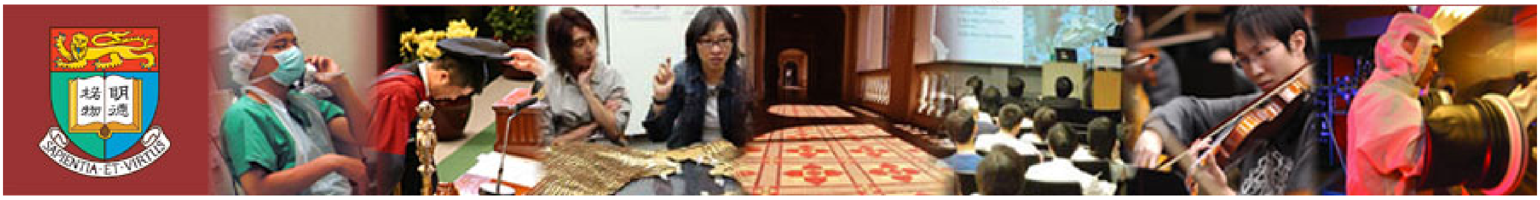

Title	A two-dimensional risk model with proportional reinsurance
Author(s)	Badescu, AL; Cheung, ECK; Rabehasaina, L
Citation	Journal Of Applied Probability, 2011, v. 48 n. 3, p. 749-765
Issued Date	2011
URL	http://hdl.handle.net/10722/143106
Rights	Creative Commons: Attribution 3.0 Hong Kong License

A two-dimensional risk model with proportional reinsurance

Andrei L. Badescu *, Eric C.K. Cheung † and Landy Rabehasaina ‡

February 18, 2011

Abstract

In this paper, we consider an extension of the two-dimensional risk model introduced by Avram et al. (2008a). To this end, we assume two insurers in which the first is subject to claims arising from two independent compound Poisson processes. The second insurer, that can be viewed as a different line of business of the same insurer or as a reinsurer, covers a proportion of the claims caused by one of these two compound Poisson processes. The Laplace transform of the time until at least one insurer is ruined is derived when the claim sizes follow a general distribution. The surplus level of the first insurer when the second one is ruined first is discussed in the end in connection with a few open questions.

Keywords: Two-dimensional risk model, Proportional reinsurance, Geometric argument, Absorbing set, Time to ruin, Deficit at ruin.

2000 Mathematics Subject Classification: Primary 60G51, Secondary 60K30, 60J75

*Department of Statistics, University of Toronto, 100 St. George Street, Toronto, Ontario, Canada. E-mail: badescu@utstat.toronto.edu

†Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam, Hong Kong. E-mail: eckc@hku.hk

‡Département de Mathématiques, Université de Franche Comté, 16 route de Gray, 25030 Besançon, France. E-mail: lrabehas@univ-fcomte.fr

1 Introduction

Multi-dimensional risk theory represents a very attractive topic that has gained a lot of popularity in recent few years. The problems that arise when studying the class of multivariate risk models involve an increased level of complexity when compared to the class of univariate risk models, mainly due to the dependence of claim severities and/or their inter-arrival times among the several lines of business under consideration. Historically, a quantity that has been mostly treated in the univariate literature is the time to ruin, or the time of default, where ruin is defined as the first passage time below a certain threshold, with level zero being the usual critical level. However, when one works with multi-dimensional collective risk theory, ruin can be defined in several ways. The first time when at least one of the risk processes falls below level zero, as well as the first time when all the risk processes are below level zero simultaneously represent some particular examples for which bounds, asymptotic results and extremely rare explicit solutions have been obtained for the ruin probabilities. Recent results pertaining to multi-dimensional risk models can be found among others in Chan et al. (2003), Cai and Li (2005, 2007), Yuen et al. (2006), Li et al. (2007), Avram et al. (2008a, b), Dang et al. (2009), Rabehasaina (2009) and Gong et al. (2010). Among these, the one that motivates the present work is Avram et al. (2008a). The authors considered the joint ruin problem for two insurance companies that divide between them in different proportions both the premium income and the aggregate claims process which is modeled through a compound Poisson process. In practice, such a problem can be interpreted as an insurer-reinsurer scenario, where the reinsurer takes over a proportion of the insurer's losses. Restricting the claims to be exponentially distributed, Avram et al. (2008a) obtained an explicit analytic expression for the Laplace transform of the time to ruin, where ruin is defined as the first time when at least one of the risk processes drops below zero. Using a geometric argument, the authors reduce the bivariate ruin problem to two distinct univariate problems, whose solutions are more easily obtainable. More specifically, Avram et al. (2008a) are able to find a (deterministic) critical time such that if ruin occurs after the critical time then it is caused by the first risk process, while if ruin occurs prior to the critical time then it is caused by the second risk process.

In this paper, we generalize the work of Avram et al. (2008a) by considering a two-dimensional insurance risk model where one of the risk processes faces claims arising from two independent compound Poisson processes, out of which only one is shared proportionally with the second risk process. As discussed in Section 2, such risk model can be viewed in real life as a model for the surplus processes of two lines of business of the same company, or as an insurer-reinsurer application. Mathematically the evolution of our two-dimensional risk process denoted by $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ is described as

$$\begin{cases} dY_t^1 = p_1\,dt - a\,dL_t - dS_t, \\ dY_t^2 = p_2\,dt - (1-a)\,dL_t, \\ (Y_0^1, Y_0^2) = (y_1, y_2), \end{cases} \tag{1.1}$$

with (y_1, y_2) the initial surpluses at time zero and (p_1, p_2) the premium rates, where $y_i \geq 0$ and $p_i > 0$, $i = 1, 2$. In addition, the aggregate claims processes $\{L_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$ are represented as $L_t = \sum_{i=1}^{N_t^L} Z_i^L$ and $S_t = \sum_{i=1}^{N_t^S} Z_i^S$. Here, $\{N_t^L\}_{t\geq 0}$ and $\{N_t^S\}_{t\geq 0}$ are Poisson processes with intensities λ_L and λ_S respectively, whereas $\{Z_i^L\}_{i=1}^{\infty}$ and $\{Z_i^S\}_{i=1}^{\infty}$ are the sequences of positive claims. It is assumed that $\{N_t^L\}_{t\geq 0}$, $\{N_t^S\}_{t\geq 0}$, $\{Z_i^L\}_{i=1}^{\infty}$ and $\{Z_i^S\}_{i=1}^{\infty}$ are all mutually independent, and $\{Z_i^L\}_{i=1}^{\infty}$ and $\{Z_i^S\}_{i=1}^{\infty}$ are independent and identically distributed sequences distributed as the generic random variables Z^L and Z^S with probability density functions (pdf) $f_L(\cdot)$ and $f_S(\cdot)$ respectively. Note that we have relaxed Avram et al. (2008a)'s exponential claim size assumption to allow for *arbitrary* claim size distributions for both Z^L and Z^S. From the dynamics of (1.1), it is clear that the first and the second insurer share the compound Poisson process $\{L_t\}_{t\geq 0}$ at a constant ratio of a and $1-a$ respectively, where $a \in (0, 1)$, whereas $\{S_t\}_{t\geq 0}$ is covered entirely by the first insurer. The positive security loading conditions for each of the processes $\{Y_t^1\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ are given by

