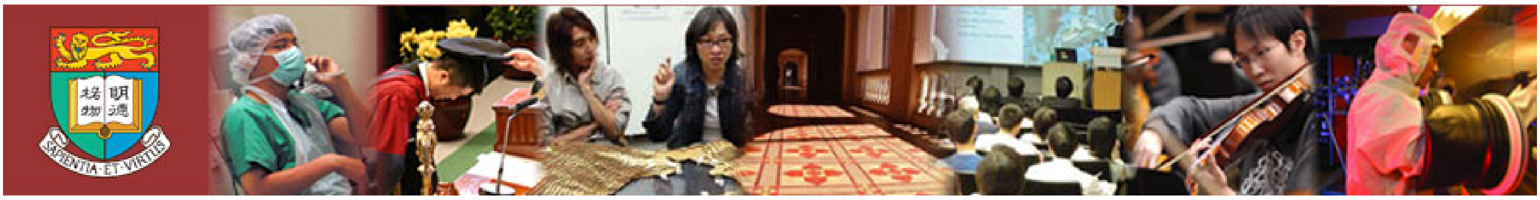

Title	On a class of stochastic models with two-sided jumps
Author(s)	Cheung, ECK
Citation	Queueing Systems, 2011, v. 69 n. 1, p. 1-28
Issued Date	2011
URL	http://hdl.handle.net/10722/145095
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Queueing Syst (2011) 69:1–28
DOI 10.1007/s11134-011-9228-z

On a class of stochastic models with two-sided jumps

Eric C.K. Cheung

Received: 22 October 2010 / Published online: 17 May 2011

Abstract In this paper a stochastic process involving two-sided jumps and a continuous downward drift is studied. In the context of ruin theory, the model can be interpreted as the surplus process of a business enterprise which is subject to constant expense rate over time along with random gains and losses. On the other hand, such a stochastic process can also be viewed as a queueing system with instantaneous work removals (or negative customers). The key quantity of our interest pertaining to the above model is (a variant of) the Gerber–Shiu expected discounted penalty function (Gerber and Shiu in N. Am. Actuar. J. 2(1):48–72, 1998) from ruin theory context. With the distributions of the jump sizes and their inter-arrival times left arbitrary, the general structure of the Gerber–Shiu function is studied via an underlying ladder height structure and the use of defective renewal equations. The components involved in the defective renewal equations are explicitly identified when the upward jumps follow a combination of exponentials. Applications of the Gerber–Shiu function are illustrated in finding (i) the Laplace transforms of the time of ruin, the time of recovery and the duration of first negative surplus in the ruin context; (ii) the joint Laplace transform of the busy period and the subsequent idle period in the queueing context; and (iii) the expected total discounted reward for a continuous payment stream payable during idle periods in a queue.

Keywords Dual risk model · Two-sided jumps · $GI/G/1$ queue · Negative customers · Gerber–Shiu function · Defective renewal equation · Time of ruin · Time of recovery · Busy period · Idle period

Mathematics Subject Classification (2000) 60K25 · 91B30 · 60J75 · 90B05 · 60K15

E.C.K. Cheung (✉)
Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam, Hong Kong, Hong Kong
e-mail: eckc@hku.hk

1 Introduction and preliminaries

Stochastic processes involving two-sided jumps have received a great deal of attention in recent years. Various first passage times, two-sided exit problems, overshoots/undershoots and some other related quantities have been studied under different model assumptions. In particular, in a compound Poisson process with linear deterministic decrease between jumps, some one-sided and two-sided exit problems were considered by [44] when the upward and/or downward jumps belong to certain classes of phase-type distributions. In the class of jump-diffusion models where the jumps form a compound Poisson process, Kou and Wang [34], Asmussen et al. [5] and Cai [17] studied models with double exponential jumps, phase-type two-sided jumps and mixed-exponential two-sided jumps, respectively. In addition, Breuer [14] studied a Markov additive process with positive jumps of phase-type, while Breuer [16] derived a quintuple law for a Markov additive process with phase-type two-sided jumps. Furthermore, for an Ornstein–Uhlenbeck process driven by a Lévy process, various models with two-sided jumps have been studied, for example by [11, 33, 43].

In the context of risk processes with two-sided jumps, various ruin-related quantities have been studied mostly in the form of the Gerber–Shiu expected discounted penalty function (or its variant) proposed by [31]. While downward jumps are interpreted as random losses suffered by the company (typically claims in insurance context), upward jumps can be regarded as random gains earned by the company (for instance lump sum premium income in insurance context). Any continuous drift represents deterministic income rate if positive and deterministic expense rate if negative. With regards to some recent literature, Jacobsen [32] studied the joint Laplace transform of the time of ruin and the deficit at ruin for a class of Markov additive risk processes where the downward jumps have rational Laplace transform, while Xing et al. [51] considered the special case of a perturbed compound Poisson risk model with phase-type downward jumps. Cai et al. [18] evaluated the Geber–Shiu function (in which the penalty function only depends on the deficit at ruin) in the compound Poisson risk model with two-sided exponential jumps, and applied it to the pricing of a perpetual American put option. Until most recently Breuer [15] considered a generalization of the Gerber–Shiu function where the penalty function further involves the minimum surplus level before ruin in a Markov additive risk process with phase-type two-sided jumps. The particular case of the compound Poisson risk model in the absence of the drift term has also been studied extensively by various authors. For example, ruin probability results can be found in [10, 41, 48]. More generally, Labbé and Sendova [36] extended the study by showing that the Gerber–Shiu function satisfies a defective renewal equation and providing detailed analysis when the upward jumps follow an Erlang(n) distribution, whereas Albrecher et al. [2] considered the Gerber–Shiu function (where the penalty depends on the deficit only) by making assumptions on either upward or downward jumps. We also refer interested readers to [7–9] for the study of risk models in which the two-sided jumps follow discrete distributions.

In queueing theory, stochastic processes with two-sided jumps can be interpreted as queueing systems with ordinary customer (or workload) arrivals and instantaneous work removal causing the upward and downward jumps, respectively. The ideas of instantaneous work removals first came from [27, 28] in which the arrival of a 'negative customer' reduces the number of customers in the queue (if any) by 1. Various variants of such a removal policy, such as batch removals, have also been proposed. The policy of individual customer removal was generalized to a random work removal policy by [12] in which the arrival of a negative customer results in instantaneous removal of a random amount of workload from the system. The quantities of interest in the above-mentioned queueing system include, for example, the equilibrium queue length (number of customers), the equilibrium workload and the Laplace transform of the busy period. Interested readers are referred to [3] for a comprehensive review regarding work removals in queueing networks.

In most of the work in the literature regarding stochastic processes with two-sided jumps, the arrivals of jumps are governed by a Poisson process or more generally a Markovian arrival process, which means that the times between jump arrivals are exponentially or phase-type distributed. However, not much work has been done regarding processes in which the arrivals of jumps follow a general renewal process. This motivates us to study models with general inter-arrival times. We remark that notable recent work allowing for such relaxation includes Labbé et al. [35] and Zhang and Yang [52], where the authors studied the Gerber–Shiu function via defective renewal equations when the drift is positive. In this paper we study a variant of the Gerber–Shiu function in ruin theory in cases where the drift is assumed to be negative. Unlike the aforementioned model with positive drift, in the present model the system can possibly become empty due to the continuous drift component or a negative jump, and these cases need to be distinguished. Nonetheless, we shall see that even if the distributions of the inter-arrival times and the two-sided jumps are left arbitrary, defective renewal equations can still be exploited to study the Gerber–Shiu function. Although in most part of the paper terminology in ruin theory context will be used, queueing applications will also be discussed.

The model of interest in this paper is described as follows. The surplus process of a business enterprise with initial surplus $U(0)=u\geq 0$ is denoted by $\{U(t)\}_{t\geq 0}$, where

$$U(t)=u-ct+\sum_{i=1}^{N(t)}Y_i,\qquad t\geq 0. \tag{1.1}$$

Here $c>0$ is the constant rate of expenses per unit time, $\{Y_i\}_{i=1}^{\infty}$ is the sequence of jumps with Y_i the size of the ith jump, and $\{N(t)\}_{t\geq 0}$ is a counting process defined via $\{V_i\}_{i=1}^{\infty}$ with V_1 being the time of the first jump and V_i the time between the $(i-1)$th and the ith jumps for $i=2,3,\dots$. In addition, $\{V_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ are assumed to be mutually independent i.i.d. (independent and identically distributed) sequences distributed as the generic r.v.'s (random variables) V and Y, respectively. We further assume that V is a positive continuous r.v. with density $k(\cdot)$ possessing finite mean, while Y is continuous with density

$$p(y)=q_+p_+(y)1\{y>0\}+q_-p_-(-y)1\{y<0\},$$

with $1\{\cdot\}$ being the indicator function. In the above expression, $p_+(\cdot)$ and $p_-(\cdot)$ are the densities of the generic positive r.v.'s Y_+ and Y_- denoting upward and downward jumps, respectively, and therefore q_+ and q_- represent the probabilities that Y is positive and negative, respectively, with $q_+ + q_- = 1$. We always assume $q_- > 0$, since the case $q_- = 0$ has been studied by [20]. We also assume $q_+ > 0$ to avoid monotonically decreasing sample paths. It is additionally assumed that both Y_+ and Y_- have finite mean. The positive security loading condition is given by

$$E[Y] = q_+ E[Y_+] - q_- E[Y_-] > cE[V], \tag{1.2}$$

which may or may not be assumed (see Remark 1). If (1.2) holds, then $\{U(t)\}_{t\geq 0}$ would approach infinity in the long run (see e.g. [45], Chap. 1). The time of ruin pertaining to the risk process $\{U(t)\}_{t\geq 0}$ is given by $T_U = \inf\{t \geq 0 : U(t) \leq 0\}$, with $T_U = \infty$ if $U(t) > 0$ for all $t \geq 0$. Note that $U(T_U) = 0$ if ruin is caused by the continuous negative drift (called 'creeping'), whereas $U(T_U) < 0$ if ruin is caused by downward jumps. Moreover, if $U(0) = 0$ then ruin occurs immediately by continuity with $T_U = 0$. We remark that it will become clear that it is necessary to distinguish between the two causes of ruin due to their different natures. The situation is like an analog to risk models perturbed by Brownian motion (see for example [25]).

In cases where $q_- = 0$, downward jumps are absent in the process (1.1). Such a model is appropriate for companies incurring expenses at a fixed rate while gains (i.e. positive jumps) occur randomly in both time and amount. According to [6], for example, these include pharmaceutical and petroleum companies, where each gain can be regarded as the net present value of future income resulting from an invention or discovery. Furthermore, Seal [47, p. 116] explained that such a process might also be suitable for annuity or pension funds, where the insurance company pays annuities and earns a portion of the reserves when a policyholder dies. More generally, with negative jumps possible, (1.1) further allows for the modeling of any unexpected random losses. These losses are evident in many business operations, and they can be, for example, corrections of previous overstatement of gains, losses resulting from defaults of contracts by counter-parties, and costs associated with a lawsuit against the company. In particular, lawsuit may arise for a pharmaceutical company when there are safety issues about its drugs, whereas petroleum companies could possibly face a lawsuit due to oil spills.

