), -(uv + \bar{u}\bar{v})).$$

The induced Bruhat Poisson structure π_∞ on S^2 is given by

$$\{x, y\} = -\frac{\varepsilon i}{4}(x-1)z, \quad \{y, z\} = -\frac{\varepsilon i}{4}(x-1)x, \quad \{z, x\} = -\frac{\varepsilon i}{4}(x-1)y,$$

and the Poisson structure π^a on S^2 is given by

$$\{x, y\} = -\frac{\varepsilon i}{4}(x+2a-1)z, \quad \{y, z\} = -\frac{\varepsilon i}{4}(x+2a-1)x, \quad \{z, x\} = -\frac{\varepsilon i}{4}(x+2a-1)y,$$

Note that π^a is symplectic when $a < 0$ or $a > 1$. When $a = 0$, it has two symplectic leaves, the point $(1, 0, 0)$ being a one-point leaf and the rest of S^2 as another leaf. Similarly for $a = 1$. When $0 < a < 1$, it has infinitely many symplectic leaves: two open leaves respectively given by $x < 1 - 2a$ and $x > 1 - 2a$, and every point on the circle $x = 1 - 2a$ as a one-point leaf.

Example 5.5 Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and $K = SU(3)$. The three positive roots are chosen to be

$$\alpha_1(x) = x_1 - x_2, \quad \alpha_2(x) = x_2 - x_3, \quad \alpha_3(x) = x_1 - x_3$$

for a diagonal matrix $x = \text{diag}(x_1, x_2, x_3)$. Take $X = S(\Sigma_+) = \{\alpha_1, \alpha_2\}$ and $X_1 = \{\alpha_1\}$. In this case

$$\check{\rho}_{X_1} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix},$$

and

$$\lambda = \begin{pmatrix} \lambda_1 + \frac{\pi i}{3} & 0 & 0 \\ 0 & \lambda_2 - \frac{\pi i}{6} & 0 \\ 0 & 0 & -(\lambda_1 + \lambda_2) - \frac{\pi i}{6} \end{pmatrix}, \quad \lambda_1 + 2\lambda_2 \neq 0.$$

Then

$$\pi_{X, X_1, \lambda} = \pi_\infty + \left(\frac{2X_{\alpha_1} \wedge Y_{\alpha_1}}{1 + e^{2(\lambda_1 - \lambda_2)}} + \frac{2X_{\alpha_2} \wedge Y_{\alpha_2}}{1 - e^{2\lambda_1 + 4\lambda_2}} + \frac{2X_{\alpha_3} \wedge Y_{\alpha_3}}{1 + e^{4\lambda_1 + 2\lambda_2}} \right)^L.$$

5.2 Connections via taking limits in λ

As noted in [E-V], the dynamical r -matrices are related to each other via taking various limits in λ . Correspondingly, the Poisson structures $\pi_{X, X_1, \lambda}$ are also related this way. We study these relations in the section.

Proposition 5.6 For any $X_1 \subset X \subset Y \subset S(\Sigma_+)$ and $\lambda = \lambda_1 + \frac{i\pi}{2}\check{\rho}_{X_1} \in \mathfrak{a}_X + \frac{i\pi}{2}\check{\rho}_{X_1}$ such that $\alpha(\lambda_1) \neq 0$ for all $\alpha \in [X]$ with $\alpha(\check{\rho}_{X_1})$ even, we have

$$\pi_{X, X_1, \lambda} = \lim_{t \rightarrow +\infty} \pi_{Y, X_1, \lambda + t\check{\rho}_{Y \setminus X}}. \quad (17)$$

In particular,

$$\pi_\infty = \lim_{t \rightarrow +\infty} \pi_{Y, \emptyset, t\check{\rho}_Y}.$$

Moreover, we also have

$$\pi_\infty = \lim_{t \rightarrow +\infty} \pi_{X, X_1, \lambda + t\check{\rho}_X}. \quad (18)$$

Proof. Set $\mu_t = \lambda + t\check{\rho}_{Y \setminus X}$ for $t > 0$. Let $\alpha \in [Y] \cap \Sigma_+$. If $\alpha \in [X]$, then $\alpha(\check{\rho}_{Y \setminus X}) = 0$ so $\alpha(\mu_t) = \alpha(\lambda)$. If $\alpha \in [Y] \setminus [X]$, then $v := \alpha(\check{\rho}_{Y \setminus X})$ is positive, so

$$\lim_{t \rightarrow \infty} \frac{1}{1 - e^{\alpha(\mu_t)}} = \lim_{t \rightarrow \infty} \frac{1}{1 - e^{tv}} = 0.$$

Hence (17) follows from the definition of $\pi_{X, X_1, \lambda}$. The limit in (18) is obvious.

Q.E.D.

5.3 The Lagrangian subalgebras of \mathfrak{g} corresponding to $\pi_{X, X_1, \lambda}$

The Lie bialgebra of the Poisson Lie group (K, π_K) is $(\mathfrak{k}, \mathfrak{a} + \mathfrak{n})$, where the pairing between \mathfrak{k} and $\mathfrak{a} + \mathfrak{n}$ is given by $\frac{2i}{\varepsilon} \text{Im} \ll, \gg$, where $\text{Im} \ll, \gg$ stands for the imaginary part of the Killing form \ll, \gg .

We will call a real subalgebra \mathfrak{l} of \mathfrak{g} a *Lagrangian algebra* if 1) $\dim \mathfrak{l} = \dim \mathfrak{k}$, and 2) $\frac{2i}{\varepsilon} \text{Im} \ll x, y \gg = 0$ for all $x, y \in \mathfrak{l}$. By a theorem of Drinfeld [D3], (K, π_K) -homogeneous Poisson structures on K/T correspond to Lagrangian subalgebras \mathfrak{l} of \mathfrak{g} with $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{t}$. In this section, we calculate the Lagrangian subalgebras $\mathfrak{l}_{X, X_1, \lambda}$ corresponding to the Poisson structures $\pi_{X, X_1, \lambda}$.

By definition [D3],

$$\mathfrak{l}_{X, X_1, \lambda} = \{x + \xi : x \in \mathfrak{k}, \xi \in \mathfrak{a} + \mathfrak{n} : \xi|_{\mathfrak{t}} = 0, \xi \lrcorner \pi_{X, X_1, \lambda}(e) = x + \mathfrak{t}\}.$$

A direct calculation gives

$$\begin{aligned} \mathfrak{l}_{X, X_1, \lambda} &= \mathfrak{t} + \text{span}_{\mathbb{R}}\{E_\beta, iE_\beta : \beta \in \Sigma_+ \setminus [X]\} \\ &\quad + \text{span}_{\mathbb{R}}\left\{\frac{1}{e^{2\alpha(\lambda)} - 1}X_\alpha + E_\alpha, \frac{1}{e^{2\alpha(\lambda)} - 1}Y_\alpha + iE_\alpha : \alpha \in [X] \cap \Sigma_+\right\}. \end{aligned}$$

On the other hand, for $\alpha \in [X]$, since $e^{2\alpha(\lambda)} \neq 1$, we have

$$\begin{aligned}\mathrm{Ad}_{e^\lambda} X_\alpha &= \mathrm{Ad}_{e^\lambda}(E_\alpha - E_{-\alpha}) = (e^{\alpha(\lambda)} - e^{-\alpha(\lambda)})\left(\frac{1}{e^{2\alpha(\lambda)} - 1}X_\alpha + E_\alpha\right) \\ \mathrm{Ad}_{e^\lambda} Y_\alpha &= \mathrm{Ad}_{e^\lambda}(iE_\alpha + iE_{-\alpha}) = (e^{\alpha(\lambda)} - e^{-\alpha(\lambda)})\left(\frac{1}{e^{2\alpha(\lambda)} - 1}Y_\alpha + iE_\alpha\right).\end{aligned}$$

Note that $e^{\alpha(\lambda)}$ is real or imaginary depending on $\alpha(\check{\rho}_{X_1})$ is even or odd. Set

$$\mathfrak{n}_X = \mathrm{span}_{\mathbb{R}}\{E_\beta, iE_\beta : \beta \in \Sigma_+ \setminus [X]\}. \quad (19)$$

Then we have proved the following proposition.