$$p_1 > aE[L_1] + E[S_1], \tag{1.2}$$

$$p_2 > (1-a)E[L_1], \tag{1.3}$$

where $E[L_1]$ and $E[S_1]$ respectively represents the average aggregate claims per unit time arising from the processes $\{L_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$. By defining the time of ruin for the i-th risk process $\tau_i = \inf\{t \geq 0 | Y_t^i < 0\}$, $i = 1, 2$, the key quantity of interest in this paper is the time at which at least one of the surplus processes $\{Y_t^1\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ becomes negative, namely

$$\tau = \inf\{t \geq 0 | \min(Y_t^1, Y_t^2) < 0\} = \min(\tau_1, \tau_2). \tag{1.4}$$

The paper is structured as follows. Using geometric arguments, in Section 2 we derive a sufficient set of constraints that will enable us to obtain an analytic solution for the Laplace transform of the time to ruin defined in (1.4). Section 3 gives two preliminary results required for the evaluation of the afore-mentioned Laplace transform whose derivation is presented in Section 4. Section 5 briefly discusses the case where ruin of $\{Y_t^2\}_{t\geq 0}$ occurs before $\{Y_t^1\}_{t\geq 0}$ and suggests a few potential open problems.

2 Model constraints via geometric interpretation

In this section we employ a similar methodology to the one in Avram et al. (2008a) to reduce the proposed two-dimensional risk model to more tractable univariate problems, for which various results in the literature can be exploited. To this end, in Figure 1 we present a sample path of the evolution of the

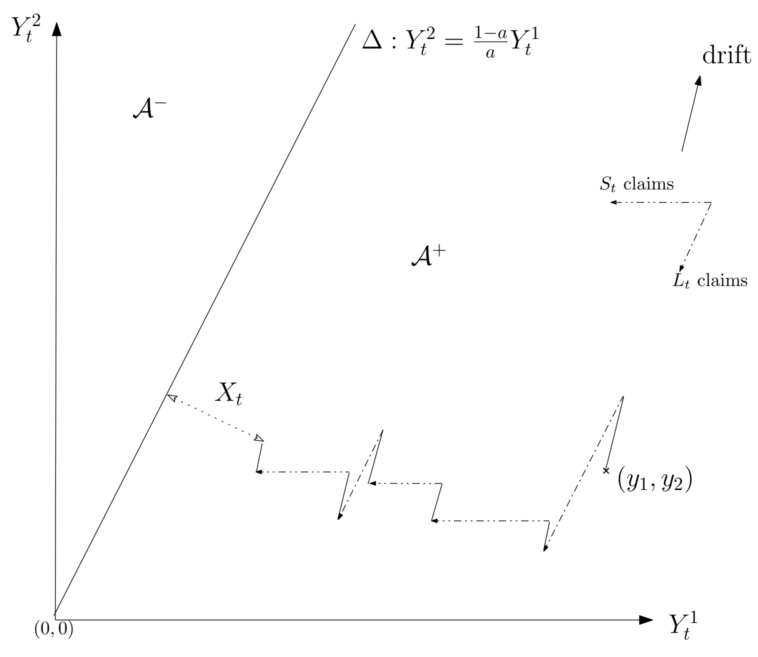

Figure 1: A sample path of $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$.

two-dimensional risk model defined in (1.1). The vertical y-axis represents the surplus process $\{Y_t^2\}_{t\geq 0}$, whereas the horizontal x-axis represents $\{Y_t^1\}_{t\geq 0}$. We denote by $\Delta \subset \mathbb{R}^2$ the line whose equation is given by $y = \dfrac{1-a}{a}x$, and we let $\{X_t\}_{t\geq 0}$ be the process defined by $X_t =< \vec{v}, (Y_t^1, Y_t^2) >$ (with $< .,. >$ denoting the usual scalar product) where $\vec{v} = (1-a, -a)$. Geometrically X_t can be seen as the algebraic distance (which is proportional to Euclidean distance) between $(Y_t^1, Y_t^2) \in \mathbb{R}^2$ and Δ , as illustrated on Figure 1. It satisfies

$$\begin{cases} dX_t = [(1-a)p_1 - ap_2]\, dt - (1-a)\, dS_t, \\ X_0 =< \vec{v}, (y_1, y_2) >= (1-a)y_1 - ay_2. \end{cases} \tag{2.1}$$

Line Δ splits $\mathbb{R}^2$ in two disjoint sets $\mathcal{A}^+$ and $\mathcal{A}^-$ defined by

$$\mathcal{A}^+ := \{\vec{x} \in \mathbb{R}^2 | \ < \vec{x}, \vec{v} > > 0\}, \quad \text{and} \quad \mathcal{A}^- := \{\vec{x} \in \mathbb{R}^2 | \ < \vec{x}, \vec{v} > < 0\}.$$

The sets are such that $X_t > 0$ is equivalent to $(Y_t^1, Y_t^2) \in \mathcal{A}^+$, and $X_t < 0$ is equivalent to $(Y_t^1, Y_t^2) \in \mathcal{A}^-$. We further introduce the time of ruin of the newly defined univariate risk process $\{X_t\}_{t\geq 0}$ and denote it by $\tau_X = \inf\{t \geq 0 | X_t < 0\}$. Without loss of generality and in order to avoid trivialities (as we will see later in the section), we let the two-dimensional risk process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ start in the set $\mathcal{A}^+$, or equivalently we let $X_0 = (1-a)y_1 - ay_2 > 0$. The process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ will then drift upwards as long as no claim occurs in any of the two individual risk processes $\{Y_t^1\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$. The occurrence of a claim in $\{L_t\}_{t\geq 0}$ will make $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ move downwards towards the origin $(0,0)$, parallel to the line Δ; whereas the occurrence of a claim in $\{S_t\}_{t\geq 0}$ will make the process move to the left towards the vertical axis in parallel to the horizontal one. The time of ruin defined in equation (1.4) can then be interpreted as the first time the two-dimensional risk process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ exits the positive quadrant. Assuming that we start in $\mathcal{A}^+$, a closer look at Figure 1 reveals a competition between the first passage times τ_X and τ_2 such that, if the process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ downcrosses the horizontal axis before time τ_X (i.e. $\tau_2 < \tau_X$) then ruin in the two-dimensional process will be caused purely by ruin in $\{Y_t^2\}_{t\geq 0}$ (i.e. $\tau = \tau_2$). A more challenging question arises in the opposite case when $\tau_X \leq \tau_2$. Under this scenario, in order to be able to analyze the time to ruin τ pertaining to the bivariate risk process defined in (1.1), we need to add an extra constraint that makes the set $\mathcal{A}^-$ an absorbing set, namely