The Gerber–Shiu function proposed by [31] not only unified but also generalized the study of various ruin-related quantities. In the present model, the Gerber–Shiu function for ruin occurring upon downward jumps is given by

$$\begin{aligned} &m_{\delta,U,-}(u) \\ &\quad = E\bigl[e^{-\delta T_U} w\bigl(U\bigl(T_U^-\bigr), \bigl|U(T_U)\bigr|\bigr) 1\bigl\{T_U < \infty, U(T_U) < 0\bigr\} | U(0) = u\bigr], \quad u \geq 0, \end{aligned} \tag{1.3}$$

where $w(\cdot,\cdot)$ is the so-called penalty function satisfying some mild integrability conditions, and $\delta \geq 0$ can either be viewed as a force of interest or a Laplace transform argument. Here the r.v.'s $U(T_U^-)$ and $|U(T_U)|$, respectively, represent the surplus just prior to ruin and the deficit at ruin which are both defined at (the neighborhood of) the time of ruin. The Laplace transform of the time of ruin due to jumps can be retrieved from (1.3) by letting $w(\cdot,\cdot) \equiv 1$. In contrast, for ruin occurring by continuity,

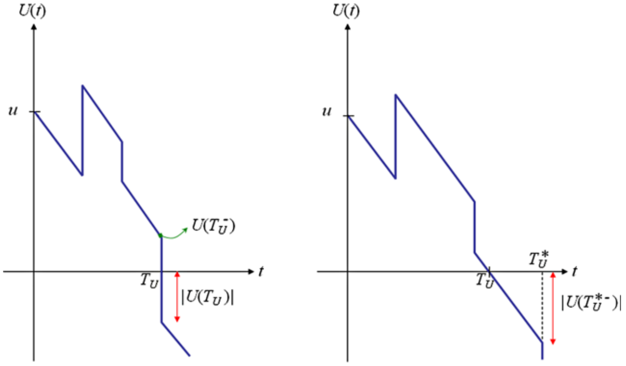

Fig. 1 (**a**) Ruin of $U(t)$ due to jumps. (**b**) Ruin of $U(t)$ due to continuity

we follow the ideas in [20] to define a Gerber–Shiu type function as follows. We first define the r.v.'s

$$T_U^* = \sum_{i=1}^{N(T_U)+1} V_i \tag{1.4}$$

and

$$\left|U\left(T_U^{*-}\right)\right| = \left|u + \sum_{i=1}^{N(T_U)} (Y_i - cV_i) - cV_{N(T_U)+1}\right| = \left|u + \sum_{i=1}^{N(T_U)} Y_i - cT_U^*\right|. \tag{1.5}$$

Clearly, T_U^* is the time of the first jump after the time of ruin T_U, while $|U(T_U^{*-})|$ is the absolute value of the amount of shortfall immediately before the first jump after ruin. Then we define the Gerber–Shiu type function for ruin occurring by continuity as

$$m_{\delta,U,0}(u) = E\left[e^{-\delta T_U^*} w_*\left(\left|U\left(T_U^{*-}\right)\right|\right) 1\left\{T_U < \infty, U(T_U) = 0\right\} | U(0) = u\right], \quad u \geq 0, \tag{1.6}$$

where $w_*(\cdot)$ satisfies some mild integrable conditions. Note that in contrast to the usual Gerber–Shiu function (1.3), the quantities T_U^* and $|U(T_U^{*-})|$ are now defined after the time of ruin. Moreover, if $U(T_U) = 0$, then $T_U = T_U^* - |U(T_U^{*-})|/c$. Therefore if we let $w_*(y) = e^{(\delta/c)y}$, then $m_{\delta,U,0}(u)$ reduces to the Laplace transform of the time of ruin by continuity. Typical sample paths for ruin by jumps and by continuity are given in Fig. 1.

In a $GI/G/1$ queue or in the theory of storage process like the dam theory, the amount of workload in the queue or the amount of water in the dam is generally non-negative. However, it is clear that the process $\{U(t)\}_{t\geq 0}$ can still be applied to analyze the above queue, as long as $0 \leq t \leq T_U$. Here an arrival of an upward (downward) jump Y_+ (Y_-) represents the random amount of workload a positive (negative) customer brings into (removes from) the system, while the constant negative drift represents the continuous removal of workload by the server. The so-called busy period of such a queue corresponds to the time of ruin T_U in the present model. See, for example, [13, 44] for more detailed descriptions of related models.

Remark 1 As far as the Gerber–Shiu functions (1.3) and (1.6) are concerned, we only need to assume the positive security loading condition (1.2) when $\delta = 0$; i.e. we do not require (1.2) to hold if $\delta > 0$. The reason is that either (1.2) or $\delta = 0$ is sufficient to ensure the strict inequality in (2.25) which in turn guarantees that the Markov renewal equation (2.22) has a unique solution (see end of Sect. 2). Note that in queueing systems it is usually assumed that the traffic intensity is less than 1, which is actually the condition (1.2) with the inequality sign reversed. The above comments mean that all the results in this paper relevant to the queueing context (see Sects. 4.2 and 4.3) hold true as long as $\delta > 0$.

Central to the analysis of the Gerber–Shiu functions (1.3) and (1.6) is the auxiliary process $\{Z(t)\}_{t\geq 0}$ defined as follows. We assume that at time 0 the process $\{Z(t)\}_{t\geq 0}$ is subject to a positive jump distributed as Y_+, after which it behaves like an independent copy of $\{U(t)\}_{t\geq 0}$. Mathematically, if the initial level of $\{Z(t)\}_{t\geq 0}$ is $Z(0^-) = z$, then

$$\{Z(t)|Z(0^-) = z\}_{t\geq 0} \stackrel{d}{=} \{U(t)|U(0) = z + Y_+\}_{t\geq 0}. \tag{1.7}$$

The time of ruin, surplus prior to ruin and deficit at ruin are similarly defined as $T_Z = \inf\{t \geq 0 : Z(t) \leq 0\}$, $Z(T_Z^-)$ and $|Z(T_Z)|$, respectively. Parallel to the r.v.'s defined by (1.4) and (1.5) for $\{U(t)\}_{t\geq 0}$, we also define $(T_Z^*|Z(0^-) = z) \stackrel{d}{=} (T_U^*|U(0) = z + Y_+)$ and $(|Z(T_Z^{*-})||Z(0^-) = z) \stackrel{d}{=} (|U(T_U^{*-})||U(0) = z + Y_+)$ pertaining to $\{Z(t)\}_{t\geq 0}$. Of course, if $Z(T_Z) = 0$ then $T_Z = T_Z^* - |Z(T_Z^{*-})|/c$. Analogous to (1.3) and (1.6), the Gerber–Shiu functions of $\{Z(t)\}_{t\geq 0}$ for ruin by downward jumps and by continuity are, respectively, given by

$$m_{\delta,Z,-}(z) = E\big[e^{-\delta T_Z} w\big(Z\big(T_Z^-\big), \big|Z(T_Z)\big|\big) 1\big\{T_Z < \infty, Z(T_Z) < 0\big\}|Z\big(0^-\big) = z\big] \tag{1.8}$$

$$= E\big[m_{\delta,U,-}(z + Y_+)\big], \quad z \geq 0, \tag{1.9}$$

and

$$m_{\delta,Z,0}(z) = E\big[e^{-\delta T_Z^*} w_*\big(\big|Z\big(T_Z^{*-}\big)\big|\big) 1\big\{T_Z < \infty, Z(T_Z) = 0\big\}|Z\big(0^-\big) = z\big] \tag{1.10}$$

$$= E\big[m_{\delta,U,0}(z + Y_+)\big], \quad z \geq 0. \tag{1.11}$$

For the remainder of the paper several operators will be used and they are introduced as follows. First, the Dickson–Hipp operator $\mathcal{T}_s$ (see [23]) is defined as, for any integrable function $f(\cdot)$ on $(0, \infty)$ and any complex number s with $\mathrm{Re}(s) \geq 0$,

$$\mathcal{T}_s f(y) = \int_y^\infty e^{-s(x-y)} f(x)\,dx = \int_0^\infty e^{-sx} f(x + y)\,dx\,, \quad y \geq 0.$$

A useful property of the Dickson–Hipp operator is given by Li and Garrido [38, Sect. 3, Property 2] as, for any complex numbers $s_1 \neq s_2$,

$$\mathcal{T}_{s_1}\mathcal{T}_{s_2} f(y) = \mathcal{T}_{s_2}\mathcal{T}_{s_1} f(y) = \frac{\mathcal{T}_{s_1} f(y) - \mathcal{T}_{s_2} f(y)}{s_2 - s_1}, \quad y \geq 0. \tag{1.12}$$

Secondly, the Laplace transform of a function $f(\cdot)$ is a special case of the Dickson–Hipp operator and will be denoted by

$$\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)\,dx = \mathcal{T}_s f(0).$$

Finally, the convolution operator '$*$' is defined such that for any functions $f_1(\cdot)$ and $f_2(\cdot)$ on $(0,\infty)$,

$$\begin{aligned}(f_1 * f_2)(y) &= \int_0^y f_1(y-x) f_2(x)\,dx = \int_0^y f_2(y-x) f_1(x)\,dx \\ &= (f_2 * f_1)(y), \quad y \geq 0.\end{aligned}$$

As a direct consequence of (1.12), the Laplace transform of the Dickson–Hipp operator of a convolution is given by, for $s \neq r$,

$$\begin{aligned}\int_0^\infty e^{-sx} \mathcal{T}_r (f_1 * f_2)(x)\,dx = \mathcal{T}_s \mathcal{T}_r (f_1 * f_2)(0) &= \frac{\mathcal{T}_s (f_1 * f_2)(0) - \mathcal{T}_r (f_1 * f_2)(0)}{r-s} \\ &= \frac{\tilde{f}_1(s)\tilde{f}_2(s) - \tilde{f}_1(r)\tilde{f}_2(r)}{r-s},\end{aligned} \tag{1.13}$$

since the Laplace transform of a convolution is the product of individual Laplace transforms.

The rest of the paper is organized as follows. In Sect. 2 the structural properties of the Gerber–Shiu functions are studied via the use of both scalar and matrix defective renewal equations, when the distributions of V, Y_+ and Y_- are all left arbitrary. The case where the upward jump Y_+ is assumed to be distributed as a combination of exponentials is studied in Sect. 3, in which the components involved in the defective renewal equations are explicitly identified. In Sect. 4 we illustrate some applications of the Gerber–Shiu functions, which include (i) the Laplace transforms of the time of ruin, the time of recovery and the duration of first negative surplus in the ruin context; (ii) the joint Laplace transform of the busy period and the subsequent idle period in the queueing context; and (iii) the expected total discounted reward for a continuous payment stream payable during idle periods in a queue. Section 5 ends the paper with some concluding remarks. Due to the large number of different but similar notations used for the Gerber–Shiu functions and related quantities, Table 1 in the Appendix provides a summary of these functions as well as their relationships.