Proposition 5.7 *Denote by $\mathfrak{l}_{X, X_1, \lambda}$ the Lagrangian subalgebra of \mathfrak{g} corresponding to the Poisson structure $\pi_{X, X_1, \lambda}$ on K/T . It is given by*

$$\begin{aligned}\mathfrak{l}_{X, X_1, \lambda} &= \mathrm{Ad}_{e^\lambda}(\mathfrak{t} + \mathfrak{n}_X + \mathrm{span}_{\mathbb{R}}\{X_\alpha, Y_\alpha : \alpha \in [X], \alpha(\check{\rho}_{X_1}) \text{ is even}\} \\ &\quad + \mathrm{span}_{\mathbb{R}}\{iX_\alpha, iY_\alpha : \alpha \in [X], \alpha(\check{\rho}_{X_1}) \text{ is odd}\}).\end{aligned}$$

Remark 5.8 Let θ be the complex conjugation on \mathfrak{g} defined by \mathfrak{k} . Let τ_{X, X_1} be the complex conjugation on \mathfrak{g} given by

$$\tau_{X, X_1} = \mathrm{Ad}_{\exp(\pi i \check{\rho}_{X_1})} \theta = \theta \mathrm{Ad}_{\exp(-\pi i \check{\rho}_{X_1})}.$$

Denote by $\mathfrak{m}_X^{\tau_{X, X_1}}$ the set of fixed points of τ_{X, X_1} in \mathfrak{m}_X , where

$$\mathfrak{m}_X = \mathfrak{h} + \mathrm{span}_{\mathbb{C}}\{E_\alpha : \alpha \in [X]\}.$$

Then

$$\mathfrak{l}_{X, X_1, \lambda} = \mathrm{Ad}_{e^\lambda}(\mathfrak{m}_X^{\tau_{X, X_1}} + \mathfrak{n}_X).$$

Remark 5.9 Let $n = \dim \mathfrak{k}$ and consider $\mathfrak{l}_{X, X_1, \lambda}$ as a point in $\mathrm{Gr}(n, \mathfrak{g})$, the Grassmannian of n -dimensional real subspaces of \mathfrak{g} . Then, corresponding to Proposition 5.6, we have, for $X_1 \subset X \subset Y \subset S(\Sigma_+)$ and for any $\lambda = \lambda_1 + \frac{i\pi}{2}\check{\rho}_{X_1} \in \mathfrak{a}_X + \frac{i\pi}{2}\check{\rho}_{X_1}$ such that $\alpha(\lambda_1) \neq 0$ for all $\alpha \in [X]$ with $\alpha(\check{\rho}_{X_1})$ even,

$$\lim_{t \rightarrow +\infty} \mathfrak{l}_{Y, X_1, \lambda + t\check{\rho}_{Y \setminus X}} = \mathfrak{l}_{X, X_1, \lambda} \quad (20)$$

in $\mathrm{Gr}(n, \mathfrak{g})$. Indeed, under the Plucker embedding of $\mathrm{Gr}(n, \mathfrak{g})$ into $\mathbb{P}^1(\wedge^n \mathfrak{g})$, the Lie subalgebra $\mathfrak{l}_{Y, X_1, \lambda}$ corresponds to the point in $\mathbb{P}^1(\wedge^n \mathfrak{g})$ defined by the vector

$$v_{Y, X_1, \lambda} := Z_0 \wedge \prod_{\alpha \in [Y] \cap \Sigma_+} \left(\frac{1}{e^{2\alpha(\lambda)} - 1} X_\alpha + E_\alpha \right) \wedge \left(\frac{1}{e^{2\alpha(\lambda)} - 1} Y_\alpha + iE_\alpha \right) \wedge \prod_{\alpha \in \Sigma_+ \setminus [Y]} E_\alpha \wedge E_{-\alpha}$$

where $Z_0 \in \wedge^{\dim \mathfrak{t}} \mathfrak{t}$ and $Z_0 \neq 0$ is fixed. Since $v_{Y, X_1, \lambda + t\check{\rho}_{Y \setminus X}} \rightarrow v_{X_1, \lambda}$ as $t \rightarrow +\infty$, we see that (20) holds in $\mathbb{P}^1(\wedge^n \mathfrak{g})$ and thus also in $\mathrm{Gr}(n, \mathfrak{g})$.

Example 5.10 When $X = X_1$ are the empty set, we have $\mathfrak{l}_{X, X_1, \lambda} = \mathfrak{t} + \mathfrak{n}$, and when $X = S(\Sigma_+)$ and X_1 is the empty set, we have $\mathfrak{l}_{X, X_1, \lambda} = \text{Ad}_{e^\lambda} \mathfrak{k}$. In general, when $X = S(\Sigma_+)$, the Lie subalgebra $\mathfrak{l}_{X, X_1, \lambda}$ is a real form of \mathfrak{g} .

5.4 Geometrical interpretation of $\pi_{X, X_1, \lambda}$

Denote by \mathcal{L} the set of all Lagrangian subalgebras of \mathfrak{g} with respect to the imaginary part of the Killing form \ll, \gg . (Here \mathfrak{g} is regarded as a real vector space.) It is an algebraic subvariety of the Grassmannian $\text{Gr}(n, \mathfrak{g})$ of n -dimensional subspaces of \mathfrak{g} , where $n = \dim \mathfrak{k}$. In [E-L2], we show that there is a smooth bivector field Π on $\text{Gr}(n, \mathfrak{g})$ such that the Schouten bracket $[\Pi, \Pi]$ vanishes at every $\mathfrak{l} \in \mathcal{L}$. More precisely, consider the G -action on $\text{Gr}(n, \mathfrak{g})$ by the Adjoint action. It defines a Lie algebra anti-homomorphism

$$\kappa : \mathfrak{g} \longrightarrow \chi^1(\text{Gr}(n, \mathfrak{g})),$$

where $\chi^1(\text{Gr}(n, \mathfrak{g}))$ is the space of vector fields on $\text{Gr}(n, \mathfrak{g})$. Denote by the same letter its multi-linear extension from $\wedge^2 \mathfrak{g}$ to the space of bi-vector fields on $\text{Gr}(n, \mathfrak{g})$. Then the bivector field Π on $\text{Gr}(n, \mathfrak{g})$ is defined to be

$$\Pi = \frac{1}{2} \kappa(R),$$

where $R \in \wedge^2 \mathfrak{g}$ is the r -matrix for \mathfrak{g} given by

$$\langle R, (x_1 + y_1) \wedge (x_2 + y_2) \rangle_\varepsilon = \langle x_1, y_2 \rangle_\varepsilon - \langle x_2, y_1 \rangle_\varepsilon \quad (21)$$

for $x_1, x_2 \in \mathfrak{k}$ and $y_1, y_2 \in \mathfrak{a} + \mathfrak{n}$ with $\langle \cdot, \cdot \rangle_\varepsilon = \frac{2i}{\varepsilon} \text{Im} \ll, \gg$. Explicitly,

$$R = -\frac{\varepsilon}{2i} \left(\sum_{j=1}^l (ih_j) \wedge h_j + \sum_{\alpha \in \Sigma_+} (-X_\alpha \wedge (iE_\alpha) + Y_\alpha \wedge E_\alpha) \right),$$

where $\{h_1, \dots, h_l\}$ is a basis for \mathfrak{a} such that $\ll h_j, h_k \gg = \delta_{jk}$. It now follows from the definition of Π that it defines a Poisson structure on every G -invariant smooth submanifold of \mathcal{L} .

