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Sharpness of Wilker and Huygens type inequalities

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Abstract

We present an elementary proof of Wilker's inequality involving trigonometric functions, and establish sharp Wilker and Huygens type inequalities.
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1. Introduction

Wilker in [1] proposed two open problems:

(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

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(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

for $0 < x < \pi/2$.

In [2], inequality (1) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x \quad \text{for } 0 < x < \frac{\pi}{2}, \tag{2}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are best possible, was also established.

Wilker type inequalities (1) and (2) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [2-13] and the references cited therein). A brief survey of some old and new inequalities associated with trigonometric functions can be found in [14]. These include (among other results) Wilker's inequality.

Another inequality which is of interest to us is Huygens [15] inequality, which asserts that

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \quad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{3}$$

Neuman and Sándor [16] have pointed out that (3) implies (1). In [17], Zhu established some new inequalities of the Huygens type for trigonometric and hyperbolic functions. Baricz and Sándor [18] pointed out that inequalities (1) and (3) are simple consequences of the arithmetic-geometric mean inequality together with the well-known Lazarević-type inequality [[19], p. 238]

$$(\cos x)^{1/3}<\frac{\sin x}{x} \quad \text{for all} \quad 0<|x|<\frac{\pi}{2}, \tag{4}$$

or equivalently,

$$\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}>1 \quad \text{for all} \quad 0<|x|<\frac{\pi}{2}.$$

Wu and Srivastava [[7], Lemma 3] established another inequality

$$\left(\frac{x}{\sin}\right)^2+\frac{x}{\tan x}>2 \ \text{ for all} \quad 0<|x|<\frac{\pi}{2}. \tag{5}$$

In [20], Chen and Cheung showed that Wilker inequality (1), Huygens inequality (3), Lazarević-type inequality (4) and Wu-Srivastava inequality (5) can be grouped into the following inequality chain:

$$\begin{aligned}\frac{(\sin x/x)^2+\tan x/x}{2}&>\frac{2(\sin x/x)+\tan x/x}{3}>\sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}>1\\&>\frac{2}{1/(\sin x/x)^2+1/(\tan x/x)}, \quad 0<|x|<\frac{\pi}{2},\end{aligned} \tag{6}$$

in terms of the arithmetic, geometric and harmonic means.

In this article, we present an elementary proof of Wilker's inequality (2), and we establish sharp Wilker and Huygens type inequalities.

The following elementary power series expansions are useful in our investigation.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \qquad |x|<\infty, \tag{7}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \qquad |x|<\infty, \tag{8}$$

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)(-1)^{n-1}B_{2n}}{(2n)!}x^{2n-1}, \qquad |x|<\frac{\pi}{2}, \tag{9}$$

$$x\cot x=\sum_{n=0}^{\infty}(-1)^nB_{2n}\frac{(2x)^{2n}}{(2n)!}, \qquad 0<|x|<\pi. \tag{10}$$

where B_n (n = 0, 1,2,...) are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The following lemma is also needed in the sequel.

Lemma 1. [21]*Let* $a_n \in \mathbb{R}$ *and* $b_n > 0$, $n = 0,1, 2,...$ *be real numbers with* $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *being strictly increasing (respectively, decreasing). If the power series* $B(x) := \sum_{n=0}^{\infty} b_n x^n$ *and* $B(x) := \sum_{n=0}^{\infty} b_n x^n$ *are convergent for* $|x| < R$*, then the function* $A(x)/B(x)$ *is strictly increasing (respectively, decreasing) on* $(0, R)$.