2 Structural properties using defective renewal equations

2.1 Defective renewal equations satisfied by $m_{\delta,U,0}(u)$ and $m_{\delta,U,-}(u)$

For the process $\{U(t)\}_{t\geq 0}$, we first analyze $m_{\delta,U,0}(u)$ by conditioning on the time and the amount of the first jump. With an initial surplus of $U(0) = u$, suppose a jump arrives at some time $t < u/c$, then there are two possibilities as follows.

1. If the jump is known to be an upward jump (with probability q_+), then $\{U(t)\}_{t\geq 0}$ essentially reverts to $\{Z(t)\}_{t\geq 0}$ with initial level $Z(0^-) = u - ct$.
2. If the jump is a downward jump (with probability q_-), then the size y of the jump should be less than $u - ct$ (otherwise ruin occurs by the jump rather than by continuity), after which the process $\{U(t)\}_{t\geq 0}$ restarts itself at level $u - ct - y$.

On the other hand, if a jump arrives at some time $t > u/c$, then ruin occurs by continuity with $T_U^{*-} = t$ and $|U(T_U^{*-})| = ct - u$. Combining the above observations, we obtain

$$\begin{aligned} m_{\delta,U,0}(u) = \int_0^{\frac{u}{c}} e^{-\delta t}\bigg[& q_+ m_{\delta,Z,0}(u-ct) \\ & + q_- \int_0^{u-ct} m_{\delta,U,0}(u-ct-y)p_-(y)\,dy\bigg]k(t)\,dt \\ & + \int_{\frac{u}{c}}^{\infty} e^{-\delta t} w_*(ct-u)k(t)\,dt. \end{aligned} \tag{2.1}$$

By defining

$$b(x) = \frac{1}{c}k\left(\frac{x}{c}\right), \quad x > 0,$$

to be the density of cV, one can rewrite (2.1) using the convolution operator as

$$\begin{aligned} m_{\delta,U,0}(u) = {} & q_-\left(\left[e^{-\frac{\delta}{c}\bullet}b(\bullet)\right] * p_- * m_{\delta,U,0}\right)(u) \\ & + q_+\left(\left[e^{-\frac{\delta}{c}\bullet}b(\bullet)\right] * m_{\delta,Z,0}\right)(u) + \alpha_{\delta,w_*}(u). \end{aligned} \tag{2.2}$$

Here the notation '$\bullet$' is used to specify the argument being convolved in the convolution, and $\alpha_{\delta,w_*}(u)$ is defined as

$$\alpha_{\delta,w_*}(u) = \int_u^{\infty} e^{-\frac{\delta}{c}x} b(x) w_*(x-u)\,dx, \quad u \geq 0. \tag{2.3}$$

If we define the proper Esscher-transformed density

$$l_\delta(x) = \frac{e^{-\frac{\delta}{c}x} b(x)}{\widetilde{b}(\frac{\delta}{c})}, \quad x > 0, \tag{2.4}$$

then (2.2) can be re-expressed as

$$m_{\delta,U,0}(u) = q_-\widetilde{b}\left(\frac{\delta}{c}\right)(f_\delta * m_{\delta,U,0})(u) + q_+\widetilde{b}\left(\frac{\delta}{c}\right)(l_\delta * m_{\delta,Z,0})(u) + \alpha_{\delta,w_*}(u), \tag{2.5}$$

where

$$f_\delta(y) = (l_\delta * p_-)(y), \quad y > 0, \tag{2.6}$$

is a proper density. Since $0 < q_-\widetilde{b}(\delta/c) \le q_- < 1$, one asserts that (2.5) is a defective renewal equation satisfied by $m_{\delta,U,0}(\cdot)$. From, for example, [46, Sect. 3.5], (2.5) has the solution

$$m_{\delta,U,0}(u) = \frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * \gamma_{\delta,0})(u) + \gamma_{\delta,0}(u), \quad u \ge 0, \tag{2.7}$$

where

$$\gamma_{\delta,0}(u) = q_+\widetilde{b}\left(\frac{\delta}{c}\right)(l_\delta * m_{\delta,Z,0})(u) + \alpha_{\delta,w_*}(u), \quad u \ge 0, \tag{2.8}$$

and $g_\delta(y) = -\overline{G}'_\delta(y)$ is the compound geometric density associated with the compound geometric tail

$$\overline{G}_\delta(y) = \left[1 - q_-\widetilde{b}\left(\frac{\delta}{c}\right)\right]\sum_{n=1}^{\infty}\left[q_-\widetilde{b}\left(\frac{\delta}{c}\right)\right]^n \overline{F}_\delta^{*n}(y), \quad y \ge 0,$$

with $F_\delta^{*n}(y) = 1 - \overline{F}_\delta^{*n}(y) = \int_0^y f_\delta^{*n}(x)\,dx$ being the c.d.f. (cumulative distribution function) corresponding to the n-fold convolution of the density $f_\delta(\cdot)$ with itself. Note that $\overline{G}_\delta(0) = q_-\widetilde{b}(\delta/c)$, and

$$\widetilde{g}_\delta(s) = \frac{[1-q_-\widetilde{b}(\frac{\delta}{c})]q_-\widetilde{b}(\frac{\delta}{c})\widetilde{f}_\delta(s)}{1-q_-\widetilde{b}(\frac{\delta}{c})\widetilde{f}_\delta(s)}. \tag{2.9}$$

It is instructive to note that the solution (2.7) depends on $m_{\delta,Z,0}(\cdot)$ via the term $\gamma_{\delta,0}(\cdot)$ defined by (2.8). This can be made more transparent by expressing (2.7) as

$$m_{\delta,U,0}(u) = q_+\widetilde{b}\left(\frac{\delta}{c}\right)\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * l_\delta * m_{\delta,Z,0})(u) + (l_\delta * m_{\delta,Z,0})(u)\right] + \vartheta_{\delta,0}(u), \tag{2.10}$$

where

$$\vartheta_{\delta,0}(u) = \frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * \alpha_{\delta,w_*})(u) + \alpha_{\delta,w_*}(u), \quad u \ge 0, \tag{2.11}$$

is independent of $m_{\delta,Z,0}(\cdot)$.

As for $m_{\delta,U,-}(u)$, using similar arguments in obtaining (2.1), we arrive at

$$\begin{aligned} m_{\delta,U,-}(u) = \int_0^{\frac{u}{c}} e^{-\delta t}\bigg[&q_+ m_{\delta,Z,-}(u-ct) \\ &+ q_-\int_0^{u-ct} m_{\delta,U,-}(u-ct-y)p_-(y)\,dy\bigg]k(t)\,dt \\ &+ \int_0^{\frac{u}{c}} e^{-\delta t}\left[q_-\int_{u-ct}^{\infty} w\big(u-ct, y-(u-ct)\big)p_-(y)\,dy\right]k(t)\,dt. \end{aligned} \tag{2.12}$$

Like (2.5), equation (2.12) is equivalent to the defective renewal equation

$$m_{\delta,U,-}(u) = q_-\widetilde{b}\left(\frac{\delta}{c}\right)(f_\delta * m_{\delta,U,-})(u) + q_+\widetilde{b}\left(\frac{\delta}{c}\right)(l_\delta * m_{\delta,Z,-})(u) + \alpha_{\delta,w}(u), \quad u \geq 0, \tag{2.13}$$

where

$$\alpha_{\delta,w}(u) = \int_0^u e^{-\frac{\delta}{c}x} b(x) q_- \omega_-(u-x)\,dx, \quad u \geq 0, \tag{2.14}$$

and

$$\omega_-(u) = \int_u^\infty w(u, y-u) p_-(y)\,dy, \quad u \geq 0.$$

The defective renewal equation (2.13) has solution

$$m_{\delta,U,-}(u) = q_+\widetilde{b}\left(\frac{\delta}{c}\right)\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * l_\delta * m_{\delta,Z,-})(u) + (l_\delta * m_{\delta,Z,-})(u)\right] + \vartheta_{\delta,-}(u), \tag{2.15}$$

where

$$\vartheta_{\delta,-}(u) = \frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * \alpha_{\delta,w})(u) + \alpha_{\delta,w}(u), \quad u \geq 0.$$

2.2 A matrix defective renewal equation

From (2.10) and (2.15), one observes that the quantities $m_{\delta,U,0}(\cdot)$ and $m_{\delta,U,-}(\cdot)$ are expressed in terms of $m_{\delta,Z,0}(\cdot)$ and $m_{\delta,Z,-}(\cdot)$, respectively. This subsection aims at exploiting additional structure via a matrix form of defective renewal equation. To begin the analysis, we note that the representation (1.10) means that $m_{\delta,Z,0}(z)$ is the expectation of a discounted penalty, which can be equivalently represented as an expectation of the penalty with respect to a 'discounted density'. Mathematically, we have

$$m_{\delta,Z,0}(z) = \int_0^\infty w_*(y) h_{\delta,Z,0}(y|z)\,dy, \quad z \geq 0, \tag{2.16}$$

where $h_{\delta,Z,0}(y|z)$ is the discounted density of $|Z(T_Z^{*-})|$ at y (discounted with respect to T_Z^* at rate δ) for ruin by continuity, given an initial level of $Z(0^-) = z$. Interested readers are referred, for example, to [30] for the notion of discounted densities.

Similarly, the representation (1.8) implies

$$m_{\delta,Z,-}(z) = \int_0^\infty \int_0^\infty w(x,y) h_{\delta,Z,-}(x,y|z)\,dx\,dy, \quad z \geq 0, \tag{2.17}$$

where $h_{\delta,Z,-}(x,y|z)$ is the discounted joint density of $(Z(T_Z^-), |Z(T_Z)|)$ at (x,y) (discounted with respect to T_Z at rate δ) for ruin upon downward jumps, given an