One particular G -invariant smooth submanifold of \mathcal{L} is the (unique) irreducible component \mathcal{L}_0 of \mathcal{L} that contains \mathfrak{k} . We show in [E-L2] that each $\mathfrak{l}_{X, X_1, \lambda} \in \mathcal{L}_0$ and that its K -orbit in \mathcal{L}_0 is a Poisson submanifold of (\mathcal{L}_0, Π) . (We also show in [E-L2] that \mathcal{L}_0 is diffeomorphic to the set of real points in the De Concini-Procesi compactification of G [D-P]). For each Poisson structure $\pi_{X, X_1, \lambda}$ on K/T , consider the map

$$P : (K/T, \pi_{X, X_1, \lambda}) \longrightarrow (\mathcal{L}_0, \Pi) : kT \longmapsto \text{Ad}_k \mathfrak{l}_{X, X_1, \lambda}.$$

It is shown in [E-L2] that P is a Poisson map. When the normalizer subgroup of $\mathfrak{l}_{X, X_1, \lambda}$ in K is T , this map is an embedding of K/T into \mathcal{L}_0 whose image is the the K -orbit of $\mathfrak{l}_{X, X_1, \lambda}$ in \mathcal{L}_0 .

In general, P is a covering map onto the K -orbit of $\iota_{X, X_1, \lambda}$ in \mathcal{L}_0 . Thus, every $(K/T, \pi_{X, X_1, \lambda})$ is a Poisson submanifold of (\mathcal{L}_0, Π) (possibly up to a covering map). This can be considered as one geometrical interpretation of $\pi_{X, X_1, \lambda}$.

Two special cases of $\pi_{X, X_1, \lambda}$ deserve more attention. The first is when $X = X_1 = \emptyset$ ($\lambda = 0$ in this case). Then $\pi_{X, X_1, \lambda} = \pi_\infty$ is the Bruhat Poisson structure. It has been the most interesting example in terms of connections to Lie theory. For its relations with the Kostant harmonic forms [Ko], see [Lu3] and [E-L1].

The second special case is when $X = S(\Sigma_+)$ and $X_1 = \emptyset$. The condition on λ is that $\lambda \in \mathfrak{a}$ is regular. We will show that $\pi_{X, X_1, \lambda}$ is symplectic in this case. In fact, we will show that $\pi_{X, X_1, \lambda}$ can be identified with the symplectic structure on a dressing orbit of K in its dual Poisson Lie group. We also remark that this symplectic structure has been used in [L-R] to give a symplectic proof of Kostant's nonlinear convexity theorem.

Recall that the Manin triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a} + \mathfrak{n}, \frac{2i}{\varepsilon} \text{Im} \ll, \gg)$ gives rise to a Poisson structure π_{AN} on the group AN making (AN, π_{AN}) into the dual Poisson Lie group of (K, π_K) . The group K acts on AN by the *(left) dressing action*:

$$K \times AN \longrightarrow AN : (k, b) \longmapsto k \cdot b := b_1, \quad \text{if } bk^{-1} = k_1 b_1 \text{ for } k_1 \in K \text{ and } b_1 \in AN.$$

The K orbits of this dressing action of K in AN , called the *dressing orbits*, are precisely all the symplectic leaves of the Poisson structure on AN and they are parametrized by a fundamental W -chamber in \mathfrak{a} . Thus each dressing orbit inherits a symplectic, and thus Poisson, structure as a symplectic leaf. Since the dressing action is Poisson [STS2] [Lu-We], these dressing orbits are examples of (K, π_K) -homogeneous Poisson spaces. Let $\lambda \in \mathfrak{a}$ be regular and consider the element $e^{-\lambda} \in A$. The stabilizer subgroup of K in AN at $e^{-\lambda}$ is T . Thus, by identifying K/T with the dressing orbit through $e^{-\lambda}$, we get a Poisson structure on K/T which is in fact symplectic.

Notation 5.11 We will use π_λ to denote the Poisson structure on K/T obtained by identifying K/T with the symplectic leaf in AN through the point $e^{-\lambda}$, and we call it the *dressing orbit Poisson structure corresponding to $e^{-\lambda} \in A$* .

Proposition 5.12 *When $X = S(\Sigma_+)$, $X_1 = \emptyset$, and $\lambda \in \mathfrak{a}$ is regular, the Poisson structure $\pi_{X, X_1, \lambda}$ on K/T is nothing but the dressing orbit Poisson structure π_λ corresponding to $e^{-\lambda}$. Explicitly, we have*

$$\pi_\lambda = -\frac{i\varepsilon}{2} \left(\sum_{\alpha \in \Sigma_+} \frac{1}{1 - e^{2\alpha(\lambda)}} X_\alpha \wedge Y_\alpha \right)^L + \pi_\infty, \quad (22)$$

where the first term is the K -invariant bi-vector field on K/T whose value at $e = eT$ is the expression given in the parenthesis.

Proof. Since $\iota_{X, X_1, \lambda}$ is given by the right hand side of (22), we only need to show that the dressing orbit Poisson structure π_λ is also given by the same formula. Denote the Poisson structure on AN by π_{AN} . Since we are identifying \mathfrak{k} with $(\mathfrak{a} + \mathfrak{n})^*$ via $\frac{2i}{\epsilon} \text{Im} \ll, \gg$, an element $x \in \mathfrak{k}$ can be regarded as a left invariant 1-form on AN which we denote by x^l . Let $p_\mathfrak{k} : \mathfrak{g} \rightarrow \mathfrak{k}$ be the projection from \mathfrak{g} to \mathfrak{k} with respect to the Iwasawa Decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. We know that (see [Lul]) for any $a \in A$,

$$\pi_{AN}(x^l, y^l)(a) = \frac{2i}{\epsilon} \text{Im} \ll \text{Ad}_a x, p_\mathfrak{k} \text{Ad}_a y \gg$$

for all $x, y \in \mathfrak{k}$. Here, Ad_a is the Adjoint action of $a \in A$ on \mathfrak{g} . Thus, when x and y run over the basis vectors $\{iH_\alpha, X_\alpha, Y_\alpha : \alpha \in \Sigma_+\}$ for \mathfrak{k} , we have $\pi_{AN}(x^l, y^l)(a) = 0$ except that

$$\begin{aligned} \pi_{AN}(X_\alpha^l, Y_\alpha^l) &= \frac{2i}{\epsilon} \text{Im} \ll \text{Ad}_a X_\alpha, p_\mathfrak{k} \text{Ad}_a Y_\alpha \gg \\ &= \frac{2i}{\epsilon} \text{Im} \ll a^\alpha E_\alpha - a^{-\alpha} E_{-\alpha}, a^{-\alpha} (iE_\alpha + iE_{-\alpha}) \gg \\ &= \frac{2i}{\epsilon} (1 - a^{-2\alpha}). \end{aligned}$$

Let σ_x be the (left)-dressing vector field on AN defined by $x \in \mathfrak{k}$, i.e., $\sigma_x = -x^l \lrcorner \pi_{AN}$. Then, taking $a = e^{-\lambda}$, we have

$$\begin{aligned} \pi_{AN}(a) &= \sum_{\alpha \in \Sigma_+} \frac{1}{\pi_{AN}(X_\alpha^l, Y_\alpha^l)} \sigma_{X_\alpha}(a) \wedge \sigma_{Y_\alpha}(a). \\ &= -\frac{i\epsilon}{2} \sum_{\alpha \in \Sigma_+} \frac{1}{1 - e^{2\alpha(\lambda)}} \sigma_{X_\alpha}(a) \wedge \sigma_{Y_\alpha}(a) \in \wedge^2 T_a(K \cdot a). \end{aligned}$$

Identify K/T with $K \cdot a$ by $kT \mapsto k \cdot a$, we get

$$\pi_\lambda(eT) = -\frac{i\epsilon}{2} \sum_{\alpha \in \Sigma_+} \frac{1}{1 - e^{2\alpha(\lambda)}} X_\alpha \wedge Y_\alpha.$$

Thus π_λ is given as by (22).

Q.E.D.