$$\frac{p_2}{p_1} > \frac{1-a}{a}, \tag{2.2}$$

which will be assumed throughout the entire paper. Geometrically, the condition (2.2) assumes that the slope of increase of the process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ (during the collection of premium income) exceeds the slope of the line Δ, ensuring that once $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ enters the set $\mathcal{A}^-$, it will never come back to $\mathcal{A}^+$. Equivalently, condition (2.2) guarantees that $\{X_t\}_{t\geq 0}$ is a strictly decreasing process which is evident from the dynamics (2.1). With (2.2) satisfied, the original two-dimensional ruin problem can be divided in two distinct univariate ruin problems as follows: $\tau_2 < \tau_X$ implies $\tau = \tau_2$, and $\tau_2 \geq \tau_X$ implies $\tau = \tau_1$. The present model generalizes the one studied in Avram et al. (2008a), but the difficulty here arises from the fact that the deterministic critical time therein is now replaced by the stochastic one τ_X. An important property that makes this approach a tractable one is due to the fact that under our construction the processes $\{X_t\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ are independent.

As previously mentioned in the introduction, we conclude this section with two practical insurance interpretations of the model defined in (1.1), together with further simplifications of the constraints introduced in (1.2), (1.3) and (2.2).

I. $\{Y_t^1\}_{t\geq 0}$ *and* $\{Y_t^2\}_{t\geq 0}$ *are regarded as two lines of business of the same insurance company*, i.e. the two lines split $\{L_t\}_{t\geq 0}$ proportionally, while $\{S_t\}_{t\geq 0}$ is taken by line 1 only. The first line possibly has different loadings $\theta_1 > 0$ and $\theta_2 > 0$ on two different types of claims, so that

$$p_1 = (1+\theta_1)aE[L_1] + (1+\theta_2)E[S_1], \tag{2.3}$$

while for the second line it is assumed a different security loading $\theta_3 > 0$, such that

$$p_2 = (1+\theta_3)(1-a)E[L_1]. \tag{2.4}$$

Because the second line only shares a proportion of the $\{L_t\}_{t\geq 0}$ compound Poisson process, we assume it has a larger loading $\theta_3 > \theta_1$. The total premium rate paid by policyholders for the aggregate claim process $\{L_t\}_{t\geq 0}$ is $[(1+\theta_1)a + (1+\theta_3)(1-a)]E[L_1]$. It is clear that under assumptions (2.3) and (2.4) the loading conditions (1.2) and (1.3) hold. Simple manipulations of (2.3) and (2.4) transform the constraint (2.2) into the condition

$$\frac{E[L_1]}{E[S_1]} > \frac{1+\theta_2}{a(\theta_3-\theta_1)} \tag{2.5}$$

that enables us to find the Laplace transform of the time to ruin τ (see Section 4). Note that under condition (2.5), the process $\{L_t\}_{t\geq 0}$ may be *generically* interpreted as a riskier process than $\{S_t\}_{t\geq 0}$ in terms of the mean. Observe that in a practical situation the right hand side of (2.5) will be larger than 1.

II. $\{Y_t^1\}_{t\geq 0}$ *and* $\{Y_t^2\}_{t\geq 0}$ *represent the surplus processes of an insurer and a reinsurer respectively.* It is then assumed that the insurer receives premiums at loadings θ_1 and θ_2 for the aggregate claim processes $\{L_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$ respectively. The insurer reinsures a proportion $1-a$ of the claims arising from $\{L_t\}_{t\geq 0}$, paying the reinsurer premiums at loading θ_3. Then

$$p_1 = (1+\theta_1)E[L_1] + (1+\theta_2)E[S_1] - (1+\theta_3)(1-a)E[L_1], \tag{2.6}$$

whereas p_2 is the same as in (2.4). It is again assumed that $\theta_3 > \theta_1$, otherwise the insurer may be tempted to reinsure the entire $\{L_t\}_{t\geq 0}$ while receiving arbitrage premium income. The premium rate paid by policyholders for $\{L_t\}_{t\geq 0}$ is just $(1+\theta_1)E[L_1]$. Note that under this possible practical interpretation, the positive security loading condition (1.2) does not necessarily hold, while (1.3) still holds. Further manipulations of (1.2) using (2.6) gives an equivalent condition

$$\frac{E[L_1]}{E[S_1]}\left[\theta_1 - \theta_3(1-a)\right] + \theta_2 > 0. \tag{2.7}$$

Using (2.4) and (2.6), the constraint condition (2.2) is transformed to

$$\frac{E[L_1]}{E[S_1]} > \frac{1+\theta_2}{\theta_3-\theta_1}. \tag{2.8}$$

As in the first practical interpretation, in practice $\frac{1+\theta_2}{\theta_3-\theta_1} > 1$ in most of the cases, which means that the condition (2.8) requires that the process $\{L_t\}_{t\geq 0}$ represents a larger risk, in terms of average, than the $\{S_t\}_{t\geq 0}$ process. Hence it makes sense for the insurer to transfer part of the risk $\{L_t\}_{t\geq 0}$ to a reinsurer through reinsurance. Under the current insurer-reinsurer interpretation, the condition (2.8) is compulsory to determine the Laplace transform of the time to ruin τ using geometric arguments, whereas relation (2.7) simply ensures that insurer will not be ruined a.s. and that the problem does make sense from a practical point of view. Suppose (2.7) is assumed. We need to distinguish between two cases as follows.

1. If $1 < \frac{\theta_3}{\theta_1} \leq \frac{1}{1-a}$, then the condition (2.7) is automatically satisfied and hence only condition (2.8) is required.

2. If $\frac{\theta_3}{\theta_1} > \frac{1}{1-a}$, then in order for both (2.7) and (2.8) to be satisfied we require

$$\frac{1+\theta_2}{\theta_3-\theta_1} < \frac{E[L_1]}{E[S_1]} < \frac{\theta_2}{\theta_3(1-a)-\theta_1}.$$

If one intends to relax the positive loading condition (2.7) (which is theoretically possible as all results obtained in the present paper apply even if (2.7) does not hold), the weaker condition $p_1 > 0$ will be required instead. Because of (2.6), such a weaker condition is translated to

$$\frac{E[L_1]}{E[S_1]}\left[\theta_3-\theta_1-(1+\theta_3)a\right] < 1+\theta_2. \tag{2.9}$$

Again two cases need to be distinguished.