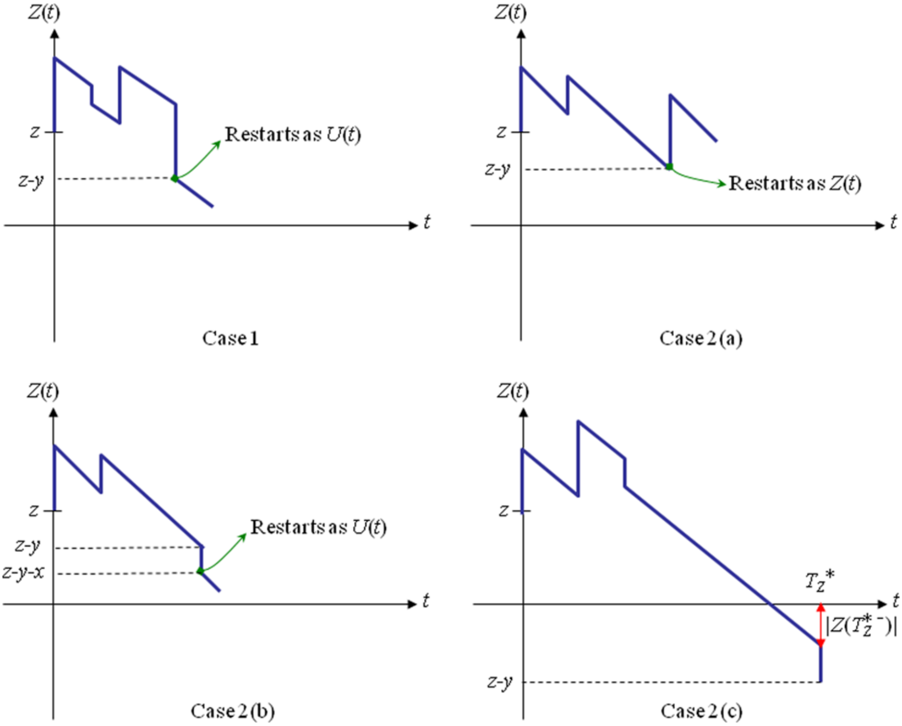

Fig. 2 Conditioning on first drop of $Z(t)$

initial level of $Z(0^-) = z$. By further defining the discounted density of $|Z(T_Z)|$ such that

$$h_{\delta,Z,-}(y|z) = \int_0^\infty h_{\delta,Z,-}(x, y|z)\, dx, \qquad y > 0, \tag{2.18}$$

one observes that if the penalty function takes on the simpler form $w(x, y) \equiv w_2(y)$, then the special case of $m_{\delta,Z,-}(z)$, namely $m_{\delta,Z,-,2}(z)$, admits the representation

$$m_{\delta,Z,-,2}(z) = \int_0^\infty w_2(y) h_{\delta,Z,-}(y|z)\, dy, \qquad z \ge 0. \tag{2.19}$$

To analyze the quantity $m_{\delta,Z,0}(z)$ for ruin by continuity pertaining to the process $\{Z(t)\}_{t\ge 0}$, we condition on the first ‘drop’ below the initial level $Z(0^-) = z$, which can be due to continuity or a downward jump. It is important to note that the amount y of such a drop is governed by the discounted densities $h_{\delta,Z,0}(y|0)$ and $h_{\delta,Z,-}(y|0)$. The possibilities are described as follows, which are also depicted in Fig. 2.

1. If the first drop is caused by a downward jump, then the amount of drop y should be less than z (otherwise ruin occurs by the jump), after which the process reverts to $\{U(t)\}_{t\ge 0}$ with initial level $U(0) = z - y$.
2. If the first drop of amount y is caused by continuity, then there are three separate cases.

(a) The drop y is less than z, and if the subsequent jump is an upward one (with probability q_+), then $\{Z(t)\}_{t\ge 0}$ restarts itself with initial level $Z(0^-) = z - y$.
(b) The drop y is less than z, and if the subsequent jump is a downward one (with probability q_-), then the jump x should be less than $z - y$ (otherwise ruin occurs by the jump), after which the process reverts to $\{U(t)\}_{t\ge 0}$ with starting level $z - y - x$.
(c) The drop y is at least z, and the shortfall $|Z(T_Z^{*-})|$ equals $y - z$.

We then arrive at

$$\begin{aligned}
m_{\delta,Z,0}(z) &= \int_0^z h_{\delta,Z,-}(y|0)m_{\delta,U,0}(z-y)\,dy + \int_0^z h_{\delta,Z,0}(y|0)q_+m_{\delta,Z,0}(z-y)\,dy \\
&\quad + \int_0^z h_{\delta,Z,0}(y|0)\left[q_-\int_0^{z-y} m_{\delta,U,0}(z-y-x)p_-(x)\,dx\right]dy \\
&\quad + \int_z^\infty h_{\delta,Z,0}(y|0)w_*(y-z)\,dy \\
&= \big(h_{\delta,Z,-}(\bullet|0) * m_{\delta,U,0}\big)(z) + q_+\big(h_{\delta,Z,0}(\bullet|0) * m_{\delta,Z,0}\big)(z) \\
&\quad + q_-\big(h_{\delta,Z,0}(\bullet|0) * p_- * m_{\delta,U,0}\big)(z) + \int_z^\infty h_{\delta,Z,0}(y|0)w_*(y-z)\,dy.
\end{aligned} \tag{2.20}$$

Similarly, for the quantity $m_{\delta,Z,-}(z)$ representing ruin by downward jumps, the same arguments lead to

$$\begin{aligned}
m_{\delta,Z,-}(z) &= \int_0^z h_{\delta,Z,0}(y|0)q_+m_{\delta,Z,-}(z-y)\,dy \\
&\quad + \int_0^z h_{\delta,Z,0}(y|0)\left[q_-\int_0^{z-y} m_{\delta,U,-}(z-y-x)p_-(x)\,dx\right]dy \\
&\quad + \int_0^z h_{\delta,Z,0}(y|0)\left[q_-\int_{z-y}^\infty w\big(z-y, x-(z-y)\big)p_-(x)\,dx\right]dy \\
&\quad + \int_0^z h_{\delta,Z,-}(y|0)m_{\delta,U,-}(z-y)\,dy \\
&\quad + \int_z^\infty \int_0^\infty h_{\delta,Z,-}(x,y|0)w(x+z, y-z)\,dx\,dy \\
&= q_+\big(h_{\delta,Z,0}(\bullet|0) * m_{\delta,Z,-}\big)(z) + q_-\big(h_{\delta,Z,0}(\bullet|0) * p_- * m_{\delta,U,-}\big)(z) \\
&\quad + q_-\big(h_{\delta,Z,0}(\bullet|0) * \omega_-\big)(z) + \big(h_{\delta,Z,-}(\bullet|0) * m_{\delta,U,-}\big)(z) \\
&\quad + \int_z^\infty \int_0^\infty h_{\delta,Z,-}(x,y|0)w(x+z, y-z)\,dx\,dy.
\end{aligned} \tag{2.21}$$

Combining the system consisting of (2.5), (2.13), (2.20) and (2.21), one can write

$$\mathbf{m}_\delta(u) = (\mathbf{r}_\delta * \mathbf{m}_\delta)(u) + \boldsymbol{\Gamma}_\delta(u), \quad u \ge 0, \tag{2.22}$$

where

$$\mathbf{m}_\delta(u) = \begin{pmatrix} m_{\delta,U,0}(u) & m_{\delta,U,-}(u) \\ m_{\delta,Z,0}(u) & m_{\delta,Z,-}(u) \end{pmatrix}, \tag{2.23}$$

$$\mathbf{r}_\delta(u) = \begin{pmatrix} q_-\widetilde{b}(\frac{\delta}{c}) f_\delta(u) & q_+\widetilde{b}(\frac{\delta}{c}) l_\delta(u) \\ q_-(h_{\delta,Z,0}(\bullet|0) * p_-)(u) + h_{\delta,Z,-}(u|0) & q_+ h_{\delta,Z,0}(u|0) \end{pmatrix},$$

and

$$\boldsymbol{\Gamma}_\delta(u) = \begin{pmatrix} \alpha_{\delta,w_*}(u) & \alpha_{\delta,w}(u) \\ \int_u^\infty h_{\delta,Z,0}(y|0) w_*(y-u)\,dy & q_-(h_{\delta,Z,0}(\bullet|0) * \omega_-)(u) + \int_u^\infty \int_0^\infty h_{\delta,Z,-}(x,y|0) w(x+u,y-u)\,dx\,dy \end{pmatrix}.$$

Here the convolution operator has been extended to matrix quantities; i.e. for $i,j=1,2$, the (i,j)th element of $(\mathbf{r}_\delta * \mathbf{m}_\delta)(u)$ is given by $[(\mathbf{r}_\delta * \mathbf{m}_\delta)(u)]_{ij} = \sum_{k=1}^2 \int_0^u [\mathbf{r}_\delta(u-y)]_{ik}[\mathbf{m}_\delta(y)]_{kj}\,dy$. Equation (2.22) is commonly known as a Markov renewal equation. For its solution, see, for example, [22, Sect. 3a], and [4, Sect. VII 4].

We would like to check whether the matrix $\int_0^\infty \mathbf{r}_\delta(y)\,dy$ is strictly sub-stochastic. First we note that with $\mathbf{1}$ a two-dimensional column vector of ones,

$$\int_0^\infty \mathbf{r}_\delta(y)\,dy\,\mathbf{1} = \begin{pmatrix} \widetilde{b}(\frac{\delta}{c}) \\ \int_0^\infty h_{\delta,Z,0}(y|0)\,dy + \int_0^\infty h_{\delta,Z,-}(y|0)\,dy \end{pmatrix}. \tag{2.24}$$

The representations (1.10) and (2.16) (at $z=0$) with $w_*(\cdot) \equiv 1$ mean that

$$\begin{aligned} \int_0^\infty h_{\delta,Z,0}(y|0)\,dy &= E\big[e^{-\delta T_Z^*} 1\{T_Z < \infty, Z(T_Z)=0\} | Z(0^-)=0\big] \\ &\le E\big[e^{-\delta T_Z} 1\{T_Z < \infty, Z(T_Z)=0\} | Z(0^-)=0\big], \end{aligned}$$

since $T_Z = T_Z^* - |Z(T_Z^{*-})|/c \le T_Z^*$ when $Z(T_Z)=0$. Similarly, (1.8) and (2.17) (at $z=0$) under $w(\cdot,\cdot)\equiv 1$ along with (2.18) imply that

$$\int_0^\infty h_{\delta,Z,-}(y|0)\,dy = E\big[e^{-\delta T_Z} 1\{T_Z < \infty, Z(T_Z) < 0\} | Z(0^-)=0\big].$$

Combining the above two expressions leads to

$$\int_0^\infty h_{\delta,Z,0}(y|0)\,dy + \int_0^\infty h_{\delta,Z,-}(y|0)\,dy \le E\big[e^{-\delta T_Z} 1\{T_Z < \infty\} | Z(0^-)=0\big] < 1, \tag{2.25}$$

where the last strict inequality holds when either $\delta > 0$ or the positive security loading condition (1.2) holds. To see this, when $\delta > 0$, one has that $E[e^{-\delta T_Z} 1\{T_Z < \infty\} | Z(0^-)=0] \le 1$, and the equality holds if and only if T_Z is a point mass at 0, which is not the case. On the other hand, if $\delta = 0$ and (1.2) holds, then

$E[e^{-\delta T_Z}1\{T_Z<\infty\}|Z(0^-)=0]=\Pr\{T_Z<\infty|Z(0^-)=0\}$ equals the ruin probability of the process $\{U(t)|U(0)=Y_+\}_{t\geq 0}$, which is clearly no larger than the ruin probability of the process $\{U(t)|U(0)=Y\}_{t\geq 0}$ via sample paths arguments. The latter ruin probability is in turn less than 1 under (1.2) by observing that the increments of the process form an i.i.d. sequence distributed as $Y-cV$ (see e.g. [45, Chap. 1]).

With (2.25), one asserts that $\int_0^\infty \mathbf{r}_\delta(y)\,dy$ is strictly sub-stochastic since the first entry in (2.24) is $\widetilde{b}(\delta/c)\leq 1$. Therefore, the Markov renewal equation (2.22) can be viewed as a matrix form of a defective renewal equation. It arises in the context of ruin theory in various (semi-)Markovian risk models (see [1, 21]). In addition, the solution is known to be unique as well (see, for example, [42]). Interested readers are also referred, for example, to [40, 42, 50] for two-sided bounds and asymptotics for the solution of a matrix defective renewal equation.