5.5 $\pi_{X, X_1, \lambda}$ as the result of Poisson induction

We now look at the general case of $\pi_{X, X_1, \lambda}$. Set

$$\mathfrak{k}_X = \mathfrak{k} + \text{span}_{\mathbb{R}}\{X_\alpha, Y_\alpha : \alpha \in [X] \cap \Sigma_+\},$$

and let $K_X \subset K$ be the connected subgroup of K with Lie algebra \mathfrak{k}_X . We will show that $\mathfrak{l}_{X, X_1, \lambda}$ can be obtained via Poisson induction (see Remark 5.15 below) from a Poisson structure on the smaller space K_X/T .

To this end, consider

$$\mathfrak{k}_X^0 = \{\xi \in \mathfrak{k}^* : \xi(x) = 0 \forall x \in \mathfrak{k}_X\}.$$

Since we are identifying \mathfrak{k}^* with $\mathfrak{a} + \mathfrak{n}$, we have $\mathfrak{k}_X^0 \cong \mathfrak{n}_X$ as real Lie algebras, where \mathfrak{n}_X is given in (19). Since $\mathfrak{n}_X \subset \mathfrak{a} + \mathfrak{n}$ is an ideal, we know that $K_X \subset K$ is a Poisson subgroup [Lu-We]. In fact, set

$$\Lambda_1 = -\frac{i\varepsilon}{2} \sum_{\alpha \in [X] \cap \Sigma_+} \frac{X_\alpha \wedge Y_\alpha}{2}, \quad \Lambda_2 = -\frac{i\varepsilon}{2} \sum_{\alpha \in \Sigma_+ \setminus [X]} \frac{X_\alpha \wedge Y_\alpha}{2}.$$

Then, we have

Proposition 5.13 1) For any $x \in \mathfrak{k}_X$, $\text{ad}_x \Lambda_2 = 0$;

2) The Poisson structure on K_X (as a Poisson submanifold of K) is given by

$$\pi_{K_X}(k_1) = R_{k_1} \Lambda_1 - L_{k_1} \Lambda_1,$$

where R_{k_1} and L_{k_1} are respectively the right and left translations on K_X by $k_1 \in K_X$.

3) The Manin triple for the Poisson Lie group (K_X, π_{K_X}) is $(\mathfrak{m}_X, \mathfrak{k}_X, \mathfrak{a} + \mathfrak{u}_X, \frac{2i}{\varepsilon} \ll, \gg)$, where \mathfrak{m}_X , given in (13), is considered as over \mathbb{R} , and $\mathfrak{u}_X = \text{span}_{\mathbb{R}}\{E_\alpha, iE_\alpha : \alpha \in [X] \cap \Sigma_+\}$.

Proof. 1) Using the embedding of $\wedge^\bullet \mathfrak{k}$ into $\wedge^\bullet \mathfrak{g}$ as a real subspace, it is enough to show that $\text{ad}_x \Lambda_2 = 0$ for $x = E_\alpha$ with $\alpha \in [X]$. Let $\alpha \in [X] \cap \Sigma_+$. Then,

$$\frac{2}{\varepsilon} \text{ad}_{E_\alpha} \Lambda = \sum_{\beta \in \Sigma_+ \setminus [X]} [E_\alpha, E_\beta] \wedge E_{-\beta} + E_\beta \wedge [E_\alpha, E_{-\beta}].$$

Set

$$Y_1 = \{\beta \in \Sigma_+ \setminus [X] : \alpha + \beta \in \Sigma\}, \quad \text{and} \quad Y_2 = \{\beta \in \Sigma_+ \setminus [X] : \beta - \alpha \in \Sigma\}.$$

Since $Y = \Sigma_+ \setminus [X]$ has the property that if $\alpha \in [X] \cap \Sigma_+$ and $\beta \in Y$ are such that $\alpha + \beta \in \Sigma$ then $\alpha + \beta \in Y$, the map $Y_1 \rightarrow Y_2 : \beta \mapsto \alpha + \beta$ is a bijection. Thus

$$\begin{aligned} \frac{2}{\varepsilon} \text{ad}_{E_\alpha} \Lambda_2 &= \sum_{\beta \in Y_1} ([E_\alpha, E_\beta] \wedge E_{-\beta} + E_{\alpha+\beta} \wedge [E_\alpha, E_{-(\alpha+\beta)}]) \\ &= \sum_{\beta \in Y_1} (N_{\alpha, \beta} + N_{\alpha, -(\alpha+\beta)}) E_{\alpha+\beta} \wedge E_{-\beta} \\ &= 0. \end{aligned}$$

Similarly, $\text{ad}_{E_{-\alpha}} \Lambda_2 = 0$. This proves 1).

2) By definition, the induced Poisson structure π_{K_X} on K_X is the restriction of π_K to K_X . Using the definition of π_K and 1), we know that π_{K_X} is as given.

3) From the general theory of Poisson Lie groups [Lu-We], we know that the induced Lie algebra structure on \mathfrak{k}_X^* is isomorphic to the quotient Lie algebra $\mathfrak{k}^*/\mathfrak{k}_X^0$. Through the identifications $\mathfrak{k}^* \cong \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{k}_X^0 \cong \mathfrak{n}_X$ via $\frac{2i}{\epsilon} \ll, \gg$, we get $\mathfrak{k}_X^* \cong \mathfrak{a} + \mathfrak{u}_X$ via $\frac{2i}{\epsilon} \ll, \gg$ which is now considered as a symmetric scalar product on \mathfrak{m}_X by restriction.

Q.E.D.

Notation 5.14 Let $X_1 \subset X$ and let $\lambda = \lambda_1 + \frac{\pi i}{2} \check{\rho}_{X_1} \in \mathfrak{a}_X + \frac{\pi i}{2} \check{\rho}_{X_1}$ be such that $\alpha(\lambda_1) \neq 0$ for any $\alpha \in [X]$ with $\alpha(\check{\rho}_{X_1})$ even. By replacing K by K_X and by regarding X as the set of all simple roots for the root system for (K_X, T) , we know that there is a (K_X, π_{K_X}) -homogeneous Poisson structure on K_X/T corresponding to X, X_1 and λ . We will denote it by $\pi_{X_1, \lambda}^X$.

We now show that the Poisson structure $\pi_{X, X_1, \lambda}$ on K/T can be obtained via Poisson induction from the Poisson structure $\pi_{X_1, \lambda}^X$ on K_X/T .

To this end, consider the product space $K \times (K_X/T)$ with the product Poisson structure $\pi_K \oplus \pi_{X_1, \lambda}^X$. Even though the diagonal (right) action of K_X on $K \times (K_X/T)$ given by $k_1 : (k, k'T) \mapsto (kk_1, k_1^{-1}k'T)$ is in general not Poisson, there is nevertheless a unique Poisson structure on the quotient space $K \times_{K_X} (K_X/T)$ such that the projection map

$$K \times (K_X/T) \longrightarrow K \times_{K_X} (K_X/T) : (k, k'T) \longmapsto [(k, k'T)]$$

is a Poisson map. We temporarily denote this Poisson structure on $K \times_{K_X} (K_X/T)$ by π_0 .

Remark 5.15 In general, suppose that K is a Poisson Lie group and $K_1 \subset K$ is a Poisson subgroup. Suppose that M is a Poisson manifold on which there is a Poisson action by K_1 . Then there is a unique Poisson structure on $K \times_{K_1} M$ such that the natural projection from $K \times M$ to $K \times_{K_1} M$ is a Poisson map. Moreover, the left action of K on $K \times_{K_1} M$ by left translations on the first factor is a Poisson action. We call this procedure of producing the Poisson K -space $K \times_{K_1} M$ from the Poisson K_1 -space M *Poisson induction*.