2. An elementary proof of Wilker's inequality (2)

Proof of (2). Consider the function

$$f(x) := \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}, \quad 0 < x < \frac{\pi}{2}.$$

By using power series expansions (7) and (8), we obtain

$$\begin{aligned}
-x^6 \sin^2 x f'(x) &= -3x^2 \sin(2x) + 5\sin^3 x \cos x + 5x - 2x^3 + 2x\cos^4 x - 7x\cos^2 x \\
&= \left(-3x^2 + \frac{5}{4}\right)\sin(2x) - \frac{5}{8}\sin(4x) - 2x^3 + \frac{9}{4}x + \frac{1}{4}x\cos(4x) - \frac{5}{2}x\cos(2x) \\
&= \sum_{n=4}^{\infty} (-1)^n u_n(x) \\
&= \frac{16}{945}x^9 - \frac{16}{1575}x^{11} + \frac{16}{7425}x^{13} - \frac{11072}{42567525}x^{15} + \sum_{n=8}^{\infty} (-1)^n u_n(x),
\end{aligned}$$

where

$$u_n(x) := \frac{((2n-9)4^{n-1} + 6n^2 - 2n)4^n}{(2n+1)!} x^{2n+1}.$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \ge 8$,

$$\begin{aligned}
\frac{u_{n+1}(x)}{u_n(x)} &= 8x^2 \frac{6n^2 + 10n + 4 + (2n-7)4^n}{(n+1)(2n+3)(24n^2 - 8n + (2n-9)4^n)} \\
&< 8\left(\frac{\pi}{2}\right)^2 \frac{6n^2 + 10n + 4 + (2n-7)4^n}{(n+1)(2n+3)(24n^2 - 8n + (2n-9)4^n)} \\
&= \frac{\pi^2}{2n+3} \frac{12n^2 + 20n + 8 + 2(2n-7)4^n}{(n+1)(24n^2 - 8n + (2n-9)4^n)}.
\end{aligned}$$

Write

$$\alpha_n := \frac{\pi^2}{2n+3} \frac{12n^2 + 20n + 8 + 2(2n-7)4^n}{(n+1)(24n^2 - 8n + (2n-9)4^n)}.$$

It is easy to see that for $n \ge 8$,

$$\frac{12n^2 + 20n + 8 + 2(2n-7)4^n}{(n+1)(24n^2 - 8n + (2n-9)4^n)} < 1.$$

Hence for all $0 < x < \pi/2$ and $n \geq 8$,

$$\frac{u_{n+1}(x)}{u_n(x)} < \alpha_n < \frac{\pi^2}{2n+3} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \mapsto u_n(x)$ is strictly decreasing with regard to $n \geq 8$. Hence, for $0 < x < \pi/2$,

$$\begin{aligned}-x^6\sin^2 x f'(x) &> \frac{16}{945}x^9 - \frac{16}{1575}x^{11} + \frac{16}{7425}x^{13} - \frac{11072}{42567525}x^{15}\\ &= x^9\left(\frac{16}{945} - \frac{16}{1575}x^2 + \frac{16}{7425}x^4 - \frac{11072}{42567525}x^6\right)\\ &> 0.\end{aligned}$$

Hence $f(x)$ is strictly decreasing on $(0, \pi/2)$. Noting that $\lim_{t\to 0^+} f(t) = \frac{8}{45}$, we have

$$\frac{16}{\pi^4} = \lim_{t\to(\pi/2)^-} f(t) < f(x) = \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3\tan x} < \lim_{t\to 0^+} f(t) = \frac{8}{45}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$, with the constants $\frac{16}{\pi^4}$ and $\frac{8}{45}$ being best possible. This completes the proof of (2).

3. Sharp Wilker's inequality

By using power series expansions (8) and (9), we have, for $0 < x < \pi/2$,

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &= \frac{1}{2x^2} = -\frac{1}{2x^2}\cos(2x) + \frac{\tan x}{x}\\ &= 2 + \sum_{k=3}^{\infty}\frac{(2(2^{2k}-1)\,|B_{2k}| - (-1)^k)2^{2k-1}}{(2k)!}x^{2k-2}.\end{aligned}$$

It is well known [[22], p. 805] that

$$\frac{2(2k)!}{(2\pi)^{2k}} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}(1-2^{1-2k})}, \quad k \geq 1. \tag{11}$$

By (11), we find that

$$2(2^{2k}-1)\,|B_{2k}| > 2(2^{2k}-1)\frac{2(2k)!}{(2\pi)^{2k}} > 1, \quad k \geq 3.$$