It is instructive to note that the matrix defective renewal equation (2.22) and hence its solution are characterized by the discounted densities $h_{\delta,Z,0}(\cdot|0)$ and $h_{\delta,Z,-}(\cdot,\cdot|0)$. In principle, once these densities are determined, a full characterization of $\mathbf{m}_\delta(u)$ is obtained. The densities $h_{\delta,Z,0}(\cdot|0)$ and $h_{\delta,Z,-}(\cdot,\cdot|0)$ are usually determined via $m_{\delta,Z,0}(0)$ and $m_{\delta,Z,-}(0)$, respectively (due to the relationships (2.16) and (2.17)) upon additional distributional assumptions on V, Y_+ and/or Y_- along with the use of (1.11) and (1.9) (all at $z=0$). This will be illustrated in the next section.

3 Example: distributional assumption on upward jumps

In this entire section we assume that the generic r.v. Y_+ representing the upward jumps follows a combination of exponentials;

$$p_+(y)=\sum_{j=1}^m A_j\beta_j e^{-\beta_j y},\qquad y>0, \tag{3.1}$$

where $\sum_{j=1}^m A_j=1$ and $\beta_j>0$ for $j=1,2,\dots,m$. The class of combinations of exponentials is known to be dense in the set of distributions on $\mathbb{R}^+$. Interested readers are referred to [24] for the fitting of this class of distributions.

3.1 Determination of $h_{\delta,Z,0}(y|0)$

Under the distributional assumption (3.1), using (1.11) followed by applications of (2.10) and the Dickson–Hipp operator leads to

$$\begin{aligned} m_{\delta,Z,0}(z) &= \int_z^\infty m_{\delta,U,0}(x)p_+(x-z)\,dx \\ &= \int_z^\infty \left\{ q_+\widetilde{b}\left(\frac{\delta}{c}\right)\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}(g_\delta * l_\delta * m_{\delta,Z,0})(x)+(l_\delta * m_{\delta,Z,0})(x)\right]\right. \\ &\quad \left. + \vartheta_{\delta,0}(x)\right\}\sum_{j=1}^m A_j\beta_j e^{-\beta_j(x-z)}\,dx \end{aligned}$$

$$= q_+\widetilde{b}\left(\frac{\delta}{c}\right)\sum_{j=1}^{m}A_j\beta_j\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\mathcal{T}_{\beta_j}(g_\delta * l_\delta * m_{\delta,Z,0})(z)\right.$$

$$\left. + \mathcal{T}_{\beta_j}(l_\delta * m_{\delta,Z,0})(z)\right] + \eta_{\delta,0}(z), \tag{3.2}$$

where

$$\eta_{\delta,0}(z) = \sum_{j=1}^{m}A_j\beta_j\mathcal{T}_{\beta_j}\vartheta_{\delta,0}(z), \quad z \geq 0. \tag{3.3}$$

Taking Laplace transforms on both sides of (3.2) and using (1.13) yields

$$\begin{aligned}
&\widetilde{m}_{\delta,Z,0}(s) \\
&\quad = q_+\widetilde{b}\left(\frac{\delta}{c}\right) \\
&\qquad \times \sum_{j=1}^{m}A_j\beta_j\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\frac{\widetilde{g}_\delta(s)\widetilde{l}_\delta(s)\widetilde{m}_{\delta,Z,0}(s)-\widetilde{g}_\delta(\beta_j)\widetilde{l}_\delta(\beta_j)\widetilde{m}_{\delta,Z,0}(\beta_j)}{\beta_j - s}\right. \\
&\qquad \left. + \frac{\widetilde{l}_\delta(s)\widetilde{m}_{\delta,Z,0}(s)-\widetilde{l}_\delta(\beta_j)\widetilde{m}_{\delta,Z,0}(\beta_j)}{\beta_j - s}\right] + \widetilde{\eta}_{\delta,0}(s).
\end{aligned}$$

Upon rearrangements, the above equation reduces to

$$\begin{aligned}
&\left\{1 - q_+\widetilde{b}\left(\frac{\delta}{c}\right)\widetilde{l}_\delta(s)\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j - s}\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(s) + 1\right]\right\}\widetilde{m}_{\delta,Z,0}(s) \\
&\quad = -q_+\widetilde{b}\left(\frac{\delta}{c}\right)\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j - s}\widetilde{l}_\delta(\beta_j)\widetilde{m}_{\delta,Z,0}(\beta_j)\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(\beta_j) + 1\right] + \widetilde{\eta}_{\delta,0}(s) \\
&\quad = -\left(\prod_{j=1}^{m}\frac{1}{\beta_j - s}\right)Q_{\delta,0}(s) + \widetilde{\eta}_{\delta,0}(s),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
Q_{\delta,0}(s) = {} & q_+\widetilde{b}\left(\frac{\delta}{c}\right)\sum_{j=1}^{m}A_j\beta_j\widetilde{l}_\delta(\beta_j)\widetilde{m}_{\delta,Z,0}(\beta_j) \\
& \times \left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(\beta_j) + 1\right]\prod_{k=1,k\neq j}^{m}(\beta_k - s)
\end{aligned}$$

is a polynomial in s of degree $m-1$ expressed in terms of the unknown constants $\widetilde{m}_{\delta,Z,0}(\beta_j)$.

In order to proceed, we further analyze the equation (in ξ)

$$1-q_+\widetilde{b}\left(\frac{\delta}{c}\right)\widetilde{l}_\delta(\xi)\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\xi}\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(\xi)+1\right]=0, \tag{3.5}$$

in connection to the left-hand side of (3.4). Using (2.4), (2.6) and (2.9), we have

$$\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(\xi)+1=\frac{q_-\widetilde{b}(\frac{\delta}{c})\widetilde{f}_\delta(\xi)}{1-q_-\widetilde{b}(\frac{\delta}{c})\widetilde{f}_\delta(\xi)}+1=\frac{1}{1-q_-\widetilde{b}(\xi+\frac{\delta}{c})\widetilde{p}_-(\xi)}. \tag{3.6}$$

Therefore, (3.5) is equivalent to

$$\begin{aligned}
&1-q_+\widetilde{b}\left(\xi+\frac{\delta}{c}\right)\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\xi}\left[\frac{1}{1-q_-\widetilde{b}(\xi+\frac{\delta}{c})\widetilde{p}_-(\xi)}\right]=0\\
\Leftrightarrow\quad & q_+\widetilde{b}\left(\xi+\frac{\delta}{c}\right)\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\xi}=1-q_-\widetilde{b}\left(\xi+\frac{\delta}{c}\right)\widetilde{p}_-(\xi)\\
\Leftrightarrow\quad & \widetilde{b}\left(\xi+\frac{\delta}{c}\right)\left[q_+\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\xi}+q_-\widetilde{p}_-(\xi)\right]=1\\
\Leftrightarrow\quad & E\left[e^{-\delta V-\xi(cV-Y)}\right]=1,
\end{aligned} \tag{3.7}$$

since

$$\begin{aligned}
\widetilde{b}\left(\xi+\frac{\delta}{c}\right)\left[q_+\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\xi}+q_-\widetilde{p}_-(\xi)\right]&=E\left[e^{-(\xi+\frac{\delta}{c})cV}\right]\widetilde{p}(-\xi)\\
&=E\left[e^{-\delta V-\xi(cV-Y)}\right].
\end{aligned}$$

Here $\widetilde{p}(\cdot)$ is the double-sided Laplace transform of the r.v. Y since Y can be positive or negative. Define $\chi_1(\xi)=[1-q_-\widetilde{b}(\xi+\delta/c)\widetilde{p}_-(\xi)]\prod_{k=1}^m(\beta_k-\xi)$ and $\chi_2(\xi)=q_+\widetilde{b}(\xi+\delta/c)\sum_{j=1}^m A_j\beta_j\prod_{k=1,k\neq j}^m(\beta_k-\xi)$. Also define the contour on the complex plane which consists of the semi-circle of radius r running anti-clockwise from $-ir$ to ir plus part of the imaginary axis from $-ir$ to ir. For $\delta>0$, one can show that $|\chi_1|>|\chi_2|$ on the contour for r sufficiently large. By Rouché 's Theorem, the number of roots of $\chi_1(\xi)=\chi_2(\xi)$ on the right-half of the complex plane must be the same as that of $\chi_1(\xi)=0$. Since $\chi_1(\xi)=\chi_2(\xi)$ is equivalent to (3.7), one concludes that (3.5) has m roots with positive real parts when $\delta>0$, which are denoted by $\{\rho_i\}_{i=1}^m$ and assumed to be distinct. The case for $\delta=0$ is similar, but the only difference is that one of these roots becomes 0 (see, for example, [39, Theorem 1]).

It is instructive to note that (3.4) is structurally identical to (4.4) in [20]. Thus, using the roots $\{\rho_i\}_{i=1}^m$, one could obtain an expression for $Q_{\delta,0}(s)$ via Lagrange interpolating polynomials. Consequently, omitting the details, the Initial Value Theorem for Laplace transforms can be applied to $\widetilde{m}_{\delta,Z,0}(s)$ to obtain

$$m_{\delta,Z,0}(0)=\sum_{i=1}^{m}\left[\frac{\prod_{k=1}^{m}(\beta_k-\rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k-\rho_i)}\right]\widetilde{\eta}_{\delta,0}(\rho_i)+\eta_{\delta,0}(0). \tag{3.8}$$

Note that the quantity $\eta_{\delta,0}(z)$ defined by (3.3) depends on the function $\vartheta_{\delta,0}(\cdot)$. By taking Laplace transforms on both sides of (2.11) along with the use of (3.6), we arrive at

$$\widetilde{\vartheta}_{\delta,0}(s)=\left[\frac{1}{1-q_-\widetilde{b}(\frac{\delta}{c})}\widetilde{g}_\delta(s)+1\right]\widetilde{\alpha}_{\delta,w_*}(s)=\frac{\widetilde{\alpha}_{\delta,w_*}(s)}{1-q_-\widetilde{b}(s+\frac{\delta}{c})\widetilde{p}_-(s)}.$$

Using the above expression, $\eta_{\delta,0}(0)$ and $\widetilde{\eta}_{\delta,0}(\rho_i)$ can be rewritten, respectively, as

$$\eta_{\delta,0}(0)=\sum_{j=1}^{m}A_j\beta_j\widetilde{\vartheta}_{\delta,0}(\beta_j)=\sum_{j=1}^{m}\frac{A_j\beta_j\widetilde{\alpha}_{\delta,w_*}(\beta_j)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}, \tag{3.9}$$

and

$$\begin{aligned}\widetilde{\eta}_{\delta,0}(\rho_i)&=\sum_{j=1}^{m}A_j\beta_j\mathcal{T}_{\rho_i}\mathcal{T}_{\beta_j}\vartheta_{\delta,0}(0)\\&=\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\rho_i}\left[\widetilde{\vartheta}_{\delta,0}(\rho_i)-\widetilde{\vartheta}_{\delta,0}(\beta_j)\right]\\&=\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\rho_i}\left[\frac{\widetilde{\alpha}_{\delta,w_*}(\rho_i)}{1-q_-\widetilde{b}(\rho_i+\frac{\delta}{c})\widetilde{p}_-(\rho_i)}-\frac{\widetilde{\alpha}_{\delta,w_*}(\beta_j)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}\right].\end{aligned} \tag{3.10}$$