Proposition 5.16 *We have $F_*\pi_0 = \pi_{X, X_1, \lambda}$, where F is the identification*

$$F : K \times_{K_X} (K_X/T) \xrightarrow{\sim} K/T : [(k, k'T)] \longmapsto kk'T.$$

Proof. Recall that $\pi_{X, X_1, \lambda}$ is the image of $\tilde{\pi}_{r_X(\lambda)} = \Lambda^R - A_X(\lambda)^L$ under the projection $p_1 : K \rightarrow K/T$, where Λ^R (resp. $A_X(\lambda)^L$) is the right (resp. left) invariant bivector field

on K with value Λ (reps. $A_X(\lambda)$) at e , and $A_X(\lambda) \in \mathfrak{k} \wedge \mathfrak{k}$ is the skew symmetric part of the r -matrix $r_X(\lambda)$ given in (12). On the other hand, π_0 is the image of $\pi_K \oplus \bar{\pi}$ under the projection

$$p_2 : K \times K_X \longrightarrow K \times_{K_X} (K_X/T) : (k, k') \longmapsto [(k, k'T)],$$

where $\bar{\pi}$ is the bi-vector field on K_X defined by $\bar{\pi} = \Lambda_1^R - \Lambda_3^L$ with

$$\Lambda_3 = -\frac{i\varepsilon}{2} \sum_{\alpha \in [X] \cap \Sigma_+} \coth \alpha(\lambda) \frac{X_\alpha \wedge Y_\alpha}{2}.$$

Because of the commutative diagram:

$$\begin{array}{ccc} K \times K_X & \xrightarrow{m} & K \\ p_2 \downarrow & & \downarrow p_1 \\ K \times_{K_X} (K_X/T) & \xrightarrow{F} & K/T, \end{array}$$

where $m : K \times K_X \longrightarrow K : (k, k') \mapsto kk'$, it is enough to show that $m_*(\pi_K \oplus \bar{\pi}) = \tilde{\pi}_{r_X(\lambda)}$, or

$$\tilde{\pi}_{r_X(\lambda)}(kk_1) = L_k \bar{\pi}(k_1) + R_{k_1} \pi_K(k), \quad \forall k \in K, k_1 \in K_X.$$

But this follows easily from the definitions and the fact that $\text{Ad}_{k_1} \Lambda_2 = \Lambda_2$ for all $k_1 \in K_X$.

Q.E.D.

We state some more properties of $\pi_{X, X_1, \lambda}$ which can be proved either by definitions or as corollaries of Proposition 5.16.

Proposition 5.17 1) *The embedding $(K_X/T, \pi_{X_1, \lambda}^X) \hookrightarrow (K/T, \pi_{X, X_1, \lambda})$ is a Poisson map;*

2) *With the Poisson structure π_K on K , the Poisson structure $\pi_{X_1, \lambda}^X$ on K_X/T and the Poisson structure $\pi_{X, X_1, \lambda}$ on K/T , the map*

$$m_1 : K \times (K_X/T) \longrightarrow K/T : (k, k'T) \longmapsto kk'T$$

is a Poisson map;

3) *Let $p_*\pi_K$ be the projection to K/K_X of π_K by $p : K \rightarrow K/K_X : k \mapsto kK_X$. Then the projection map $(K/T, \pi_{X, X_1, \lambda}) \rightarrow (K/K_X, p_*\pi_K)$ is a Poisson map.*

Remark 5.18 The Poisson structure $p_*\pi_K$ on K/K_X is known as the Bruhat-Poisson structure, because its symplectic leaves are exactly the Bruhat cells in K/K_X . See [Lu-We].

5.6 The symplectic leaves of $\pi_{X, X_1, \lambda}$

In this section, we first describe the symplectic leaves of $\pi_{X, X_1, \lambda}$ for any $X \subset S(\Sigma_+)$ but $X_1 = \emptyset$. The description of symplectic leaves for general $\pi_{X, X_1, \lambda}$ is somewhat complicated, and we will leave it to the future. However, we will show that each $\pi_{X, X_1, \lambda}$, for any X, X_1 and λ , has at least one open symplectic leaf.

Notation 5.19 We will use $\pi_{X, \emptyset, \lambda}$ to denote the Poisson structure $\pi_{X, X_1, \lambda}$ when X_1 is the empty set.

We first recall that the space K/T has the well-known Bruhat decomposition: Because of the Iwasawa decomposition $G = KAN$ of G , the natural map $K/T \rightarrow G/B : kT \mapsto kB$ is a diffeomorphism. Its inverse map is $G/B \rightarrow K/T : gB \mapsto kT$ if $g = kan$ is the Iwasawa decomposition of g . Thus we have

$$K/T \cong G/B = \bigcup_{w \in W} NwB$$

as a disjoint union. The set NwB is called the *Bruhat (or Schubert) cell* corresponding to $w \in W$. We denote it by Σ_w . For $w \in W$, set

$$\Phi_w = (-w\Sigma_+) \cap \Sigma_+ = \{\alpha \in \Sigma_+ : w^{-1}\alpha \in -\Sigma_+\}.$$

Set $\mathfrak{n}_w = \text{span}_{\mathbb{C}}\{E_\alpha : \alpha \in \Phi_w\}$ and $N_w = \exp \mathfrak{n}_w$. Then Σ_w is parametrized by N_w by the map

$$j_w : N_w \longrightarrow \Sigma_w : n \longmapsto nwB.$$

Define

$$\begin{aligned} j_1 &= G \longrightarrow K : g = kb \longmapsto k \quad \text{for } k \in K, b \in AN; \\ j_2 &= G \longrightarrow K : g = bk \longmapsto k \quad \text{for } k \in K, b \in AN. \end{aligned}$$

Then we have a left action of G on K by

$$G \times K \longrightarrow K : (g, k) \longmapsto g \circ k := j_1(gk),$$

and a right action of G on K :

$$K \times G \longrightarrow K : (k, g) \longmapsto k^g := j_2(kg).$$

The parametrization of Σ_w by N_w is then also given by

$$j_w : N_w \longrightarrow \Sigma_w : n \longmapsto (n \circ \dot{w})T,$$

where $\dot{w} \in K$ is any representative of w in K .

Notation 5.20 For $k \in K$ and a subgroup $G_1 \subset G$, we set

$$G_1 \circ k = \{g \circ k : g \in G_1\}, \quad k^{G_1} = \{k^g : g \in G_1\}.$$

It is easy to show that $(AN) \circ k = k^{AN}$ for any $k \in K$. This set is the symplectic leaf of π_K in K through the point k (see [So] [Lu-We]). Since $K_X \subset K$ is a Poisson submanifold, we know that $(AN) \circ k = k^{AN} \subset K_X$ for $k \in K_X$. Moreover, if $w \in W$ and if $\dot{w} \in K$ is a representative of w in K , set

$$C_{\dot{w}} = (AN) \circ \dot{w} \subset K.$$

Then

$$C_{\dot{w}} = (AN) \circ \dot{w} = N \circ \dot{w} = N_w \circ \dot{w} = \dot{w}^{AN} = \dot{w}^N = \dot{w}^{N_{w^{-1}}}. \quad (23)$$

Its image under the projection $K \rightarrow K/T$ is the Bruhat cell Σ_w , which is also the symplectic leaf of the Bruhat Poisson structure π_∞ in K/T . See [So] [Lu-We].

Let $X \subset S(\Sigma_+)$. Denote by W_X the subgroup of W generated by the simple reflections corresponding to elements in X . It is the Weyl group for $(\mathfrak{m}_X, \mathfrak{h})$. Introduce the subset W^X of W :

$$W^X = \{w \in W : \Phi_{w^{-1}} \subset \Sigma_+ \setminus [X]\}.$$

It follows from the definition that $w \in W^X$ if and only if $w([X] \cap \Sigma_+) \subset \Sigma_+$. Moreover, we have $C_{\dot{w}_1} = \dot{w}_1^{N_X}$ for $w_1 \in W^X$ because $N_{w_1^{-1}} \subset N_X$, where $N_X = \exp \mathfrak{n}_X$ with \mathfrak{n}_X given by (19). The following Lemma says that each $w_1 \in W^X$ is the minimal length representative for the coset $w_1 W_X$, and that the set W^X is a ‘‘cross section’’ for the canonical projection from W to the coset space W/W_X . For a proof of the Lemma, see [Ko], Prop. 5.13.