Hence, we have

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &> 2 + \sum_{k=3}^{n}\frac{(2(2^{2k}-1)\,|B_{2k}| - (-1)^k)2^{2k-1}}{(2k)!}x^{2k-2}\\ &= 2 + \frac{8}{45}x^4 + \frac{16}{315}x^6 + \frac{104}{4725}x^8 + \frac{592}{66825}x^{10}\\ &\quad + \cdots + \frac{(2(2^{2n}-1)\,|B_{2n}| - (-1)^n)2^{2n-1}}{(2n)!}x^{2n-2}.\end{aligned} \tag{12}$$

Motivated by (12), we are now in a position to establish our first main result.

Theorem 1. (i) *For* $0 < x < \pi/2$, *we have*

$$2+\frac{8}{45}x^4+\frac{16}{315}x^5\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\frac{8}{45}x^4+\left(\frac{2}{\pi}\right)^6x^5\tan x. \qquad (13)$$

The constants $\frac{16}{315}$ *and* $\left(\frac{2}{\pi}\right)^6$ *are best possible.*

(ii) *For* $0 < x < \pi/2$, *we have*

$$\begin{aligned}2+\frac{8}{45}x^4+\frac{16}{315}x^6+\frac{104}{4725}x^7\tan x&<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\\&<2+\frac{8}{45}x^4+\frac{16}{315}x^6+\left(\frac{2}{\pi}\right)^8x^7\tan x.\end{aligned} \qquad (14)$$

The constants $\frac{104}{4725}$ *and* $\left(\frac{2}{\pi}\right)^8$ *are best possible.*

Proof. We only prove inequality (13). The proof of (14) is analogous.
Consider the function

$$\begin{aligned}g(x):&=\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2-\frac{8}{45}x^4}{x^5\tan x}\\&=\frac{\sin(2x)}{2x^7}+\frac{1}{x^6}-\frac{2\cot x}{x^5}-\frac{8\cot x}{45x}, \quad 0<x<\frac{\pi}{2}.\end{aligned}$$

By using power series expansions (7) and (10), we find that

$$g(x)=\sum_{n=3}^{\infty}\beta_n 2^{2n-1}x^{2n-6},$$

where

$$\beta_n:=\frac{4\cdot|B_{2n}|}{(2n)!}+\frac{\left|B_{2(n-2)}\right|}{45\cdot(2n-4)!}+(-1)^n\frac{2}{(2n+1)!}, \quad n\ge 3.$$

By (11), we obtain

$$\begin{aligned}\frac{4\cdot|B_{2n}|}{(2n)!}+\frac{\left|B_{2(n-2)}\right|}{45\cdot(2n-4)!}&>\frac{4}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}+\frac{1}{45\cdot(2n-4)!}\frac{2(2n-4)!}{(2\pi)^{2n-4}}\\&=\frac{180+2\cdot(2\pi)^4}{45\cdot(2\pi)^{2n}}.\end{aligned}$$

By induction on n, it is easy to see that

$$\frac{180+2\cdot(2\pi)^4}{45\cdot(2\pi)^{2n}}>\frac{2}{(2n+1)!} \quad \text{for all } n\ge 3.$$

Hence $\beta_n > 0$ for $n \geq 3$, and we have

$$g'(x) = \sum_{n=4}^{\infty} b_n 2^{2n-1}(2n-6)x^{2n-7} > 0, \quad 0 < x < \frac{\pi}{2}.$$

Therefore, $g(x)$ is strictly increasing on $(0, \pi/2)$. Noting that $\lim_{t \to (\pi/2)^-} g(t) = \left(\frac{2}{\pi}\right)^6$, we have

$$\frac{16}{315} = \lim_{t \to 0^+} g(t) < g(x) = \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 - \frac{8}{45}x^4}{x^5 \tan x} < \lim_{t \to (\pi/2)^-} g(t) = \left(\frac{2}{\pi}\right)^6$$

for all $x \in \left(0, \frac{\pi}{2}\right)$, with the constants $\frac{16}{315}$ and $\left(\frac{2}{\pi}\right)^6$ being best possible. This completes the proof of (13).

