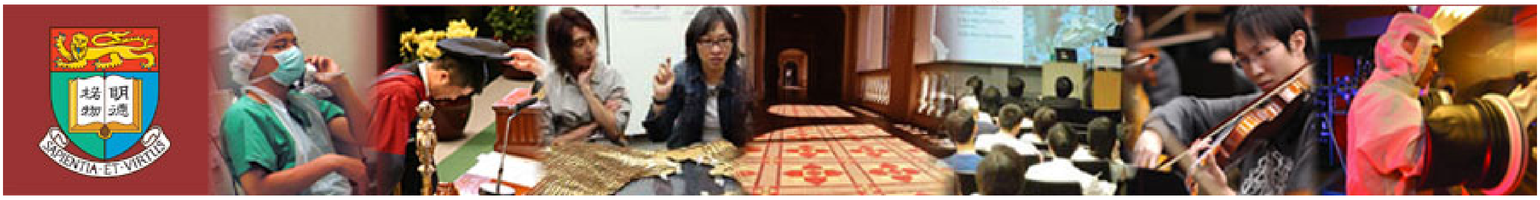

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ON THE POSITIVITY OF POLYNOMIALS ON THE COMPLEX UNIT DISC VIA LMIS

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ABSTRACT

Investigating positivity of polynomials over the complex unit disc is a relevant problem in electrical and computer engineering. This paper provides two sufficient and necessary conditions for solving this problem via linear matrix inequalities (LMIs). These conditions are obtained by exploiting trigonometric transformations, a key tool for the representation of polynomials, and results from the theory of positive polynomials. Some numerical examples illustrate the proposed conditions.

***Index Terms*—** Linear system, Discrete time, Transfer function, Positivity, LMI.

1. INTRODUCTION

It is well-known that an useful tool for studying a linear system consists of investigating its frequency response, i.e. the restriction of the transfer function of the system onto the stability boundary in the complex plane, which is the imaginary axis in the case of continuous-time systems or the unit disc in the case of discrete-time systems, see e.g. [1] and references therein. Numerous results based on the frequency response have been provided in the literature through the years, for stability and performance analysis, control synthesis, etc. Some of these results involve the positivity of a polynomial over the stability boundary, for instance because this can be used to detect and impose stability.

This paper proposes two conditions based on linear matrix inequalities (LMIs) for investigating positivity of polynomials over the complex unit disc. First, a sufficient and necessary condition is proposed for the case of a generic polynomial, which requires to solve a convex optimization problem, in particular an eigenvalue problem (also known as semidefinite program). Second, another sufficient and necessary condition is proposed for the case of polynomials with real coefficients, which requires to solve another eigenvalue problem in a smaller number of variables. These conditions are obtained by exploiting trigonometric transformations, a key tool for the representation of polynomials, and results from the theory of positive polynomials. Some numerical examples illustrate the proposed conditions.

The paper is organized as follows. Section 2 introduces the problem formulation and some preliminaries on the representation of polynomials. Section 3 describes the proposed results. Section 4 presents some illustrative examples. Lastly, Section 5 concludes the paper with some final remarks.

2. PRELIMINARIES

2.1. Problem Formulation

The notation used throughout the paper is as follows: $\mathbb{R}$ ($\mathbb{C}$): space of real (complex) numbers; j: imaginary unit, i.e. $j = \sqrt{-1}$; $\Re(a)$ ($\Im(a)$): real (imaginary) part of a; a^*: complex conjugate of a, i.e. $a^* = \Re(a) - j\Im(a)$; A': transpose of A; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A; $\star$: symmetric entry.

Let $f : \mathbb{C} \to \mathbb{C}$ be defined as

$$f(z) = \sum_{k=-d}^{d} a_k z^k \tag{1}$$

where d is a nonnegative integer that defines the degree of $f(z)$ and $a_{-d}, \ldots, a_d \in \mathbb{C}$ are the coefficients of $f(z)$. We assume that these coefficients satisfy

$$a_{-k} = a_k^* \;\; \forall k = 0, \ldots, d \tag{2}$$

hence implying that $f(z)$ is real on the complex unit disc.