Lemma 5.21 *For any $w \in W$, there exists a unique $w_1 \in W^X$ and $w_2 \in W_X$ such that $w = w_1 w_2$. Moreover,*

$$\Phi_{w^{-1}} = \Phi_{w_2^{-1}} \cup w_2^{-1} \Phi_{w_1^{-1}}$$

is a disjoint union, and the components on the right hand side are the respective intersections of $\Phi_{w^{-1}}$ with $[X]$ and $\Sigma_+ \setminus [X]$. Hence, $l(w) = l(w_1) + l(w_2)$.

We can now describe the symplectic leaves of $\pi_{X,\emptyset,\lambda}$ in K/T .

Theorem 5.22 *1) For each $w_1 \in W^X$, the union $\bigcup_{w_2 \in W_X} \Sigma_{w_1 w_2}$ is the symplectic leaf of $\pi_{X,\emptyset,\lambda}$ in K/T through the point $w_1 \in K/T$.*

2) These are all the symplectic leaves of $\pi_{X,\emptyset,\lambda}$ in K/T .

Proof. Set

$$L_{X,\lambda} = e^\lambda K_X e^{-\lambda} N_X = N_X e^\lambda K_X e^{-\lambda}.$$

It is the connected subgroup of G with Lie algebra

$$\mathfrak{l}_{X,\lambda} = \text{Ad}_{e^\lambda}(\mathfrak{n}_X + \mathfrak{k}_X).$$

Notice that each $l \in L_{X,\lambda}$ can be written as a unique product $l = n_X e^\lambda k e^{-\lambda}$ for $n_X \in N_X$ and $k \in K_X$.

Denote by S_{w_1} the symplectic leaf of $\pi_{X,\emptyset,\lambda}$ through the point $w_1 \in K/T$. Pick a representative \dot{w}_1 of w_1 in K . By Theorem 7.2 of [Lu2] (see also [Ka1]), the symplectic leaf S_{w_1} is the image of the set $\dot{w}_1^{L_{X,\lambda}}$ under the projection $K \rightarrow K/T$. We define a map

$$M : L_{X,\lambda} \longrightarrow N_{w_1^{-1}} \times K_X$$

as follows: For $l = n_X e^\lambda k e^{-\lambda} \in L_{X,\lambda}$, write $ke^{-\lambda} = bk'$, where $b \in AU_X$ with $U_X = \exp u_X$ and $k' \in K_X$, so that $l = n_X e^\lambda bk'$. Since the map $N_{w_1^{-1}} \rightarrow C_{\dot{w}_1} : n \mapsto \dot{w}_1^n$ is a diffeomorphism, there exists a unique $n' \in N_{w_1^{-1}}$ such that $\dot{w}_1^{n'} = \dot{w}_1^{n_X} e^{\lambda b}$. Now define $M(l) = (n', k')$. It is easy to see that the map M is onto and that $\dot{w}_1^l = \dot{w}_1^{n'} k' \in C_{\dot{w}_1} K_X$. This shows that

$$\dot{w}_1^{L_{X,\lambda}} = C_{\dot{w}_1} K_X.$$

It is easy to show that the map

$$C_{\dot{w}_1} \times K_X \longrightarrow C_{\dot{w}_1} K_X : (c, k) \longmapsto ck$$

is a diffeomorphism, and that the image of $C_{\dot{w}_1} K_X$ to K/T under the projection $K \rightarrow K/T$ is the union $\bigcup_{w_2 \in W_X} \Sigma_{w_1 w_2}$, which is thus the symplectic leaf of the Poisson structure $\pi_{X,\emptyset,\lambda}$ through the point $w_1 \in K/T$. Now since

$$K/T = \bigcup_{w_1 \in W^X} S_{w_1}$$

is already a disjoint union, we conclude that the collection $\{S_{w_1} : w_1 \in W^X\}$ is that of all symplectic leaves of $\pi_{X,\emptyset,\lambda}$ in K/T .

Q.E.D.

Let $w_1 \in W^X$. The following proposition identifies the symplectic manifold $S_{w_1} = \bigcup_{w_2 \in W_X} \Sigma_{w_1 w_2}$, as a symplectic leaf of $\pi_{X,\emptyset,\lambda}$ in K/T , with the product of two symplectic manifolds. Recall that for $w \in W$ with a representative \dot{w} in K , the set $C_{\dot{w}} \subset K$ is the symplectic leaf of π_K through the point \dot{w} . Recall also from Notation 5.14 the definition of the Poisson structure $\pi_{\emptyset,\lambda}^X$ on K_X/T . Note that it is symplectic by Proposition 5.12.

Proposition 5.23 *Let $w_1 \in W^X$ and let \dot{w}_1 be a representative of w_1 in K . Equip $C_{\dot{w}_1}$ with the symplectic structure as a symplectic leaf of π_K in K ; Equip K_X/T with the symplectic structure $\pi_{\emptyset, \lambda}^X$, and finally, equip S_{w_1} with the symplectic structure as a symplectic leaf of $\pi_{X, \emptyset, \lambda}$. Then the map*

$$m_1 : C_{\dot{w}_1} \times K_X/T \longrightarrow S_{w_1} : (k, k'T) \longmapsto kk'T$$

is a diffeomorphism between symplectic manifolds.

Proof. This is a direct consequence of 2) in Proposition 5.17.

Q.E.D.

Among all the elements in W^X , there is one which is the longest. We denote this element by w^X , so $l(w^X) \geq l(w_1)$ for all $w_1 \in W^X$.

Proposition 5.24 *The symplectic leaf S_{w^X} of $\pi_{X, \emptyset, \lambda}$ in K/T through the point w^X is open and dense.*

Proof. Consider the projection $K/T \rightarrow K/K_X : kT \mapsto kK_X$. The image of $\Sigma_{w^X} \subset K/T$ under this projection is an open dense subset (in fact a cell) in K/K_X . Since $K/T \rightarrow K/K_X$ is a fibration, we know that S_{w^X} is open and dense in K/T .

Q.E.D.

Corollary 5.25 *Each Poisson structure $\pi_{X, \emptyset, \lambda}$ has a finite number of symplectic leaves with at least one of them open and dense.*

Remark 5.26 Note that the statement in Corollary 5.25 may not be true if $X_1 \neq \emptyset$, as is seen from case 3 of Example 5.4.

The description of the symplectic leaves of $\pi_{X, X_1, \lambda}$ in general is somewhat complicated. However, we have

Proposition 5.27 *The Poisson structure $\pi_{X, X_1, \lambda}$ for $X = S(\Sigma_+)$ (and $X_1 \subset X$ arbitrary) is non-degenerate at every element in the Weyl group W of (K, T) considered as a point in K/T . Consequently, the symplectic leaves of $\pi_{X, X_1, \lambda}$ through these points are open.*

Proof. Let $w \in W$ and let $\dot{w} \in K$ be a representative of w in K . Recall from the definition of $\pi_{X, X_1, \lambda}$ that $\pi_{X, X_1, \lambda} = p_* \tilde{\pi}_1$, where $p : K \rightarrow K/T$ is the natural projection and $\tilde{\pi}_1$ is the bi-vector field on K defined by

$$\tilde{\pi}_1 = \Lambda^R - A^L,$$

with $\Lambda = -\frac{i\varepsilon}{4} \sum_{\alpha \in \Sigma_+} X_\alpha \wedge Y_\alpha$ and

$$A = -\frac{i\varepsilon}{4} \sum_{\alpha \in \Sigma_+} \frac{e^{2\alpha(\lambda)} + 1}{e^{2\alpha(\lambda)} - 1} X_\alpha \wedge Y_\alpha.$$