Since the Laplace transform of (2.3) can be expressed as

$$\begin{aligned}\widetilde{\alpha}_{\delta,w_*}(s)&=\int_0^\infty e^{-sx}\int_0^\infty e^{-\frac{\delta}{c}(y+x)}b(y+x)w_*(y)\,dy\,dx\\&=\int_0^\infty e^{-\frac{\delta}{c}y}w_*(y)\mathcal{T}_{s+\frac{\delta}{c}}b(y)\,dy,\end{aligned}$$

substitution of (3.9) and (3.10) into (3.8) yields

$$\begin{aligned}h_{\delta,Z,0}(y|0)=e^{-\frac{\delta}{c}y}\Bigg\{&\sum_{i=1}^{m}\left[\frac{\prod_{k=1}^{m}(\beta_k-\rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k-\rho_i)}\right]\\&\times\sum_{j=1}^{m}\frac{A_j\beta_j}{\beta_j-\rho_i}\left[\frac{\mathcal{T}_{\rho_i+\frac{\delta}{c}}b(y)}{1-q_-\widetilde{b}(\rho_i+\frac{\delta}{c})\widetilde{p}_-(\rho_i)}\right.\end{aligned}$$

$$- \frac{\mathcal{T}_{\beta_j+\frac{\delta}{c}} b(y)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}\Bigg]$$

$$+ \sum_{j=1}^{m} \frac{A_j \beta_j \mathcal{T}_{\beta_j+\frac{\delta}{c}} b(y)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}\Bigg\}, \quad y>0, \tag{3.11}$$

owing to the representation (2.16) at $z=0$. Using the identity (see, for example, [19])

$$\sum_{i=1}^{m} \frac{\prod_{k=1,k\neq j}^{m}(\beta_k-\rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k-\rho_i)} = 1, \tag{3.12}$$

(3.11) reduces to

$$h_{\delta,Z,0}(y|0) = e^{-\frac{\delta}{c}y} \sum_{i=1}^{m} \left[\frac{\prod_{k=1}^{m}(\beta_k-\rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k-\rho_i)}\right] \times \sum_{j=1}^{m} \frac{A_j\beta_j}{\beta_j-\rho_i}\left[\frac{\mathcal{T}_{\rho_i+\frac{\delta}{c}} b(y)}{1-q_-\widetilde{b}(\rho_i+\frac{\delta}{c})\widetilde{p}_-(\rho_i)}\right], \quad y>0, \tag{3.13}$$

which is simply a linear combination of terms of the form $e^{-(\delta/c)y}\mathcal{T}_{\rho_i+\delta/c} b(y)$.

3.2 Determination of $h_{\delta,Z,-}(x,y|0)$

The determination of $h_{\delta,Z,-}(x,y|0)$ follows in an identical manner as in Sect. 3.1 and hence the details are omitted. Analogous to (3.8), one has

$$m_{\delta,Z,-}(0) = \sum_{i=1}^{m} \left[\frac{\prod_{k=1}^{m}(\beta_k-\rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k-\rho_i)}\right] \widetilde{\eta}_{\delta,-}(\rho_i) + \eta_{\delta,-}(0), \tag{3.14}$$

where

$$\eta_{\delta,-}(0) = \sum_{j=1}^{m} \frac{A_j\beta_j \widetilde{\alpha}_{\delta,w}(\beta_j)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}, \tag{3.15}$$

and

$$\widetilde{\eta}_{\delta,-}(\rho_i) = \sum_{j=1}^{m} \frac{A_j\beta_j}{\beta_j-\rho_i}\left[\frac{\widetilde{\alpha}_{\delta,w}(\rho_i)}{1-q_-\widetilde{b}(\rho_i+\frac{\delta}{c})\widetilde{p}_-(\rho_i)} - \frac{\widetilde{\alpha}_{\delta,w}(\beta_j)}{1-q_-\widetilde{b}(\beta_j+\frac{\delta}{c})\widetilde{p}_-(\beta_j)}\right], \tag{3.16}$$

with $\alpha_{\delta,w}(\cdot)$ given by (2.14). Thus,

$$\begin{aligned}\widetilde{\alpha}_{\delta,w}(s) &= \int_0^\infty e^{-sv}\int_0^v e^{-\frac{\delta}{c}(v-x)} b(v-x)\left[q_-\int_0^\infty w(x,y)p_-(x+y)\,dy\right]dx\,dv \\ &= q_-\int_0^\infty\int_0^\infty w(x,y)p_-(x+y)\left[\int_x^\infty e^{-sv}e^{-\frac{\delta}{c}(v-x)}b(v-x)\,dv\right]dx\,dy\end{aligned}$$

$$= q_- \int_0^\infty \int_0^\infty w(x,y) p_-(x+y) e^{-sx} \widetilde{b}\left(s + \frac{\delta}{c}\right) dx\, dy. \tag{3.17}$$

Because of (2.17) at $z = 0$, combining (3.14)–(3.17) along with the use of (3.12) leads to

$$h_{\delta,Z,-}(x,y|0) = q_- p_-(x+y) \sum_{i=1}^{m} \left[\frac{\prod_{k=1}^{m}(\beta_k - \rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k - \rho_i)} \right]$$
$$\times \sum_{j=1}^{m} \frac{A_j \beta_j}{\beta_j - \rho_i} \left[\frac{e^{-\rho_i x} \widetilde{b}(\rho_i + \frac{\delta}{c})}{1 - q_- \widetilde{b}(\rho_i + \frac{\delta}{c}) \widetilde{p}_-(\rho_i)} \right], \quad x, y > 0.$$

Hence, (2.18) at $z = 0$ gives

$$h_{\delta,Z,-}(y|0) = q_- \sum_{i=1}^{m} \left[\frac{\prod_{k=1}^{m}(\beta_k - \rho_i)}{\prod_{k=1,k\neq i}^{m}(\rho_k - \rho_i)} \right]$$
$$\times \sum_{j=1}^{m} \frac{A_j \beta_j}{\beta_j - \rho_i} \left[\frac{\widetilde{b}(\rho_i + \frac{\delta}{c}) \mathcal{T}_{\rho_i} p_-(y)}{1 - q_- \widetilde{b}(\rho_i + \frac{\delta}{c}) \widetilde{p}_-(\rho_i)} \right], \quad y > 0, \tag{3.18}$$

which is a linear combination of $\mathcal{T}_{\rho_i} p_-(y)$.

Remark 2 Although throughout we assume $q_- > 0$, the discounted densities given by (3.13) and (3.18) hold true even $q_- = 0$. When $q_- = 0$, (3.13) agrees with equation (4.12) of [20] (which can also be simplified using (3.12)), whereas (3.18) becomes 0 which is expected because ruin can only occur by continuity in the absence of downward jumps.

4 Applications of the Gerber–Shiu functions

4.1 Laplace transforms of time of ruin, time of recovery and duration of first negative surplus

In the ruin theory context, [26, 29] commented that the probability of ruin is extremely small for certain businesses. If ruin occurs, the business can usually obtain funds to survive negative surplus for some time. This naturally leads to the study of quantities such as the time of recovery and the duration of first negative surplus. We shall separate the study of the present model with negative drift and its reflection (i.e. with positive drift).

To aid our upcoming analysis, we define the reflection of the process $\{U(t)\}_{t\geq 0}$, namely $\{X(t)\}_{t\geq 0}$, such that

$$X(t) = x + ct - \sum_{i=1}^{N^*(t)} Y_i^*, \quad t \geq 0, \tag{4.1}$$

where $X(0)=x\geq 0$ is the initial level, and $\{Y_i^*\}_{i=1}^{\infty}$ and $\{N^*(t)\}_{t\geq 0}$ are mutually independent copies of $\{Y_i\}_{i=1}^{\infty}$ and $\{N(t)\}_{t\geq 0}$, respectively. The time of ruin for the process $\{X(t)\}_{t\geq 0}$ is $T_X=\inf\{t\geq 0: X(t)<0\}$, and ruin can only occur by jumps (but not by continuity). In the case of ruin, the deficit at ruin is given by $|X(T_X)|$. The Gerber–Shiu function for which the penalty function depends on the deficit only is defined by

$$m_{\delta,X}(x)=E\big[e^{-\delta T_X}w_X\big(\big|X(T_X)\big|\big)1\{T_X<\infty\}|X(0)=x\big],\quad x\geq 0,\tag{4.2}$$

with $w_X(\cdot)$ satisfying some mild integrable conditions. The Laplace transform of T_X can be retrieved from $m_{\delta,X}(x)$ by letting $w_X(\cdot)\equiv 1$, and is denoted by

$$L_{\delta,X}(x)=E\big[e^{-\delta T_X}1\{T_X<\infty\}|X(0)=x\big],\quad x\geq 0.\tag{4.3}$$

We remark that $m_{\delta,X}(x)$ is a special case of the usual Gerber–Shiu function studied by [35], and will be regarded as a known quantity for the rest of the paper.

4.1.1 Present model with negative drift

Formally, in cases of ruin, the time of recovery pertaining to the surplus process $\{U(t)\}_{t\geq 0}$ is defined by $T_U^R=\inf\{t>T_U: U(t)>0\}$; i.e. T_U^R is the first time the surplus becomes positive after ruin. The duration of first negative surplus is then given by $T_U^R-T_U$. Furthermore, the condition (1.2) is assumed to hold in order to ensure recovery occurs almost surely given ruin occurs. We are then interested in the quantities

$$L^R_{\delta_1,\delta_2,U,0}(u)=E\big[e^{-\delta_1 T_U-\delta_2(T_U^R-T_U)}1\big\{T_U<\infty,U(T_U)=0\big\}|U(0)=u\big],\quad u\geq 0,\tag{4.4}$$

and

$$L^R_{\delta_1,\delta_2,U,-}(u)=E\big[e^{-\delta_1 T_U-\delta_2(T_U^R-T_U)}1\big\{T_U<\infty,U(T_U)<0\big\}|U(0)=u\big],\quad u\geq 0,\tag{4.5}$$

for ruin occurring by continuity and by downward jumps, respectively. The above two functions obviously contain the marginal Laplace transforms of the time of ruin T_U, the time of recovery T_U^R and the duration of first negative surplus $T_U^R-T_U$, as well as their joint Laplace transform, as special cases by suitable choices of δ_1 and δ_2.

It is instructive to note that under the current assumption of (1.2), we have $T_X<\infty$ almost surely. If $\delta>0$, it is known from [35] that $L_{\delta,X}(x)$ is the tail of a compound geometric distribution. In contrast, if $\delta=0$, it is clear that $L_{0,X}(x)=1$.