Remark 1. *Inequality* (14) *is sharper than* (13). *On the other hand, there is no strict comparison between inequalities* (2) *and* (13). *There is no strict comparison between inequalities* (2) *and* (14) *either.*

In view of inequalities (13) and (14), we propose the following conjecture.

Conjecture 1. For $0 < x < \pi/2$ *and* $n \geq 3$, *we have*

$$\begin{aligned}
&2 + \sum_{k=3}^{n} \frac{(2(2^{2k}-1)\,|B_{2k}| - (-1)^k)2^{2k-1}}{(2k)!}x^{2k-2} \\
&\quad + \frac{(2(2^{2n+2}-1)\,|B_{2n+2}| - (-1)^{n+1})2^{2n+1}}{(2n+2)!}x^{2n-1}\tan x \\
&< \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \\
&< 2 + \sum_{k=3}^{n} \frac{(2(2^{2k}-1)\,|B_{2k}| - (-1)^k)2^{2k-1}}{(2k)!}x^{2k-2} + \left(\frac{2}{\pi}\right)^{2n} x^{2n-1}\tan x.
\end{aligned}$$

4. Sharp the Wu-Srivastava inequality

By using power series expansion (10), we obtain for $0 < x < \pi/2$,

$$\csc^2 x = -(\cot x)' = \frac{1}{x^2} + \sum_{k=1}^{\infty} \frac{2^{2k}(2k-1\,|B_{2k}|)}{(2k)!}x^{2k-2}. \tag{15}$$

Hence for $0 < x < \pi/2$,

$$\begin{aligned}
\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} &= x^2\csc^2 x + x\cot x \\
&= 2 + \sum_{k=2}^{\infty} \frac{2^{2k+1}(k-1)\,|B_{2k}|}{(2k)!}x^{2k} \\
&= 2 + \frac{2}{45}x^4 + \frac{8}{945}x^6 + \frac{2}{1575}x^8 + \frac{16}{93555}x^{10} + \cdots .
\end{aligned} \tag{16}$$

Motivated by (16), we establish our second main result:

Theorem 2. (i) *For* $0 < x < \pi/2$, *we have*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3 \tan x. \tag{17}$$

The constant $\frac{2}{45}$ *is best possible.*

(ii) *For* $0 < x < \pi/2$, *we have*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^4 + \frac{8}{945}x^5 \tan x. \tag{18}$$

The constant $\frac{8}{945}$ *is best possible.*

Proof. We only prove inequality (18). The proof of (17) is analogous.

Consider the function

$$G(x) := \frac{\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \frac{2}{45}x^4}{x^5 \tan x} = \frac{A(x)}{B(x)},$$

where

$$\begin{aligned} A(x) : &= \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \frac{2}{45}x^4 \\ &= \sum_{n=1}^{\infty} \frac{2^{2n+5}(n+1)\left|B_{2(n+2)}\right|}{(2n+4)!}x^{2n+4} = \sum_{n=1}^{\infty} a_n x^{2n+4} \end{aligned}$$

with

$$a_n := \frac{2^{2n+5}(n+1)\left|B_{2(n+2)}\right|}{(2n+4)!},$$

and

$$B(x) := x^5 \tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n+4} = \sum_{n=1}^{\infty} b_n x^{2n+4}$$

with

$$b_n := \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}.$$

We claim that the function $G(x)$ is strictly decreasing on $(0, \pi/2)$. By Lemma 1, it suffices to show that

$$\frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}}, \quad n \geq 1. \tag{19}$$

It is known [[23], p. 96] that

$$\frac{2 \cdot (2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{\pi^2(2n)!}{3(2\pi)^{2n}}, \quad n \geq 1. \tag{20}$$