Thus

$$\begin{aligned} l_{\dot{w}^{-1}} \tilde{\pi}_1(\dot{w}) &= \text{Ad}_{\dot{w}^{-1}} \Lambda - A \\ &= -\frac{i\varepsilon}{4} \left(\sum_{\alpha \in \Sigma_+} (X_{w^{-1}\alpha} \wedge Y_{w^{-1}\alpha}) + \sum_{\alpha \in \Sigma_+} \left(\frac{e^{2\alpha(\lambda)} + 1}{e^{2\alpha(\lambda)} - 1} X_\alpha \wedge Y_\alpha \right) \right) \\ &= -\frac{i\varepsilon}{4} \sum_{\alpha \in \Sigma_+, w\alpha < 0} \left(1 + \frac{e^{2\alpha(\lambda)} + 1}{e^{2\alpha(\lambda)} - 1} \right) X_\alpha \wedge Y_\alpha \\ &\quad - \frac{i\varepsilon}{4} \sum_{\alpha \in \Sigma_+, w\alpha > 0} \left(-1 + \frac{e^{2\alpha(\lambda)} + 1}{e^{2\alpha(\lambda)} - 1} \right) X_\alpha \wedge Y_\alpha. \end{aligned}$$

Since $\frac{e^{2\alpha(\lambda)} + 1}{e^{2\alpha(\lambda)} - 1} \neq \pm 1$, $l_{\dot{w}^{-1}} \pi_{X, X_1, \lambda}(\dot{w}T) = p_* l_{\dot{w}^{-1}} \tilde{\pi}_1(\dot{w}) \in \wedge^2 T_e(K/T)$ is non-degenerate. Hence $\pi_{X, X_1, \lambda}$ is non-degenerate at $w = \dot{w}T \in K/T$.

Q.E.D.

Corollary 5.28 *For any X, X_1 and λ , the Poisson structure $\pi_{X, X_1, \lambda}$ on K/T has at least one open symplectic leaf.*

Proof. We use Proposition 5.16 which says that $\pi_{X, X_1, \lambda}$ can be obtained via Poisson induction from the Poisson structure $\pi_{X_1, \lambda}^X$ on K_X/T . Recall the definition of $\pi_{X_1, \lambda}^X$ from Notation 5.14. Since X is the set of all simple roots for the root systems for (K_X, T) , we know from Proposition 5.27 that $\pi_{X_1, \lambda}^X$ is non-degenerate at every Weyl group element in W_X , regarded as points in K_X/T . Let $w_2 \in W_X$. Recall that w^X is the longest element in the set W^X . Let \dot{w}^X be any representative of w^X in K . Recall that $C_{\dot{w}^X}$ is the symplectic leaf of π_K in K through \dot{w}^X . By Proposition 5.17, the map

$$(C_{\dot{w}^X}, \pi_K) \times (K_X/T, \pi_{X_1, \lambda}^X) \longrightarrow (K/T, \pi_{X, X_1, \lambda}) : (k, k'T) \longmapsto kk'T$$

is a Poisson map. But this map is a diffeomorphism onto its image which is open because it is the inverse image under the natural projection $K/T \rightarrow K_X/T$ of the biggest cell in K_X/T . Thus the symplectic leaf of $\pi_{X, X_1, \lambda}$ through the point $\dot{w}^X w_2 \in K/T$ is open.

Q.E.D.

Note that the proof of Corollary 5.28 shows that $\pi_{x,x_1,\lambda}$ is open at every point in the coset $w^x W_x \subset K/T$.

Example 5.29 Corollary 5.28 can be checked directly for the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ by looking at the explicit formulas in Example 5.4.

5.7 The modular vector fields and the leaf-wise moment maps for the T -actions

For an orientable Poisson manifold (P, π) and a given volume form μ on P , the modular vector field of π associated to μ is defined to be the vector field v_μ on P satisfying $v_\mu \lrcorner \mu = d(\pi \lrcorner \mu)$. It measures how Hamiltonian flows on P fail to preserve μ . More details can be found in [W].

Coming back to (K, π_K) -homogeneous Poisson structures on K/T , we set $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha$ for the choice of Σ_+ in the definition of π_K . Then we have $iH_\rho \in \mathfrak{t}$. We use σ_{iH_ρ} to denote the infinitesimal generator of the T action on K/T by left translations in the direction of iH_ρ .

Proposition 5.30 *For the Poisson structure π_K on K defined by (14) with Λ given in (15), all (K, π_K) -homogeneous Poisson structures on K/T , and in particular all the $\pi_{x,x_1,\lambda}$'s, have the same modular vector field v , namely $v = -i\varepsilon\sigma_{iH_\rho}$, with respect to a (and thus any) K -invariant volume form on K/T .*

Remark 5.31 Proposition 5.30 is a statement about any Poisson Lie group structure on K since the Poisson structure π_K on K defined by (14) with Λ given in (15) is the most general form of such structures.

Proof of Proposition 5.30. Let π be an arbitrary (K, π_K) -homogeneous Poisson structure. Then we know that π is the sum

$$\pi = \pi(e)^L + p_*\pi_K,$$

where $\pi(e)^L$ is the K -invariant bi-vector field on K/T whose value at $e = eT$ is $\pi(e)$, and $p_*\pi_K$ is the projection of π_K from K to K/T by $p : K \rightarrow K/T : k \mapsto kT$ (it is the Bruhat Poisson structure π_∞ when $u = 0$ in the definition of Λ). Let μ be a K -invariant volume form on K/T . Let b_μ be the degree -1 operator on $\chi^\bullet(K/T)$ defined by $b_\mu(U) = (-1)^{|U|} d(U \lrcorner \mu)$, so that $v = b_\mu(\pi)$ [E-L-W]. Then $b_\mu(\pi) = b_\mu(\pi(e)^L) + b_\mu(p_*\pi_K)$. Since μ is K -invariant, the operator b_μ maps a K -invariant multi-vector field to another such. Hence $b_\mu(\pi(e)^L)$ must be a K -invariant (1-)vector field so it must be zero. Thus $b_\mu(\pi) = b_\mu(p_*\pi_K)$. It is proved in [E-L-W] that $b_\mu(p_*\pi_K) = -i\varepsilon\sigma_{iH_\rho}$, which is therefore the modular vector field for any π .

Q.E.D.

The modular vector field is always a Poisson vector field [W], but it is not necessarily Hamiltonian in general. For the rest of this section, we study this problem for the modular vector field $v = -i\varepsilon\sigma_{iH_\rho}$ for the Poisson structure $\pi_{X,\emptyset,\lambda}$. We will show that although v is not globally Hamiltonian unless $X = S(\Sigma_+)$, it is leaf-wise, and we describe its Hamiltonian function on each leaf. In fact, since every $\pi_{X,\emptyset,\lambda}$ is T -invariant (for the T -action on K/T by left translations), we will describe the moment map for the T -action on each symplectic leaf of $\pi_{X,\emptyset,\lambda}$. We are particularly interested in the behavior of these moment maps when λ goes infinity in various directions as in Section 5.2.

We first look at the Bruhat Poisson structure π_∞ corresponding to $X = \emptyset$. This case (when $\varepsilon = i$) is studied in detail in [Lu3]. We recall the results there. Denote by

$$P_A : G = KAN \longrightarrow A : g = kan \longmapsto a,$$

where $G = KAN$ is the Iwasawa decomposition of G (as a real Lie group). For each $w \in W$, choose a representative $\dot{w} \in K$ of w in K , and use

$$j_w : N_w \longrightarrow \Sigma_w : n \longmapsto (n \circ \dot{w})T$$

to parametrize the Bruhat cell Σ_w . For $n \in N_w$, let $a_w(n) = P_A(n\dot{w}) \in A$. The element $a_w(n)$ is independent of the choice of \dot{w} , so we have a well-defined map

$$a_w : N_w \longrightarrow A : n \longmapsto a_w(n).$$

Denote by Ω_w the symplectic structure on Σ_w as a symplectic leaf of π_∞ . Then each (Σ_w, Ω_w) is a Hamiltonian T -space. The following fact is proved in [Lu3].

Proposition 5.32 *The map*

$$\phi_w : \Sigma_w \longrightarrow \mathfrak{t}^* : \langle \phi_w, x \rangle(kT) = \frac{2i}{\varepsilon} \text{Im} \ll \text{Ad}_{\dot{w}} \log a_w(j_w^{-1}(kT)), x \gg, \quad x \in \mathfrak{t}$$

is the moment map for the T -action on (Σ_w, Ω_w) such that $\phi_w(w) = 0$.