First, to determine the quantity $L^R_{\delta_1,\delta_2,U,0}(u)$ for ruin by continuity, we condition on $(|U(T_U^{*-})|, T_U^*)$ and the subsequent jump $Y_{N(T_U)+1}$. The time of ruin T_U is always given by $T_U=T_U^*-|U(T_U^{*-})|/c$. In contrast, we need to distinguish between the following cases for the variable $T_U^R-T_U$.

1. If $Y_{N(T_U)+1}$ is an upward jump greater than $|U(T_U^{*-})|$, then $T_U^R-T_U=|U(T_U^{*-})|/c$.
2. If $Y_{N(T_U)+1}$ is an upward jump less than or equal to $|U(T_U^{*-})|$, then $T_U^R-T_U$ is distributed as an independent sum of $|U(T_U^{*-})|/c$ and $(T_X|X(0)=|U(T_U^{*-})|-Y_+)$.

3. If $Y_{N(T_U)+1}$ is a downward jump, then $T_U^R - T_U$ is distributed as an independent sum of $|U(T_U^{*-})|/c$ and $(T_X|X(0) = |U(T_U^{*-})| + Y_-)$.

Therefore, similar to [20, Sect. 3.4], one concludes that $L^R_{\delta_1,\delta_2,U,0}(u)$ can indeed be retrieved from $m_{\delta_1,U,0}(u)$ with the choice of penalty function

$$w_*(y) = e^{(\frac{\delta_1-\delta_2}{c})y}\left\{q_+\left[\overline{P}_+(y) + \int_0^y L_{\delta_2,X}(y-v)p_+(v)\,dv\right]\right.$$
$$\left. + q_-\int_0^\infty L_{\delta_2,X}(y+v)p_-(v)\,dv\right\}, \quad y > 0, \tag{4.6}$$

where $\overline{P}_+(.)$ is the survival probability associated to the generic r.v. Y_+.

Remark 3 If the upward jump Y_+ has density (3.1) as in Sect. 3, then following the ideas as in [35, 37, 49], for $\delta_2 > 0$, $L_{\delta_2,X}(\cdot)$ is seen to be a linear combination of exponential terms. Hence, all quantities on the right-hand side of (4.6) can be explicitly evaluated, and the choice of penalty function $w_*(\cdot)$ is also a linear combination of exponential terms.

As for the quantity $L^R_{\delta_1,\delta_2,U,-}(u)$ for ruin by jumps, by conditioning on the pair $(|U(T_U)|, T_U)$, one observes that $T_U^R - T_U$ is distributed as an independent copy of $(T_X|X(0) = |U(T_U)|)$. Therefore, $L^R_{\delta_1,\delta_2,U,-}(u)$ is simply a special case of $m_{\delta_1,U,-}(u)$ with the choice of penalty function $w(x, y) \equiv L_{\delta_2,X}(y)$.

4.1.2 Insurance risk model with positive drift

From the discussion in Sect. 1, the process $\{X(t)\}_{t\geq 0}$ can serve as an insurance risk model in which $c > 0$ is the constant premium income per unit time, whereas Y_+ and Y_- are now interpreted as positive claim and negative claim (or lump sum gain), respectively. For such a model, the positive security loading condition is opposite to the model $\{U(t)\}_{t\geq 0}$ and is given by

$$E[Y] = q_+E[Y_+] - q_-E[Y_-] < cE[V]. \tag{4.7}$$

The above condition is assumed to ensure ruin (i.e. $T_X < \infty$) is not a certain event for $\{X(t)\}_{t\geq 0}$. Given that ruin occurs, the time of recovery is defined by $T_X^R = \inf\{t > T_X : X(t) \geq 0\}$, and the duration of first negative surplus is $T_X^R - T_X$. The condition (4.7) guarantees recovery occurs almost surely given $T_X < \infty$. Note that recovery can possibly occur due to positive drift (resulting in $X(T_X^R) = 0$) or negative claims (resulting in $X(T_X^R) > 0$), respectively, resulting in the quantities

$$L^R_{\delta_1,\delta_2,X,0}(x) = E\big[e^{-\delta_1 T_X - \delta_2(T_X^R - T_X)}1\big\{T_X < \infty, X\big(T_X^R\big) = 0\big\}|X(0) = x\big], \quad x \geq 0, \tag{4.8}$$

and

$$L^R_{\delta_1,\delta_2,X,+}(x) = E\big[e^{-\delta_1 T_X - \delta_2(T_X^R - T_X)}1\big\{T_X < \infty, X\big(T_X^R\big) > 0\big\}|X(0) = x\big], \quad x \geq 0. \tag{4.9}$$

Due to the condition (4.7), we have to restrict $\delta_2 > 0$ so that the results in Sects. 2 and 3 are applicable. One sees that $L^R_{\delta_1,\delta_2,X,0}(x)$ and $L^R_{\delta_1,\delta_2,X,+}(x)$ are both special cases of $m_{\delta_1,X}(x)$ with the choices of penalty $w_X(y) = E[e^{-\delta_2 T_U}1\{U(T_U)=0\}|U(0)=y]$ and $w_X(y) = E[e^{-\delta_2 T_U}1\{U(T_U)<0\}|U(0)=y]$, respectively, which can in turn be retrieved from $m_{\delta_2,U,0}(y)$ with $w_*(y) = e^{(\delta_2/c)y}$ and $m_{\delta_2,U,-}(y)$ with $w(\cdot,\cdot)\equiv 1$. Here the indicator for the event $\{T_U < \infty\}$ is removed since it occurs almost surely because of condition (4.7). Condition (4.7) also guarantees that $m_{\delta_1,X}(\cdot)$ satisfies a defective renewal equation and the results of [35] apply.

4.2 Joint Laplace transform of the first busy period and subsequent idle period

As mentioned in Sect. 1, in queueing systems or storage processes the amount of workload is usually non-negative. Therefore, the process $\{U(t)\}_{t\geq 0}$ has to be modified such that it is bounded below by 0 so as to retrieve the workload process. Denoting the workload process by $\{U^Q(t)\}_{t\geq 0}$, we have

$$U^Q(t) = U(t) - \min\bigl(0, U(t)\bigr), \quad t \geq 0. \tag{4.10}$$

While the first busy period of $\{U^Q(t)\}_{t\geq 0}$ coincides with T_U, its subsequent idle period is given by $T^I_U - T_U$ where $T^I_U = \inf\{t > T_U : U^Q(t) > 0\} = \inf\{t > T_U : Y_{N(t)} > 0\}$. See Fig. 3. In queueing systems, the condition (4.7) is assumed so that the queue must become empty eventually. The joint Laplace transforms for the pair $(T_U, T^I_U - T_U)$ with an initial workload of $U^Q(0) = U(0) = u \geq 0$ are given by

$$L^Q_{\delta_1,\delta_2,U,0}(u) = E\bigl[e^{-\delta_1 T_U - \delta_2(T^I_U - T_U)}1\bigl\{U(T_U)=0\bigr\}|U(0)=u\bigr], \quad u \geq 0, \tag{4.11}$$

and

$$L^Q_{\delta_1,\delta_2,U,-}(u) = E\bigl[e^{-\delta_1 T_U - \delta_2(T^I_U - T_U)}1\bigl\{U(T_U)<0\bigr\}|U(0)=u\bigr], \quad u \geq 0, \tag{4.12}$$

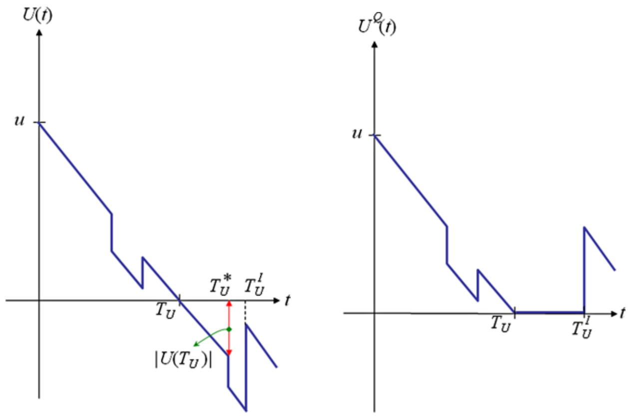

Fig. 3 (**a**) Risk process. (**b**) Workload process

respectively, for termination of busy period by continuous workload removal and by the arrival of a negative customer. We always assume $\delta_1 > 0$ at the expense of the condition (4.7).

For $L^Q_{\delta_1,\delta_2,U,0}(u)$ which is due to continuity, one always has $T_U = T^*_U - |U(T^{*-}_U)|/c$. On the other hand, $T^I_U - T_U$ is distributed as an independent sum of $|U(T^{*-}_U)|/c$ and a compound geometric r.v. with primary probability mass function $q^n_- q_+$ for $n = 0, 1, \ldots$ and secondary density $k(\cdot)$. Therefore, we arrive at

$$L^Q_{\delta_1,\delta_2,U,0}(u) = E\big[e^{-\delta_1 T^*_U} e^{(\frac{\delta_1-\delta_2}{c})|U(T^{*-}_U)|} 1\big\{U(T_U) = 0\big\} | U(0) = u\big] \frac{q_+}{1 - q_- \widetilde{k}(\delta_2)}, \tag{4.13}$$

with the expectation term a special case of $m_{\delta_1,U,0}(u)$ under the penalty function $w_*(y) = e^{[(\delta_1-\delta_2)/c]y}$.

By a similar argument, one concludes that

$$L^Q_{\delta_1,\delta_2,U,-}(u) = E\big[e^{-\delta_1 T_U} 1\big\{U(T_U) < 0\big\} | U(0) = u\big] \frac{q_+ \widetilde{k}(\delta_2)}{1 - q_- \widetilde{k}(\delta_2)}, \tag{4.14}$$

where the expectation term is simply $m_{\delta_1,U,-}(u)$ with $w(\cdot,\cdot) \equiv 1$.

Remark 4 For any subsequent pair of busy period and idle period, the busy period starts with an upward jump distributed as Y_+ and therefore it is sufficient to consider the process $\{Z(t)|Z(0^-) = 0\}_{t\geq 0}$. Hence, such joint Laplace transforms for the termination of busy period by continuous workload removal and by the arrival of a negative customer are simply

$$\begin{aligned} L^Q_{\delta_1,\delta_2,Z,0} &= E\big[L^Q_{\delta_1,\delta_2,U,0}(Y_+)\big] \\ &= E\big[e^{-\delta_1 T^*_Z} e^{(\frac{\delta_1-\delta_2}{c})|Z(T^{*-}_Z)|} 1\big\{Z(T_Z) = 0\big\} | Z\big(0^-\big) = 0\big] \frac{q_+}{1 - q_- \widetilde{k}(\delta_2)}, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} L^Q_{\delta_1,\delta_2,Z,-} &= E\big[L^Q_{\delta_1,\delta_2,U,-}(Y_+)\big] \\ &= E\big[e^{-\delta_1 T_Z} 1\big\{Z(T_Z) < 0\big\} | Z\big(0^-\big) = 0\big] \frac{q_+ \widetilde{k}(\delta_2)}{1 - q_- \widetilde{k}(\delta_2)}, \end{aligned} \tag{4.16}$$

respectively, with the expectation terms in the above two equations being the special cases of $m_{\delta_1,Z,0}(0)$ and $m_{\delta_1,Z,-}(0)$ with the corresponding choices of penalty functions $w_*(y) = e^{[(\delta_1-\delta_2)/c]y}$ and $w(\cdot,\cdot) \equiv 1$.