By using (20), we obtain

$$\frac{a_n}{b_n} = \frac{2^5(n+1)\left|B_{2(n+2)}\right|}{(2n+4)!} \cdot \frac{(2n)!}{(2^{2n}-1)\left|B_{2n}\right|} > \frac{192(n+1)}{\pi^2(2\pi)^4(4^n-1)}$$

and

$$\frac{a_{n+1}}{b_{n+1}} = \frac{2^5(n+2)\left|B_{2(n+3)}\right|}{(2n+6)!} \cdot \frac{(2n+2)!}{(2^{2n+2}-1)\left|B_{2(n+1)}\right|} < \frac{16\pi^2(n+2)}{3(2\pi)^4(4^{n+1}-1)}.$$

So (19) is a consequence of the elementary inequality

$$\frac{192(n+1)}{\pi^2(2\pi)^4(4^n-1)} > \frac{16\pi^2(n+2)}{3(2\pi)^4(4^{n+1}-1)},$$

which is equivalent to

$$\frac{36(n+1)}{4^n-1} > \frac{\pi^4(n+2)}{4^{n+1}-1}, \quad n \geq 1. \tag{21}$$

The proof of the inequality (21) is not difficult, and is left with the readers. This proves the claim.

Noting that $\lim_{t\to 0^+} G(t) = \frac{8}{945}$, we have

$$G(x) < \lim_{t\to 0^+} G(t) = \frac{8}{945} \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right)$$

with the constant $\frac{8}{945}$ being best possible. This completes the proof of (18).

In view of inequalities (17) and (18), we propose the following conjecture.

Conjecture 2. *For* $0 < x < \pi/2$ *and* $n \geq 1$,

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \sum_{k=2}^{n} \frac{(k-1)\cdot 2^{2k+1}\left|B_{2k}\right|}{(2k)!}x^{2k} + \frac{n \cdot 2^{2n+3}\left|B_{2(n+1)}\right|}{(2n+2)!}x^{2n+1}\tan x. \tag{22}$$

5. Skarp Huygens inequality

By using power series expansions (7) and (9), for $0 < x < \pi/2$, we have

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} = 3 + \sum_{k=3}^{\infty}\left(\frac{2^{2k}(2^{2k}-1)\left|B_{2k}\right|}{4k} - (-1)^k\right)\frac{2x^{2k-2}}{(2k-1)!}.$$

By (11), we find that

$$\frac{2^{2k}(2^{2k}-1)\left|B_{2k}\right|}{4k} > \frac{2^{2k}(2^{2k}-1)}{4k}\frac{2(2k)!}{(2\pi)^{2k}} = \frac{(2^{2k}-1)\cdot(2k-1)!}{\pi^{2k}} > 1, \quad k \geq 3.$$

Hence we have

$$\begin{aligned}2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x} &> 3+\sum_{k=3}^{n}\left(\frac{2^{2k}(2^{2k}-1)\,|B_{2k}|}{4k}-(-1)^k\right)\frac{2x^{2k-2}}{(2k-1)!}\\ &= 3+\frac{3}{20}x^4+\frac{3}{56}x^6+\frac{7}{320}x^8+\frac{3931}{443520}x^{10}\\ &\quad+\cdots+\left(\frac{(2^{2n}(2^{2n}-1)\,|B_{2n}|)}{4n}-(-1)^n\right)\frac{2x^{2n-2}}{(2n-1)!}.\end{aligned}\tag{23}$$

Motivated by (23), we establish our third main result:

Theorem 3. (i) *For* $0 < x < \pi/2$, *we have*

$$3+\frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x} < 3+\left(\frac{2}{\pi}\right)^4 x^3\tan x.\tag{24}$$

The constants $\frac{3}{20}$ *and* $\left(\frac{2}{\pi}\right)^4$ *are best possible.*

(ii) *For* $0 < x < \pi/2$, *we have*

$$3+\frac{3}{20}x^4+\frac{3}{56}x^5\tan x < 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x} < 3+\frac{3}{20}x^4+\left(\frac{2}{\pi}\right)^6 x^5\tan x.\tag{25}$$

The constants $\frac{3}{56}$ *and* $\left(\frac{2}{\pi}\right)^6$ *are best possible.*

Proof. We only prove inequality (25). The proof of (24) is analogous.