In [Lu3], we have written down an explicit formula for ϕ_w in certain Bott-Samelson type coordinates $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_{l(w)}, \bar{z}_{l(w)}\}$. It takes the form

$$\langle \phi_w, x \rangle = -\frac{1}{\varepsilon} \sum_{j=1}^{l(w)} \frac{2\alpha_j(x)}{\ll \alpha_j, \alpha_j \gg} \log(1 + |z_j|^2)$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_{l(w)}\} = \Sigma_+ \cap (-w\Sigma_+)$. In particular, let $x = -i\varepsilon(iH_\rho) = \varepsilon H_\rho$, we get a Hamiltonian function for the vector field $v = -i\varepsilon\sigma_{iH_\rho}$ on (Σ_w, Ω_w) as

$$\langle \phi_w, \varepsilon H_\rho \rangle = - \sum_{j=1}^{l(w)} \frac{2 \ll \rho, \alpha_j \gg}{\ll \alpha_j, \alpha_j \gg} \log(1 + |z_j|^2).$$

This function goes to $-\infty$ as $|z_j| \rightarrow \infty$ which corresponds to the boundary of Σ_w . Thus, the modular vector field v can not be globally Hamiltonian on K/T .

Next, we look at the case when $X = S(\Sigma_+)$, so $\pi_{X, \emptyset, \lambda} = \pi_\lambda$ is the the symplectic structure on K/T obtained by identifying K/T with the dressing orbit in the group AN through the point $e^{-\lambda}$ (see Proposition 5.12). Since K/T is simply connected, The T -action on K/T is Hamiltonian. The following fact is proved in [L-R].

Proposition 5.33 *The moment map for the T -action on $(K/T, \pi_\lambda)$ is given by*

$$\Phi_\lambda : K/T \longrightarrow \mathfrak{t}^* : \langle \Phi_\lambda, x \rangle(kT) = \frac{2i}{\varepsilon} \text{Im} \ll \log(P_A(ke^{-\lambda}k^{-1})), x \gg, \quad x \in \mathfrak{t}.$$

Remark 5.34 This fact plays the key role in the symplectic proof of Kostant's nonlinear convexity theorem given in [L-R].

Corresponding to the fact that $\lim_{t \rightarrow +\infty} \pi_{\lambda+t\check{\rho}} = \pi_\infty$, where $\check{\rho}$ is the sum of all fundamental coweights, the two moment maps are related as follows.

Proposition 5.35 *For any $\lambda \in \mathfrak{a}, w \in W$ and $kT \in \Sigma_w$,*

$$\begin{aligned} \lim_{t \rightarrow +\infty} (\Phi_{\lambda+t\check{\rho}}(kT) - \Phi_{\lambda+t\check{\rho}}(w)) &= \phi_w(kT) \\ \lim_{t \rightarrow +\infty} d\Phi_{\lambda+t\check{\rho}}(kT) &= d\phi_w(kT). \end{aligned}$$

Proof. Using the parametrization of Σ_w by N_w , we regard both $\Phi_{\lambda+t\check{\rho}}|_{\Sigma_w}$ and ϕ_w as (\mathfrak{t}^* -valued) functions on N_w . Let $n \in N_w$ with $k = n \circ \dot{w}$. Write

$$n\dot{w} = ka_w(n)m(n)$$

with $m(n) \in N_w$. Then

$$e^{-\lambda}k^{-1} = (e^{-\lambda}a_w(n)m(n)a_w(n)^{-1}e^{\lambda}\dot{w}^{-1})(\dot{w}e^{-\lambda}a_w(n)\dot{w}^{-1})n^{-1}.$$

Thus, for any $x \in \mathfrak{t}$,

$$\begin{aligned} &\langle \Phi_{\lambda+t\check{\rho}}(n) - \Phi_{\lambda+t\check{\rho}}(e) - \phi_w(n), x \rangle \\ &= \frac{2i}{\varepsilon} \text{Im} \ll \log P_A(e^{-\lambda-t\check{\rho}}a_w(n)m(n)a_w(n)^{-1}e^{\lambda+t\check{\rho}}\dot{w}^{-1}), x \gg, \end{aligned}$$

where $e \in N_w$ is the identity element. Consider now the map

$$\psi_t : N_w \longrightarrow N_w : m \longmapsto e^{-\lambda-t\tilde{\rho}} m e^{\lambda+t\tilde{\rho}}.$$

Under the identification of \mathfrak{n}_w with N_w by the exponential map of N_w , this is the linear map $\text{Ad}_{-\lambda-t\tilde{\rho}}$ on \mathfrak{n}_w , which goes to 0 as $t \rightarrow +\infty$. Thus

$$\lim_{t \rightarrow +\infty} \psi_t(m) = 0, \quad \text{and} \quad \lim_{t \rightarrow +\infty} d\psi_t(m) = 0$$

for all $m \in N_w$. But we have the composition of maps

$$\langle \Phi_{\lambda+t\tilde{\rho}}(n) - \Phi_{\lambda+t\tilde{\rho}}(e) - \phi_w(n), x \rangle = \eta_x(\psi_t(\xi(n))),$$

where $\eta_x : N_w \rightarrow \mathbb{R} : m \mapsto \frac{2i}{\varepsilon} \text{Im} \ll \log P_A(m\dot{w}^{-1}), x \gg$ and $\xi : N_w \rightarrow N_w : n \mapsto a_w(n)m(n)a_w(n)^{-1}$. Thus the two limits in Proposition 5.35 hold.

Q.E.D.

Now consider the general case of $\pi_{x,\theta,\lambda}$. Recall that the symplectic leaves of $\pi_{x,\theta,\lambda}$ in K/T are indexed by elements in W^X . We keep the notation in Proposition 5.23, in which we have used the map m_1 to identify the symplectic leaf S_{w_1} of $\pi_{x,\theta,\lambda}$ in K/T with the product symplectic manifold $C_{\dot{w}_1} \times K_X/T$. We use the projection map $C_{\dot{w}_1} \rightarrow \Sigma_{w_1} : k \mapsto kT$ to identify $C_{\dot{w}_1}$ and Σ_{w_1} . This identification is T -equivariant if we equip $C_{\dot{w}_1}$ with the T -action

$$T \times C_{\dot{w}_1} \longrightarrow C_{\dot{w}_1} : t \cdot k \longmapsto tk(\dot{w}_1^{-1}t^{-1}\dot{w}_1).$$

Equip $C_{\dot{w}_1} \times K_X/T$ with the T -action

$$T \times (C_{\dot{w}_1} \times K_X/T) \longrightarrow C_{\dot{w}_1} \times K_X/T : t \cdot (k, k'T) \longmapsto (tk(\dot{w}_1^{-1}t^{-1}\dot{w}_1), \dot{w}_1^{-1}t\dot{w}_1k'T).$$

Then the map m_1 in Proposition 5.23 is T -equivariant. Denote by $\Phi_{\lambda,x}$ the moment map for the T -action on $(K_X/T, \pi_{\theta,\lambda}^x)$. Then the moment map for the T -action on $S_{w_1} \cong C_{\dot{w}_1} \times K_X/T$ is given by

$$\langle \phi_{\lambda,x,w_1}(k, k'T), x \rangle = \langle \phi_{w_1}(kT), x \rangle + \langle \Phi_{\lambda,x}(k'T), \text{Ad}_{\dot{w}_1^{-1}}x \rangle$$

for all $x \in \mathfrak{t}$.

Remark 5.36 There remain many problems to be addressed concerning the Poisson structures $\pi_{x,x_1,\lambda}$. Other than the description of their symplectic leaves in the general case, one can try to compute its Poisson cohomology according to the theory developed in [Lu2]. One can also study the K -invariant Poisson harmonic forms [E-L1] of $\pi_{x,x_1,\lambda}$. Another problem is to construct the symplectic groupoids for $\pi_{x,x_1,\lambda}$. We hope to treat these problems in the future.

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