Problem. The problem that we consider in this paper consists of establishing whether $f(z)$ is positive on the complex unit disc, i.e.

$$f\left(e^{j\omega}\right) > 0 \;\; \forall \omega \in \mathbb{R}. \tag{3}$$

2.2. Representation of Polynomials

Before proceeding we briefly introduce a key tool that will be exploited in the next sections. Let $p : \mathbb{R}^n \to \mathbb{R}$ a polynomial

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of degree less than or equal to $2m$. Then, $p(x)$ can be expressed via the square matrix representation (SMR) [2] (also known as Gram matrix method) as

$$p(x) = x^{\{m\}'} (P + L(\alpha)) x^{\{m\}} \tag{4}$$

where $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ is a vector containing all monomials of degree equal to m in x, where $\sigma(n,m)$ is the number of such monomials given by

$$\sigma(n,m) = \frac{(n+m-1)!}{(n-1)!m!}, \tag{5}$$

$P \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a symmetric matrix, and $L(\alpha) \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a linear parametrization of the set

$$\mathcal{L}(n,m) = \{L = L' : \ x^{\{m\}'} L x^{\{m\}} = 0\} \tag{6}$$

where $\alpha \in \mathbb{R}^{\tau(n,m)}$ is a vector of free parameters, being $\tau(n,m)$ the dimension of $\mathcal{L}(n,m)$ given by

$$\tau(n,m) = \frac{1}{2}\sigma(n,m)(\sigma(n,m)+1) - \sigma(n,2m). \tag{7}$$

The representation (4) was introduced in [2] in order to establish whether $p(x)$ is a sum of squares of polynomials (SOS) through linear matrix inequalities (LMIs). Specifically, $p(x)$ is SOS if there exist polynomials $p_1(x), p_2(x), \ldots$ such that

$$p(x) = \sum_i p_i(x)^2, \tag{8}$$

and by using the SMR one has that $p(x)$ is SOS if and only if there exists α such that

$$P + L(\alpha) \geq 0. \tag{9}$$

Condition (9) is a linear matrix inequality (LMI) feasibility test, which amounts to solving a convex optimization problem. See e.g. [3] and references therein for details about SOS polynomials. See also [4] for algorithms for the construction of the matrices P and $L(\alpha)$.

3. PROPOSED RESULTS

The first condition proposed in this paper is obtained by recalling that

$$e^{j\omega} = \frac{1+jt}{1-jt}, \ \ t = \tan\frac{\omega}{2} \tag{10}$$

whenever $\omega \neq \pi + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$. Thus, it follows that

$$f\left(e^{j\omega}\right) = \sum_{k=-d}^{d} a_k \left(\frac{1+jt}{1-jt}\right)^k. \tag{11}$$

Consequently, after some calculations we obtain that

$$f\left(e^{j\omega}\right) = \frac{q(t)}{(1+t^2)^d} \tag{12}$$

where

$$q(t) = a_0\left(1+t^2\right)^d + r(t) \tag{13}$$

with

$$r(t) = \sum_{k=1}^{d}\sum_{i=0}^{2k}\sum_{l=0}^{d-k} b_{k,i}\frac{(d-k)!}{l!(d-k-l)!}t^{i+2l} \tag{14}$$

and

$$b_{k,i} = \begin{cases} 2\Re(a_k)\dfrac{(-1)^{i/2}(2k)!}{i!(2k-i)!} & \text{if } i \text{ even} \\ 2\Im(a_k)\dfrac{(-1)^{(i+1)/2}(2k)!}{i!(2k-i)!} & \text{if } i \text{ odd.} \end{cases} \tag{15}$$

Next, let us express $q(t)$ via the SMR as

$$q(t) = t^{\{d\}'} (Q + L_q(\alpha)) t^{\{d\}} \tag{16}$$

where Q is a symmetric matrix and $L_q(\alpha)$ is a linear parametrization of the subspace $\mathcal{L}(1,d)$, see Section 2.2 and references therein for details about their construction. Let O_q be a matrix satisfying

$$1 = t^{\{d\}'} O_q t^{\{d\}}. \tag{17}$$

The following result provides a sufficient and necessary condition for establishing whether (3) holds.

Theorem 1 *Define the optimization problem*

$$\begin{array}{rcl} \mu^* & = & \sup\limits_{\alpha,\mu} \ \mu \\ & & \textit{s.t.} \ \ Q + L_q(\alpha) - \mu O_q \geq 0. \end{array} \tag{18}$$

The condition (3) *is satisfied if and only if* $f(-1) > 0$ *and* $\mu^* > 0$.

Proof. "$\Leftarrow$" Suppose that $f(-1) > 0$ and $\mu^* > 0$. Let α^* be the optimal value of α in (18). One has that

$$Q + L_q(\alpha^*) - \mu^* O_q \geq 0.$$

Pre- and post-multiply this constraint by $t^{\{d\}'}$ and $t^{\{d\}}$, respectively. One gets:

$$\begin{array}{rcl} 0 & \leq & t^{\{d\}'} (Q + L_q(\alpha^*) - \mu^* O_q) t^{\{d\}} \\ & = & q(t) - \mu^* \end{array}$$

i.e. μ^* is a lower bound of $q(t)$. Therefore, $q(t)$ is positive, and from (12) and $f(-1) > 0$ we conclude that (3) is satisfied.