Consider the function

$$\begin{aligned}h(x) := \frac{2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}-3-\frac{3}{20}x^4}{x^5\tan x}\\ = \frac{2\cos x}{x^6}+\frac{1}{x^6}-\frac{3\cot x}{x^5}-\frac{3\cot x}{20x},\quad 0<x<\frac{\pi}{2}.\end{aligned}$$

By using power series expansions (8) and (10), we find that

$$h(x)=\sum_{n=3}^{\infty}c_n x^{2n-6},$$

where

$$c_n := \frac{3\cdot 2^{2n}\,|B_{2n}|}{(2n)!}+\frac{3\cdot 2^{2n-4}\left|B_{2(n-2)}\right|}{20\cdot(2n-4)!}+(-1)^n\frac{2}{(2n)!},\quad n\geq 3.$$

By (11), we obtain

$$\begin{aligned}\frac{3\cdot 2^{2n}\,|B_{2n}|}{(2n)!}+\frac{3\cdot 2^{2n-4}\left|B_{2(n-2)}\right|}{20\cdot(2n-4)!} &> \frac{3\cdot 2^{2n}}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}+\frac{3\cdot 2^{2n-4}}{20\cdot(2n-4)!}\frac{2(2n-4)!}{(2\pi)^{2n-4}}\\ &= \frac{60+3\cdot\pi^4}{10\cdot\pi^{2n}}.\end{aligned}$$

By induction on n, it is easy to show that

$$\frac{60 + 3 \cdot \pi^4}{10 \cdot \pi^{2n}} > \frac{2}{(2n)!} \quad \text{for all } n \geq 3.$$

Hence $c_n > 0$ for $n \geq 3$, and we have

$$h'(x) = \sum_{n=4}^{\infty} c_n (2n - 6) x^{2n-7} > 0 \quad \text{for all } 0 < x < \frac{\pi}{2}.$$

Therefore, $h(x)$ is strictly increasing on $(0, \pi/2)$. Noting that $\lim_{t \to 0^+} h(t) = \frac{3}{56}$ and $\lim_{t \to (\pi/2)-} h(t) = \left(\frac{2}{\pi}\right)^6$, we have

$$\frac{3}{56} = \lim_{t \to 0^+} h(t) < h(x) = \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3 - \frac{3}{20}x^4}{x^5 \tan x} < \lim_{t \to (\pi/2)^-} h(t) = \left(\frac{2}{\pi}\right)^6$$

for all $x \in \left(0, \frac{\pi}{2}\right)$ with the constants $\frac{3}{56}$ and $\left(\frac{2}{\pi}\right)^6$ being possible. This completes the proof of (25).

Remark 2. *There is no strict comparison between inequalities* (24) *and* (25).

In view of inequalities (24) and (25), we propose the following conjecture.

Conjecture 3. *For* $0 < x < \pi/2$ *and* $n \geq 2$, *we have*

$$\begin{aligned}
& 3 + \sum_{k=3}^{n} \left(\frac{2^{2k}(2^{2k} - 1) |B_{2k}|}{4k} - (-1)^k \right) \frac{2}{(2k-1)!} x^{2k-2} \\
& + \left(\frac{2^{2n+2}(2^{2n+2} - 1) |B_{2n+2}|}{4(n+1)} - (-1)^{n+1} \right) \frac{2}{(2n+1)!} x^{2n-1} \tan x \\
& < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} \\
& < 3 + \sum_{k=3}^{n} \left(\frac{2^{2k}(2^{2k} - 1) |B_{2k}|}{4k} - (-1)^k \right) \frac{2}{(2k-1)!} x^{2k-2} + \left(\frac{2}{\pi}\right)^{2n} x^{2n-1} \tan x.
\end{aligned} \tag{26}$$

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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