1. If $0 < \theta_3 - \theta_1 \leq (1+\theta_3)a$, then (2.9) is satisfied automatically and therefore one only needs (2.8).

2. If $\theta_3 - \theta_1 > (1+\theta_3)a$, then (2.8) together with (2.9) yields the condition

$$\frac{1+\theta_2}{\theta_3-\theta_1} < \frac{E[L_1]}{E[S_1]} < \frac{1+\theta_2}{\theta_3-\theta_1-(1+\theta_3)a}.$$

In the next section, we analyze two important quantities for the independent processes $\{X_t\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ that will help us to obtain the desired Laplace transform of the time to ruin τ.

3 Preliminary results

As discussed in the previous section, the time to ruin τ_X in the process $\{X_t\}_{t\geq 0}$ plays a key role in the analysis. If ruin in the $\{X_t\}_{t\geq 0}$ process occurs prior to ruin in the $\{Y_t^2\}_{t\geq 0}$ process (i.e. $\tau_X \leq \tau_2$), then the knowledge of the surplus level $Y_{\tau_X}^1$ is of crucial importance because ruin of the bivariate process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ will be due to ruin of $\{Y_t^1\}_{t\geq 0}$ (i.e. $\tau = \tau_1$). It is easy to see that under such scenario one has the relationship

$$Y_{\tau_X}^1 = \frac{a}{1-a}Y_{\tau_X}^2 - \frac{1}{1-a}|X_{\tau_X}|, \tag{3.1}$$

which expresses the surplus levels of the process $\{Y_t^1\}_{t\geq 0}$ in terms of those of $\{Y_t^2\}_{t\geq 0}$ and $\{X_t\}_{t\geq 0}$, all evaluated at time τ_X. Driven by equation (3.1), in this section we derive two preliminary results as follows.

On one hand we are interested in the analysis of the time of ruin and the deficit at ruin in the $\{X_t\}_{t\geq 0}$ process. Under the current assumptions, $\{X_t\}_{t\geq 0}$ can be viewed as a risk process with positive initial surplus, negative drift (see the sufficient condition (2.2)) and downward jumps. Thus, ruin in the $\{X_t\}_{t\geq 0}$ process will occur almost surely, in two possible ways: due to jumps (i.e. $X_{\tau_X} < 0$) or due to continuity (i.e. $X_{\tau_X} = 0$). As a consequence, an important quantity that will be analyzed in the following subsection is the joint distribution of the time to ruin τ_X and the level of the deficit at ruin $|X_{\tau_X}|$.

On the other hand, if $\tau_X \leq \tau_2$, the level of the process $\{Y_t^2\}_{t\geq 0}$ at ruin time τ_X is mandatory in the evaluation of (3.1). Thus, in Section 3.2, we present the distribution of the surplus process $\{Y_t^2\}_{t\geq 0}$ at a given time t avoiding ruin enroute, namely $\mathrm{P}_{y_2}\left(\inf_{s\leq t} Y_s^2 > 0, Y_t^2 \in du\right)$. The notation $\mathrm{P}_y(\cdot)$, represents the conditional probability given the initial surplus level y at time zero.

3.1 The joint density of the time and deficit at ruin in $\{X_t\}_{t\geq 0}$

Starting with (2.1), we rewrite its first equation as

$$dX_t = -c\,dt - dS_t^a, \tag{3.2}$$

where $c = ap_2 - (1-a)p_1 > 0$ and $S_t^a = (1-a)S_t$. Note that $\{S_t^a\}_{t\geq 0}$ is still a compound Poisson process with the same arrival rate λ_S as $\{S_t\}_{t\geq 0}$, but with scaled secondary generic random variable $Z^{a,S} = (1-a)Z^S$ having associated pdf $f_{a,S}(\cdot)$. Furthermore, we denote by $h_C(t|x)$ the density of τ_X at t $(0 < t < \frac{x}{c})$ for ruin by continuity given that $X_0 = x$, and by $h_J(z,t|x)$ the joint density of $(|X_{\tau_X}|, \tau_X)$ at (z,t) $(0 < t < \frac{x}{c},\ z > 0)$ for ruin by jumps given $X_0 = x$. The following proposition gives the above densities in explicit form, for generally distributed claim sizes.

Proposition 1 *In the risk process $\{X_t\}_{t\geq 0}$ defined by (3.2) with $X_0 = x > 0$, the joint distribution at the time of ruin τ_X and the deficit at ruin $|X_{\tau_X}|$ consists of the following contributions.*

1. ***Due to continuity*** *(i.e.* $|X_{\tau_X}| = 0$*)*

   (a) *A point mass at* $\tau_X = \frac{x}{c}$ *with probability* $P_x\left(\tau_X = \frac{x}{c}, |X_{\tau_X}| = 0\right) = e^{-\frac{\lambda_S}{c}x}$.

   (b) *The density part of* τ_X *given by*

$$h_C(t|x) = c\sum_{n=1}^{\infty} e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!} f_{a,S}^{*n}(x-ct), \tag{3.3}$$

   for $0 < t < \frac{x}{c}$, *where* $f_{a,S}^{*n}(\cdot)$ *is the n-fold convolution of* $f_{a,S}(\cdot)$ *with itself.*

2. ***Due to jumps*** *(i.e.* $|X_{\tau_X}| > 0$*)*

$$h_J(z,t|x) = \lambda_S e^{-\lambda_S t} f_{a,S}(z+x-ct) + \lambda_S \sum_{n=1}^{\infty} e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}\int_0^{x-ct} f_{a,S}(z+y) f_{a,S}^{*n}(x-ct-y)\,dy, \tag{3.4}$$

   for $0 < t < \frac{x}{c}$ *and* $z > 0$.

Proof.