"$\Rightarrow$" Suppose that (3) is satisfied. This means that $f(-1) > 0$ and $q(t) > 0$ for all $t \in \mathbb{R}$. Hence, it follows that $\mu^\# > 0$ where

$$\mu^\# = \inf_t \ q(t).$$

Since $q(t) - \mu^{\#}$ is nonnegative and t is a scalar, it follows that $q(t) - \mu^{\#}$ is SOS, see e.g. [3] and references therein, and hence there exists α such that

$$Q + L_q(\alpha) - \mu^{\#} O_q \geq 0.$$

Therefore, for such an α and for $\mu = \mu^{\#}$, the constraint in (18) holds, i.e. $\mu^* \geq \mu^{\#} > 0$. □

Theorem 1 provides a sufficient and necessary condition for establishing whether (3) holds based on the solution of the optimization problem (18). This optimization problem belongs to the class of eigenvalue problems [5] (also known as semidefinite programs) and is a convex optimization problem since the cost function is convex (linear in this case) and the feasible set is convex (feasible set of an LMI in this case). The number of scalar variables in (18) is given by

$$n_1 = \frac{d^2 - d + 2}{2}. \tag{19}$$

The second condition proposed in this paper assumes that the coefficients of $f(z)$ are real, i.e.

$$\Im(a_k) = 0 \ \ \forall k = -d, \ldots, d. \tag{20}$$

This clearly implies that

$$f\left(e^{-j\omega}\right) = f\left(e^{j\omega}\right) \ \ \forall \omega \in \mathbb{R}. \tag{21}$$

By using the fundamental relation

$$e^{j\omega} = \cos\omega + j\sin\omega \tag{22}$$

it follows that

$$f\left(e^{j\omega}\right) = \sum_{k=-d}^{d} \Re(a_k)\cos(\omega k) - \Im(a_k)\sin(\omega k) \tag{23}$$

i.e.

$$f\left(e^{j\omega}\right) = a_0 + \sum_{k=1}^{d} 2\left(\Re(a_k)\cos(\omega k) - \Im(a_k)\sin(\omega k)\right). \tag{24}$$

By defining

$$x = \cos\omega \tag{25}$$

and using trigonometric formulas, one has that

$$\cos(\omega k) = \sum_{l\geq 0,\ 2l\leq k} \frac{(-1)^l k! x^{k-2l}(1-x^2)^l}{(2l)!(k-2l)!}. \tag{26}$$

Consequently, we obtain that

$$f\left(e^{j\omega}\right) = s(x) \tag{27}$$

where

$$s(x) = a_0 + \sum_{k=1}^{d} \sum_{l\geq 0,\ 2l\leq k} c_{k,l} x^{k-2l}(1-x^2)^l \tag{28}$$

and

$$c_{k,l} = 2\Re(a_k)\frac{(-1)^l k!}{(2l)!(k-2l)!}. \tag{29}$$

Next, let $u(x)$ be an auxiliary polynomial, and let us define

$$v(x) = s(x) + u(x)(x^2 - 1). \tag{30}$$

Let us express $u(x)$ and $v(x)$ as

$$\begin{array}{rcl} u(x) & = & x^{\{d_1-1\}'} U x^{\{d_1-1\}} \\ v(x) & = & x^{\{d_1\}'} \left(V(U) + L_v(\beta)\right) x^{\{d_1\}} \end{array} \tag{31}$$

where

$$d_1 = \left\lceil \frac{d}{2} \right\rceil, \tag{32}$$

U and $V(U)$ are symmetric matrices with $V(U)$ depending affine linearly on U, and $L_v(\beta)$ is a linear parametrization of the subspace $\mathcal{L}(1, d_1)$, see Section 2.2 and references therein for details about their construction. Let O_v be a matrix satisfying

$$1 = x^{\{d_1\}'} O_v x^{\{d_1\}}. \tag{33}$$

The following result provides a sufficient condition for establishing whether (3) holds.

Theorem 2 *Assume that $d > 0$ and that (20) holds. Define the optimization problem*

$$\begin{array}{rcl} \nu^* & = & \sup\limits_{U,\beta,\nu} \ \nu \\ & & \text{s.t.} \ \left\{ \begin{array}{l} U \geq 0 \\ V(U) + L_v(\beta) - \nu O_v \geq 0. \end{array} \right. \end{array} \tag{34}$$

The condition (3) is satisfied if and only if $\nu^* > 0$.

Proof. "$\Leftarrow$" Suppose that $\nu^* > 0$. Let U^* and β^* be the optimal values of U and β in (34). One has that

$$\begin{array}{l} U^* \geq 0 \\ V(U^*) + L_v(\beta^*) - \nu^* O_v \geq 0. \end{array}$$

Pre- and post-multiply the first constraint by $x^{\{d_1-1\}'}$ and $x^{\{d_1-1\}}$, respectively. One gets:

$$\begin{array}{rcl} 0 & \leq & x^{\{d_1-1\}'} U^* x^{\{d_1-1\}} \\ & = & u(x) \end{array}$$

i.e. $u(x)$ is nonnegative. Similarly, from the second constraint one gets

$$0 \leq s(x) + u(x)(x^2 - 1) - \nu^*.$$

These two facts imply that, for any $x \in [-1, 1]$,

$$s(x) \geq \nu^*$$

i.e. ν^* is a lower bound of $s(x)$ over $[-1, 1]$. Therefore, from (27) we conclude that (3) is satisfied.