4.3 Expected total discounted reward payable during idle periods

In this subsection, we consider the same queueing model $\{U^Q(t)\}_{t\geq 0}$ as in Sect. 4.2, with the assumption of condition (4.7). We are interested in evaluating the expected total discounted value for a dollar payable continuously during the idle periods of the queue, or equivalently, the expected total discounted reward earned by a server at a rate of 1 whenever there is no work stored in the queue. With an initial workload of $U^Q(0) = U(0) = u \geq 0$ and a force of interest of $\delta > 0$, such an expectation is denoted by $DR_{\delta,U}(u)$. Since the first payment stream is payable from time T_U to T_U^I, its expected present value (at time 0), namely $DR^*_{\delta,U}(u)$, is given by

$$\begin{aligned} DR^*_{\delta,U}(u) &= E\left[\int_{T_U}^{T_U^I} e^{-\delta y}\,dy \Big| U(0)=u\right] \\ &= \frac{1}{\delta}\left\{E\left[e^{-\delta T_U}|U(0)=u\right] - E\left[e^{-\delta T_U^I}|U(0)=u\right]\right\} \\ &= \frac{1}{\delta}\left\{\left[L^Q_{\delta,0,U,0}(u) + L^Q_{\delta,0,U,-}(u)\right]\right. \\ &\quad \left. - \left[L^Q_{\delta,\delta,U,0}(u) + L^Q_{\delta,\delta,U,-}(u)\right]\right\}, \quad u \geq 0, \end{aligned} \tag{4.17}$$

where notations defined in (4.11) and (4.12) have been used. It is instructive to note that the expected total discounted reward $DR_{\delta,U}(u)$ consists of the expected present value $DR^*_{\delta,U}(u)$ plus the expected discounted reward from time T_U^I onwards. Since at time T_U^I the process restarts at level 0 with an upward jump distributed as Y_+, we have

$$\begin{aligned} DR_{\delta,U}(u) &= DR^*_{\delta,U}(u) + E\left[e^{-\delta T_U^I}|U(0)=u\right]DR^*_{\delta,Z} \\ &= DR^*_{\delta,U}(u) + \left[L^Q_{\delta,\delta,U,0}(u) + L^Q_{\delta,\delta,U,-}(u)\right]DR^*_{\delta,Z}, \quad u \geq 0, \end{aligned} \tag{4.18}$$

where (4.11) and (4.12) are used again. In addition, $DR^*_{\delta,Z}$ is the expected present value (at time T_U^I) of future reward satisfying

$$\begin{aligned} DR^*_{\delta,Z} &= E\left[DR^*_{\delta,U}(Y_+)\right] + E\left[L^Q_{\delta,\delta,U,0}(Y_+) + L^Q_{\delta,\delta,U,-}(Y_+)\right]DR^*_{\delta,Z} \\ &= \frac{1}{\delta}\left[\left(L^Q_{\delta,0,Z,0} + L^Q_{\delta,0,Z,-}\right) - \left(L^Q_{\delta,\delta,Z,0} + L^Q_{\delta,\delta,Z,-}\right)\right] \\ &\quad + \left(L^Q_{\delta,\delta,Z,0} + L^Q_{\delta,\delta,Z,-}\right)DR^*_{\delta,Z} \end{aligned}$$

with the use of (4.15)–(4.17). Solving the above equation for $DR^*_{\delta,Z}$ yields

$$DR^*_{\delta,Z} = \frac{1}{\delta}\frac{(L^Q_{\delta,0,Z,0} + L^Q_{\delta,0,Z,-}) - (L^Q_{\delta,\delta,Z,0} + L^Q_{\delta,\delta,Z,-})}{1 - (L^Q_{\delta,\delta,Z,0} + L^Q_{\delta,\delta,Z,-})},$$

leading to a complete characterization of $DR_{\delta,U}(u)$ via (4.18).

5 Concluding remark

This paper studies the Gerber–Shiu function in a stochastic model involving two-sided jumps and a continuous downward drift. With arbitrary distributions of jump sizes and inter-arrival times, various general structures are exploited using defective renewal equations. This is in contrary to most of the traditional analysis, in which processes with stationary and independent increment or phase-type inter-arrival time V are used. Although our solutions in Sect. 2 involve convolutions, in cases where both V and Y_- are distributed as, for example, a combination of exponentials, these convolutions can be explicitly evaluated due to the nice form of (3.13) and (3.18).

We also remark that in the process $\{U(t)\}_{t\geq 0}$ defined by (1.1), the arrivals of jumps, no matter positive or negative, are assumed to be generated by the same renewal process $\{N(t)\}_{t\geq 0}$, and therefore the process $\{U(t)\}_{t\geq 0}$ restarts after a jump. An alternative model could have been proposed by assuming that the arrivals of upward and downward jumps follow independent renewal processes $\{N_+(t)\}_{t\geq 0}$ and $\{N_-(t)\}_{t\geq 0}$, respectively, resulting in

$$U(t) = u - ct + \sum_{i=1}^{N_+(t)} Y_{+,i} - \sum_{i=1}^{N_-(t)} Y_{-,i}, \quad t \geq 0, \tag{5.1}$$

where $\{Y_{+,i}\}_{i=1}^{\infty}$ and $\{Y_{-,i}\}_{i=1}^{\infty}$ are independent i.i.d. sequences distributed as Y_+ and Y_-, respectively, both independent of $\{N_+(t)\}_{t\geq 0}$ and $\{N_-(t)\}_{t\geq 0}$. Of course, if $\{N(t)\}_{t\geq 0}$ in (1.1) is a Poisson process with rate $\lambda > 0$, then (1.1) and (5.1) are equivalent if $\{N_+(t)\}_{t\geq 0}$ and $\{N_-(t)\}_{t\geq 0}$ are Poisson processes with rates λq_+ and λq_-, respectively. In general, the process (5.1) does not restart after a jump, and therefore the arguments used in this paper are not applicable.

Acknowledgements Part of the work was completed during the author's visit to the Department of Actuarial Science at the University of Lausanne, and the host's hospitality is greatly appreciated. The author would like to thank the anonymous referees and Professor Hansjörg Albrecher for helpful comments and suggestions as well as Professor Kristina Sendova for use of various references. Support for the author from a start-up fund provided by the Faculty of Science and the Department of Statistics and Actuarial Science as well as the Seed Funding for Basic Research (Project number: 201103159001) provided by the University Research Committee at the University of Hong Kong is also gratefully acknowledged.

Appendix

Throughout the paper, a large number of notations are used for various Gerber–Shiu type functions and their related quantities. This Appendix aims at providing a summary of their definitions as well as their relationships. As a general rule of thumb:

1. The letter 'm' represents Gerber–Shiu functions, whereas 'L' denotes (joint) Laplace transforms of random times.
2. The subscripts 'U', 'Z' and 'X' are related to the processes (1.1), (1.7) and (4.1), respectively.
3. The subscripts '$-$' and '0', respectively, correspond to 'jumps' and 'continuity' as the cause of ruin.

Table 1 List of major notations

Notation	Definition
$m_{\delta,U,-}(u)$	GSF defined by (1.3) for ruin of $\{U(t)\}_{t\ge 0}$ by jumps
$m_{\delta,U,0}(u)$	GSF defined by (1.6) for ruin of $\{U(t)\}_{t\ge 0}$ by continuity
$m_{\delta,Z,-}(z)$	GSF defined by (1.8) for ruin of $\{Z(t)\}_{t\ge 0}$ by jumps; related to $m_{\delta,U,-}(\cdot)$ via (1.9); can be represented as (2.17)
$m_{\delta,Z,0}(z)$	GSF defined by (1.10) for ruin of $\{Z(t)\}_{t\ge 0}$ by continuity; related to $m_{\delta,U,0}(\cdot)$ via (1.11); can be represented as (2.16)
$\mathbf{m}_\delta(u)$	Matrix GSF containing above four elements; defined by (2.23) with solution uniquely determined by the Markov renewal equation (2.22)
$m_{\delta,Z,-,2}(z)$	Special case of $m_{\delta,Z,-}(z)$ under $w(x,y)\equiv w_2(y)$; can be represented as (2.19)
$m_{\delta,X}(x)$	GSF defined by (4.2) for ruin of $\{X(t)\}_{t\ge 0}$; known from [35]
$L_{\delta,X}(x)$	LT of the time of ruin of $\{X(t)\}_{t\ge 0}$ defined by (4.3); special case of $m_{\delta,X}(x)$ under $w_X(\cdot)\equiv 1$
$L^R_{\delta_1,\delta_2,U,0}(u)$	Joint LT defined by (4.4) for ruin of $\{U(t)\}_{t\ge 0}$ by jumps; special case of $m_{\delta_1,U,0}(u)$ under (4.6)
$L^R_{\delta_1,\delta_2,U,-}(u)$	Joint LT defined by (4.5) for ruin of $\{U(t)\}_{t\ge 0}$ by continuity; special case of $m_{\delta_1,U,-}(u)$ under $w(x,y)\equiv L_{\delta_2,X}(y)$
$L^R_{\delta_1,\delta_2,X,0}(x)$	Joint LT defined by (4.8) for recovery of $\{X(t)\}_{t\ge 0}$ by continuity; special case of $m_{\delta_1,X}(x)$ under $w_X(y)=E[e^{-\delta_2 T_U}1\{U(T_U)=0\}\vert U(0)=y]$
$L^R_{\delta_1,\delta_2,X,+}(x)$	Joint LT defined by (4.9) for recovery of $\{X(t)\}_{t\ge 0}$ by jumps; special case of $m_{\delta_1,X}(x)$ under $w_X(y)=E[e^{-\delta_2 T_U}1\{U(T_U)<0\}\vert U(0)=y]$
$L^Q_{\delta_1,\delta_2,U,0}(u)$	Joint LT defined by (4.11) for termination of busy period of $\{U^Q(t)\}_{t\ge 0}$ by continuous workload removal; with solution (4.13)
$L^Q_{\delta_1,\delta_2,U,-}(u)$	Joint LT defined by (4.12) for termination of busy period of $\{U^Q(t)\}_{t\ge 0}$ by arrival of a negative customer; with solution (4.14)
$L^Q_{\delta_1,\delta_2,Z,0}$	Given by (4.15) in relation to $L^Q_{\delta_1,\delta_2,U,0}(\cdot)$
$L^Q_{\delta_1,\delta_2,Z,-}$	Given by (4.16) in relation to $L^Q_{\delta_1,\delta_2,U,-}(\cdot)$

4. The superscripts (for the L functions) 'R' and 'Q' denote functions for 'recovery' and 'queues' (see (4.10)), respectively.

In the following Table, 'GSF' and 'LT' are abbreviations for 'Gerber–Shiu function' and 'Laplace transform', respectively.

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