1. In order to derive the joint distribution of the time to ruin and the deficit at ruin for $\{X_t\}_{t\geq 0}$, we first consider the Laplace transform of τ_X due to continuity, namely

$$\phi_{\beta,C}(x) = E_x\left[e^{-\beta\tau_X}\mathbf{1}_{\{X_{\tau_X}=0\}}\right],$$

where $\beta \geq 0$ is the Laplace transform argument and E_x represents the conditional expectation given the initial surplus level x at time zero. By conditioning on the time and the size of the first claim, and using the Markov property of $\{X_t\}_{t\geq 0}$ followed by a change of variable, we obtain

$$\begin{aligned}
\phi_{\beta,C}(x) &= \int_{\frac{x}{c}}^{\infty} e^{-\beta\left(\frac{x}{c}\right)}\lambda_S e^{-\lambda_S t}\,dt + \int_0^{\frac{x}{c}} e^{-\beta t}\lambda_S e^{-\lambda_S t}\int_0^{x-ct}\phi_{\beta,C}(x-ct-y) f_{a,S}(y)\,dy\,dt \\
&= e^{-\frac{\lambda_S+\beta}{c}x} + \frac{\lambda_S}{c}\int_0^x e^{-\frac{\lambda_S+\beta}{c}v}\int_0^{x-v}\phi_{\beta,C}(x-v-y) f_{a,S}(y)\,dy\,dv \\
&= e^{-\frac{\lambda_S+\beta}{c}x} + \frac{\lambda_S}{c}\left(e^{-\frac{\lambda_S+\beta}{c}\cdot} * \phi_{\beta,C} * f_{a,S}\right)(x),
\end{aligned}$$

where $*$ is the convolution operator such that $(a_1 * a_2)(y) = \int_0^x a_1(x-y)a_2(y)\,dy$ $(x \geq 0)$ for any functions $a_1(\cdot)$ and $a_2(\cdot)$ on $(0,\infty)$. If we define the Laplace transforms $\widetilde{\phi}_{\beta,C}(s) = \int_0^\infty e^{-sx}\phi_{\beta,C}(x)\,dx$ and

$\widetilde{f}_{a,S}(s) = \int_0^\infty e^{-sx} f_{a,S}(x)\,dx$, further taking an extra Laplace transform with respect to x in the above equation and then solving for $\widetilde{\phi}_{\beta,C}(s)$ leads to

$$\widetilde{\phi}_{\beta,C}(s) = \frac{\frac{1}{\frac{\lambda_S+\beta}{c}+s}}{1 - \frac{\lambda_S}{c}\frac{1}{\frac{\lambda_S+\beta}{c}+s}\widetilde{f}_{a,S}(s)} = \frac{1}{\frac{\lambda_S+\beta}{c}+s} + \sum_{n=1}^{\infty}\left(\frac{\lambda_S}{c}\right)^n \frac{1}{\left(\frac{\lambda_S+\beta}{c}+s\right)^{n+1}}[\widetilde{f}_{a,S}(s)]^n.$$

Therefore, via Laplace transform inversion with respect to s we arrive at

$$\begin{aligned} \phi_{\beta,C}(x) &= e^{-\frac{\lambda_S+\beta}{c}x} + \sum_{n=1}^{\infty}\left(\frac{\lambda_S}{c}\right)^n \int_0^x \frac{y^n e^{-\frac{\lambda_S+\beta}{c}y}}{n!} f_{a,S}^{*n}(x-y)\,dy \\ &= e^{-\frac{\lambda_S+\beta}{c}x} + c\sum_{n=1}^{\infty}\frac{\lambda_S^n}{n!}\int_0^{\frac{x}{c}} e^{-(\lambda_S+\beta)t} t^n f_{a,S}^{*n}(x-ct)\,dt. \end{aligned} \tag{3.5}$$

An extra Laplace transform inversion with respect to β yields the desired result in *1.* Probabilistically the parts *(a)* and *(b)* can be interpreted as follows.

(a) The process $\{X_t\}_{t\geq 0}$ decreases continuously for a length of time $\frac{x}{c}$ without any claim in the interim.

(b) The density part (3.3) arises when there is at least one jump (i.e. $n \geq 1$) before ruin by continuity. The term $e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}$ is the probability that there are n jumps until time t, and the sum of the n jumps should be exactly $x - ct$ so that ruin occurs by continuity at time t, giving rise to the term $f_{a,S}^{*n}(x-ct)$. The factor c reflects a change in unit.

2. Similarly, we consider the quantity for ruin of $\{X_t\}_{t\geq 0}$ due to jumps defined by

$$\phi_{\beta,J}(x) = E_x\left[e^{-\beta\tau_X} w(|X_{\tau_X}|)\mathbf{1}_{\{X_{\tau_X}<0\}}\right],$$

where $w(\cdot)$ is a penalty function that depends only on the deficit at ruin $|X_{\tau_X}|$. Conditioning again on the time and the size of the first claim, we obtain

$$\begin{aligned} \phi_{\beta,J}(x) &= \int_0^{\frac{x}{c}} e^{-\beta t}\lambda_S e^{-\lambda_S t}\int_0^{x-ct}\phi_{\beta,J}(x-ct-y) f_{a,S}(y)\,dy\,dt \\ &\quad + \int_0^{\frac{x}{c}} e^{-\beta t}\lambda_S e^{-\lambda_S t}\int_{x-ct}^{\infty} w(y-(x-ct)) f_{a,S}(y)\,dy\,dt \\ &= \frac{\lambda_S}{c}\left(e^{-\frac{\lambda_S+\beta}{c}\cdot} * \phi_{\beta,J} * f_{a,S}\right)(x) + \frac{\lambda_S}{c}\left(e^{-\frac{\lambda_S+\beta}{c}\cdot} * \omega_a\right)(x), \end{aligned}$$

where

$$\omega_a(x) = \int_x^\infty w(y-x) f_{a,S}(y)\, dy. \tag{3.6}$$

Taking further Laplace transforms yields

$$\widetilde{\phi}_{\beta,J}(s) = \frac{\frac{\lambda_S}{c}\frac{1}{\frac{\lambda_S+\beta}{c}+s}\widetilde{\omega}_a(s)}{1-\frac{\lambda_S}{c}\frac{1}{\frac{\lambda_S+\beta}{c}+s}\widetilde{f}_{a,S}(s)} = \frac{\lambda_S}{c}\widetilde{\omega}_a(s)\widetilde{\phi}_{\beta,C}(s),$$

where $\widetilde{\phi}_{\beta,J}(s) = \int_0^\infty e^{-sx}\phi_{\beta,J}(x)\, dx$ and $\widetilde{\omega}_a(s) = \int_0^\infty e^{-sx}\omega_a(x)\, dx$. Hence, inversion with respect to β leads to

$$\phi_{\beta,J}(x) = \frac{\lambda_S}{c}\int_0^x \omega_a(y)\phi_{\beta,C}(x-y)\, dy = \int_0^\infty w(z)\left(\frac{\lambda_S}{c}\int_0^x f_{a,S}(z+y)\phi_{\beta,C}(x-y)\, dy\right) dz,$$

where the second equality follows from substitution of (3.6). Using (3.5) followed by some manipulations, one finds

$$\begin{aligned}
&\frac{\lambda_S}{c}\int_0^x f_{a,S}(z+y)\phi_{\beta,C}(x-y)\, dy \\
&= \lambda_S\int_0^{\frac{x}{c}} e^{-(\lambda_S+\beta)t} f_{a,S}(z+x-ct)\, dt + \lambda_S\sum_{n=1}^\infty \frac{\lambda_S^n}{n!}\int_0^{\frac{x}{c}} e^{-(\lambda_S+\beta)t}t^n \int_0^{x-ct} f_{a,S}(z+y) f_{a,S}^{*n}(x-ct-y)\, dy\, dt.
\end{aligned}$$