"⇒" Suppose that (3) is satisfied. Define the minimum of $s(x)$ over $[-1,1]$ as

$$\nu^{\#} = \inf_{x\in[-1,1]} s(x).$$

This means that $\nu^{\#} > 0$. Moreover, $s(x)-\nu^{\#}$ is nonnegative over $[-1,1]$. Hence, there exists a SOS polynomial $s(x)$ of degree $2(d_1-1)$ such that $s(x)+u(x)(x^2-1)-\nu^{\#}$ is also SOS, see e.g. [3] and references therein, and hence there exist U and β such that

$$V(U)+L_v(\beta)-\nu^{\#}O_v \geq 0.$$

Therefore, for such a U and β and for $\nu=\nu^{\#}$, the constraints in (34) hold, i.e. $\nu^* \geq \nu^{\#} > 0$. □

Theorem 2 provides a sufficient and necessary condition for establishing whether (3) holds in the case that the coefficients of $f(z)$ are real. This condition is based on the optimization problem (34) which belongs to the class of eigenvalue problems similarly to the one in Theorem 1. The number of scalar variables in (34) is given by

$$n_2 = \begin{cases} \dfrac{d^2+4}{4} & \text{if } d \text{ is even} \\ \dfrac{d^2+2d+5}{4} & \text{if } d \text{ is odd} \end{cases} \quad (35)$$

4. EXAMPLES

This section provides two illustrative examples of the proposed conditions. The eigenvalue problems (18) and (34) are solved with the toolbox SeDuMi for Matlab [6].

4.1. Example 1

Let us consider

$$f(z) = (1+2j)z^{-2}+z^{-1}+5+z+(1-2j)z^2.$$

We have that

$$f\left(e^{j\omega}\right) = 5+2\cos\omega+2\cos(2\omega)-4\sin(2\omega).$$

Let us use Theorem 1. The polynomial $q(t)$ is given by

$$q(t) = 9+16t-2t^2-16t^3+5t^4.$$

By choosing $t^{\{d\}} = (1,t,t^2)$, the matrices in (18) are

$$Q = \begin{pmatrix} 9 & 8 & -1 \\ \star & 0 & -8 \\ \star & \star & 5 \end{pmatrix}, \quad L_q(\alpha) = \begin{pmatrix} 0 & 0 & -\alpha \\ \star & 2\alpha & 0 \\ \star & \star & 0 \end{pmatrix}$$

$$O_q = \begin{pmatrix} 1 & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix}.$$

It turns out that $\mu^* = -19.61$, and hence from Theorem 1 we conclude that (3) does not hold.

4.2. Example 2

Let us consider

$$f(z) = 3z^{-15}+2z^{-2}-z^{-1}+11-z+2z^2+3z^{15}.$$

We have that

$$f\left(e^{j\omega}\right) = 11-2\cos\omega+4\cos(2\omega)+6\cos(15\omega).$$

Since (20) holds we can use Theorem 2. The polynomial $s(x)$ is given by

$$\begin{aligned} s(x) = {} & 7-92x+8x^2+3360x^3-36288x^5+172800x^7 \\ & -422400x^9+552960x^{11}-368640x^{13}+98304x^{15}. \end{aligned}$$

It turns out that $\nu^* = 0.878$, and hence from Theorem 2 we conclude that (3) holds. The same conclusion can be found by using Theorem 1. However, the number of scalar variables in (18) is $n_1 = 106$, while the number of scalar variables in (34) is just $n_2 = 65$.

5. CONCLUSIONS

This paper has provided two sufficient and necessary conditions for investigating positivity of polynomials over the complex unit disc. These conditions are based on eigenvalue problems, which belong to the class of convex optimization problems, and have been obtained by exploiting trigonometric transformations, a key tool for the representation of polynomials, and results from the theory of positive polynomials.

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7. REFERENCES

[1] L. Qiu and K. Zhou, *An Introduction to Feedback Control*, Prentice Hall, 2010.

[2] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *European Control Conf.*, Karlsruhe, Germany, 1999.

[3] G. Chesi, "LMI techniques for optimization over polynomials in control: a survey," *IEEE Trans. on Automatic Control*, vol. 55, no. 11, pp. 2500–2510, 2010.

[4] G. Chesi, *Domain of Attraction: Analysis and Control via SOS Programming*, Springer, 2011.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[6] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11-12, pp. 625–653, 1999.