Therefore one finally observes that $\phi_{\beta,J}(x)$ admits the representation

$$\phi_{\beta,J}(x) = \int_0^\infty \int_0^{\frac{x}{c}} e^{-\beta t} w(z) h_J(z,t|x)\, dt\, dz,$$

where $h_J(z,t|x)$ is the joint density of $(|X_{\tau_X}|, \tau_X)$ at (z,t) given in part *2*. Probabilistically, the density (3.4) can be interpreted as follows. The first term $\lambda_S e^{-\lambda_S t} f_{a,S}(z+x-ct)$ is the case where only one jump causes ruin. Such a jump occurs at time t, and the size of the jump should be $z+x-ct$ in order to result in a deficit of z. The second term reflects the cases where $n+1$ jumps cause ruin ($n \geq 1$). In this case, there are n jumps by time t with probability $e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}$, and the sum of these n jumps should be $x-ct-y$ for some $0 < y < x-ct$, contributing to the term $f_{a,S}^{*n}(x-ct-y)$. At this moment the process is at level y, then the $(n+1)$-th jump occurs at rate λ_S and it should be of size $z+y$ to cause a deficit of z. □

Remark 1 *The convolution terms and hence the integrals in (3.3) and (3.4) can be explicitly evaluated when the density $f_{a,S}(\cdot)$ belongs to, for example, the class of mixed Erlang distributions (see Dickson and Willmot (2005)). Such a class is not only dense in the set of positive continuous distributions (Tijms (1995, pp.163–164)) but also contains many other distributions, some of which are non-trivial, as special cases (Willmot and Woo (2007)).*

3.2 The density of $\{Y_t^2\}_{t\geq 0}$ avoiding ruin enroute

In a similar way in which we rewrite the process $\{X_t\}_{t\geq 0}$ in (3.2), we write

$$dY_t^2 = p_2\, dt - dL_t^a, \tag{3.7}$$

where $L_t^a = (1-a)L_t$. $\{L_t^a\}_{t\geq 0}$ is a compound Poisson process with the same arrival rate λ_L as in the original $\{L_t\}_{t\geq 0}$ process but with scaled secondary generic random variable $Z^{a,L} = (1-a)Z^L$ having associated pdf $f_{a,L}(\cdot)$. The following proposition recovers the density of the process $\{Y_t^2\}_{t\geq 0}$ avoiding ruin enroute.

Proposition 2 *In the risk process $\{Y_t^2\}_{t\geq 0}$ defined by (3.7), the distribution of the surplus process $\{Y_t^2\}_{t\geq 0}$ at level u, at a given time t, avoiding ruin enroute consists of the following contributions.*

1. *A point mass given by*

$$P_{y_2}\left(\inf_{s\leq t} Y_s^2 > 0, Y_t^2 = u\right) = e^{-\lambda_L t}, \tag{3.8}$$

for $u = y_2 + p_2 t$.

2. *The density part given by*

$$P_{y_2}\left(\inf_{s\leq t} Y_s^2 > 0, Y_t^2 \in du\right) = \zeta(y_2, t, u)\, du, \tag{3.9}$$

for $u < y_2 + p_2 t$, *where*

$$\zeta(y_2,t,u)$$

$$= \begin{cases} e^{-\lambda_L t}\left(b(y_2,p_2t,u) + \sum_{n=1}^{\infty}\frac{(\lambda_L t)^n}{n!}\int_0^{p_2 t}\frac{z}{p_2 t}f_{a,L}^{*n}(p_2t-z)b(y_2,z,u)\,dz\right), & u \le y_2; t>0, \\ e^{-\lambda_L t}\left[b(y_2,p_2t,u) + \sum_{n=1}^{\infty}\frac{(\lambda_L t)^n}{n!}\left(\frac{u-y_2}{p_2 t}f_{a,L}^{*n}(p_2t-(u-y_2)) + \int_{u-y_2}^{p_2 t}\frac{z}{p_2 t}f_{a,L}^{*n}(p_2t-z)b(y_2,z,u)\,dz\right)\right], & \\ & u>y_2; t>\frac{u-y_2}{p_2}, \end{cases} \tag{3.10}$$

$$b(y_2,z,u) = \begin{cases} \sum_{n=1}^{\infty}\left(\frac{\lambda_L}{p_2}\right)^n\int_0^{z\wedge u}\xi_n(y_2-u+v,z-v)\,dv, & u\le y_2; z>0, \\ \sum_{n=1}^{\infty}\left(\frac{\lambda_L}{p_2}\right)^n\int_0^{y_2+((z-u)\wedge 0)}\xi_n(v,y_2+z-u-v)\,dv, & u>y_2; z>u-y_2, \end{cases} \tag{3.11}$$

and

$$\xi_n(y_2,z) = \frac{y_2^{n-1}}{(n-1)!}f_{a,L}^{*n}(z+y_2) + \sum_{j=1}^{n-1}\binom{n}{j}\frac{(-1)^j}{(n-1)!}\int_0^{y_2}v^{n-1}f_{a,L}^{*j}(y_2-v)f_{a,L}^{*(n-j)}(z+v)\,dv. \tag{3.12}$$

Proof. Our goal here to perform Laplace transform inversion to the expression

$$\int_{t=0}^{\infty}e^{-\beta t}\mathrm{P}_{y_2}\left(\inf_{s\le t}Y_s^2>0, Y_t^2\in du\right)dt = \left[e^{-\rho u}W^{(\beta)}(y_2) - \mathbf{1}_{\{y_2\ge u\}}W^{(\beta)}(y_2-u)\right]du. \tag{3.13}$$

See Suprun (1976) and Bertoin (1997, Lemma 1). Here $W^{(\beta)}(\cdot)$ is the β-scale function of $\{Y_t^2\}_{t\ge 0}$ with Laplace transform

$$\int_0^{\infty}e^{-sx}W^{(\beta)}(x)\,dx = \frac{1}{p_2 s - (\lambda_L+\beta) + \lambda_L\widetilde{f}_{a,L}(s)}, \qquad s>\rho, \tag{3.14}$$

where $\widetilde{f}_{a,L}(s) = \int_0^{\infty}e^{-sx}f_{a,L}(x)\,dx$, and the quantity ρ appearing in (3.13) is the unique non-negative root to the equation (in ξ)

$$p_2\xi - (\lambda_L+\beta) + \lambda_L\widetilde{f}_{a,L}(\xi) = 0.$$

Next, we denote by $g_\beta(u|y_2)$ the so-called discounted density of the surplus prior to ruin $Y^2_{\tau_2-}$ at u for the process $\{Y^2_t\}_{t\geq 0}$, given initial capital of $Y^2_0 = y_2$, which is such that

$$g_\beta(u|y_2)\,du = \int_{t=0}^{\infty} e^{-\beta t}\mathrm{P}_{y_2}(Y^2_{\tau_2-} \in du, \tau_2 \in dt).$$

By integrating out the second argument of equation (41) of Cheung and Landriault (2010), we have that

$$g_\beta(u|y_2) = \frac{\lambda_L}{p_2}\left[e^{-\rho u}v_\beta(y_2) - \mathbf{1}_{\{y_2\geq u\}}v_\beta(y_2-u)\right]\overline{F}_{a,L}(u), \tag{3.15}$$

where the function $v_\beta(\cdot)$ is related to $W^{(\beta)}(\cdot)$ via $v_\beta(\cdot) = p_2 W^{(\beta)}(\cdot)$ (which is evident by comparing equation (11) of Cheung and Landriault (2010) with (3.14)), and $\overline{F}_{a,L}(\cdot)$ is the survival function of the generic random variable $Z^{a,L}$. Thus, using (3.13), one can express the above equation as

$$g_\beta(u|y_2)\,du = \lambda_L\left[\int_{t=0}^{\infty} e^{-\beta t}\mathrm{P}_{y_2}\left(\inf_{s\leq t} Y^2_s > 0, Y^2_t \in du\right)\,dt\right]\overline{F}_{a,L}(u). \tag{3.16}$$

On the other hand, Corollary 1 of Landriault and Willmot (2009) implies

$$g_\beta(u|y_2) = \lambda_L\left[e^{-\beta\left(\frac{u-y_2}{p_2}\right)}\left(\frac{1}{p_2}e^{-\lambda_L\left(\frac{u-y_2}{p_2}\right)}\mathbf{1}_{\{u>y_2\}}\right) + \int_0^{\infty} e^{-\beta t}\zeta(y_2,t,u)\,dt\right]\overline{F}_{a,L}(u), \tag{3.17}$$

where $\zeta(y_2,t,u)$ is defined through (3.10), (3.11) and (3.12). Hence, by the uniqueness of Laplace transforms, a comparison between (3.16) and (3.17) yields the desired result in (3.8) and (3.9). □

Remark 2 *Instead of drawing connections between the existing results (3.13) and (3.15) to prove (3.16), we also provide a probabilistic proof as follows. In order to ensure a surplus prior to ruin $Y^2_{\tau_2-}$ to be u, the process $\{Y^2_t\}_{t\geq 0}$ has to first reach level u from level y_2 at some time t without ruin occurring in the interim, which is explained by the term $P_{y_2}\left(\inf_{s\leq t} Y^2_s > 0, Y^2_t \in du\right)$. Being at level u, if a claim occurs at the next instant (with probability $\lambda_L\,dt$) and such a claim is larger then u (with probability $\overline{F}_{a,L}(u)$), then the time of ruin of $\{Y^2_t\}_{t\geq 0}$ is t. Since discounted density is concerned here, we need to multiply by $e^{-\beta t}$ and integrate with respect to t, resulting in the expression (3.16).*

Remark 3 *The quantity $\zeta(y_2,t,u)$ can be explicitly evaluated when $f_{a,L}(\cdot)$ follows mixed Erlang distribution using the results in Landriault and Willmot (2009, Section 4).*

4 The Laplace transform of the joint ruin time

In this section, we give the main result of the paper regarding the Laplace transform of the time to ruin τ, namely $E_{(y_1,y_2)}\left[e^{-\beta\tau}\mathbf{1}_{\{\tau<\infty\}}\right]$, where $E_{(y_1,y_2)}$ is the conditional expectation given the initial surplus levels $(Y_0^1, Y_0^2) = (y_1, y_2)$.

Proposition 3 *In the bivariate risk process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ defined by (1.1) with $x = (1-a)y_1 - ay_2 > 0$, the Laplace transform of the time to ruin is given by*

$$E_{(y_1,y_2)}\left[e^{-\beta\tau}\mathbf{1}_{\{\tau<\infty\}}\right] = E_{(y_1,y_2)}\left[e^{-\beta\tau_2}\mathbf{1}_{\{\tau_X>\tau_2\}}\right] + E_{(y_1,y_2)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_X\leq\tau_2,\tau_1<\infty\}}\right]. \tag{4.1}$$

The first part in (4.1) can be evaluated as

$$E_{(y_1,y_2)}\left[e^{-\beta\tau_2}\mathbf{1}_{\{\tau_X>\tau_2\}}\right] = \int_0^{\frac{x}{c}} e^{-\beta t} P_x(\tau_X > t) P_{y_2}(\tau_2 \in dt), \tag{4.2}$$

where for $0 < t < \frac{x}{c}$,

$$P_x(\tau_X > t) = \sum_{n=0}^{\infty} e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!} F_{a,S}^{*n}(x - ct), \tag{4.3}$$

whereas $P_{y_2}(\tau_2 \in dt)$ is simply the density of the time of ruin τ_2 of $\{Y_t^2\}_{t\geq 0}$ and is given by Dickson and Willmot (2005). Here $F_{a,S}^{*n}(\cdot)$ is the cumulative distribution function corresponding to the pdf $f_{a,S}^{*n}(\cdot)$ with the usual convention that $F_{a,S}^{*0}(v) = \mathbf{1}_{\{v\geq 0\}}$.

Next, we denote by $E_{y_1}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right]$ the Laplace transform of the time of ruin τ_1 for $\{Y_t^1\}_{t\geq 0}$, which is known to be the tail of a compound geometric distribution under $\beta > 0$ or the positive loading condition (1.2) holds (Lin and Willmot (1999, Section 2)). Then the second part of (4.1), namely $E_{(y_1,y_2)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_X\leq\tau_2,\tau_1<\infty\}}\right]$, is the sum of six contributions given below.

1. ***Ruin in $\{X_t\}_{t\geq 0}$ by continuity***

   (a) No claims from $\{S_t\}_{t\geq 0}$, no claims from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$e^{-(\beta+\lambda_S+\lambda_L)\frac{x}{c}} E_{\frac{a}{1-a}\left(y_2+p_2\frac{x}{c}\right)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right]. \tag{4.4}$$

(b) No claims from $\{S_t\}_{t\geq 0}$, at least one claim from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$\int_0^{y_2+p_2\frac{x}{c}} e^{-(\beta+\lambda_S)\frac{x}{c}} E_{\frac{a}{1-a}u}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right]\zeta\left(y_2, \frac{x}{c}, u\right)\, du. \tag{4.5}$$

(c) At least one claim from $\{S_t\}_{t\geq 0}$, no claims from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$\int_0^{\frac{x}{c}} e^{-(\beta+\lambda_L)t} E_{\frac{a}{1-a}(y_2+p_2t)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right] h_C(t|x)\, dt. \tag{4.6}$$

(d) At least one claim from $\{S_t\}_{t\geq 0}$, at least one claim from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$\int_0^{\frac{x}{c}} e^{-\beta t}\left(\int_0^{y_2+p_2t} E_{\frac{a}{1-a}u}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right]\zeta(y_2, t, u)\, du\right) h_C(t|x)\, dt. \tag{4.7}$$

2. ***Ruin in*** $\{X_t\}_{t\geq 0}$ ***by jumps***

(a) No claims from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$\int_0^{\frac{x}{c}} e^{-(\beta+\lambda_L)t}\left(\int_0^{a(y_2+p_2t)} E_{\frac{a}{1-a}(y_2+p_2t)-\frac{1}{1-a}z}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right] h_J(z, t|x)\, dz + \int_{a(y_2+p_2t)}^{\infty} h_J(z, t|x)\, dz\right) dt. \tag{4.8}$$

(b) At least one claim from $\{L_t\}_{t\geq 0}$ in $(0, \tau_X]$:

$$\int_0^{\frac{x}{c}} e^{-\beta t}\int_0^{y_2+p_2t}\left(\int_0^{au} E_{\frac{a}{1-a}u-\frac{1}{1-a}z}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}\right] h_J(z, t|x)\, dz + \int_{au}^{\infty} h_J(z, t|x)\, dz\right)\zeta(y_2, t, u)\, du\, dt. \tag{4.9}$$

Proof. The decomposition in (4.1) is a direct consequence of the fact that $\tau_X > \tau_2$ implies $\tau = \tau_2$ whereas $\tau_X \leq \tau_2$ implies $\tau = \tau_1$. Equation (4.2) follows by conditioning on τ_2 along with the independence of the processes $\{X_t\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$. Equation (4.3) can be argued probabilistically as follows. The term $e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}$ represents the probability that there are n jumps until time t. In order for $\{Y_t^2\}_{t\geq 0}$ to survive at time t $(0 < t < \frac{x}{c})$, the sum of the n jumps should be no larger than $x - ct$, resulting in the term $F_{a,S}^{*n}(x - ct)$. Summing over all possible n (i.e. $n \geq 0$) yields the desired result.

For the second term $E_{(y_1,y_2)}\left[e^{-\beta\tau_1}\mathbf{1}_{\{\tau_X\leq\tau_2,\tau_1<\infty\}}\right]$ in (4.1), we note that when $\tau_X \leq \tau_2$, at time τ_X the process $\{Y_t^1\}_{t\geq 0}$ is either on the borderline Δ of the absorbing set $\mathcal{A}^-$ (i.e. $\{X_t\}_{t\geq 0}$ ruins due to continuity) or inside the set $\mathcal{A}^-$ (i.e. $\{X_t\}_{t\geq 0}$ ruins due to jumps). Moreover, to keep track of the the ruin times τ_X and τ_2 we also recall that $\{X_t\}_{t\geq 0}$ is only subject to claims from $\{S_t\}_{t\geq 0}$ whereas $\{Y_t^2\}_{t\geq 0}$ is only subject to claims from $\{L_t\}_{t\geq 0}$. First, for ruin of $\{X_t\}_{t\geq 0}$ due to continuity, one has $Y^1_{\tau_X} = \frac{a}{1-a} Y^2_{\tau_X}$. This case is analyzed based on the number of claims arising from $\{S_t\}_{t\geq 0}$ and $\{L_t\}_{t\geq 0}$ respectively in the interval $(0, \tau_X]$ and it further consists of four scenarios. In Scenario *1(a)*, there are no claims at all. Therefore the process $\{X_t\}_{t\geq 0}$ decreases linearly over a length of time $\frac{x}{c}$, and at time τ_X the process $\{Y_t^1\}_{t\geq 0}$ is at level $\frac{a}{1-a}(y_2 + p_2 \frac{x}{c})$. This explains equation (4.4). Equation (4.5) in Scenario *1(b)* arises when within $(0, \tau_X]$ there are no claims from $\{S_t\}_{t\geq 0}$ (and hence we still have $\tau_X = \frac{x}{c}$) but there is at least one claim from $\{L_t\}_{t\geq 0}$. Therefore we should make use of the density of $\{Y_t^2\}_{t\geq 0}$ at u avoiding ruin given by Item *2* in Proposition 2 at $t = \frac{x}{c}$. For equation (4.6) in Scenario *1(c)*, there is at least one claim from $\{S_t\}_{t\geq 0}$ and hence we require the density in Item *1(b)* in Proposition 1; and this has to be combined with the point mass in Item *1* in Proposition 2 since there are no claims from $\{L_t\}_{t\geq 0}$. Equation (4.7) also follows from a similar arguments simply using the densities in Item *1(b)* of Proposition 1 and Item *2* of Proposition 2.

Second, for ruin of $\{X_t\}_{t\geq 0}$ due to jumps we have $Y^1_{\tau_X} = \frac{a}{1-a} Y^2_{\tau_X} - \frac{1}{1-a}|X_{\tau_X}|$. Note that this case involves at least one claim from $\{S_t\}_{t\geq 0}$, and therefore it is analyzed based on the number of claims arising from $\{L_t\}_{t\geq 0}$ only in the interval $(0, \tau_X]$ and it further consists of two scenarios. The detailed explanations of (4.8) and (4.9) are omitted here, but one just has to apply Item *2* instead of Item *1* of Proposition 1. □

Remark 4 *In principle $E_{(y_1,y_2)}\left[e^{-\beta\tau}\mathbf{1}_{\{\tau<\infty\}}\right]$ can be obtained explicitly via straightforward but tedious integration for mixed Erlang claims.*

Remark 5 *An alternative expression for $P_x(\tau_X > t)$ can also be obtained from a direct application of Proposition 1, and is given as, for $0 < t < \frac{x}{c}$,*

$$P_x(\tau_X > t) = \int_t^{\frac{x}{c}} h_C(v|x)\, dv + \int_t^{\frac{x}{c}} \int_0^{\infty} h_J(z, v|x)\, dz\, dv + P_x\left(\tau_X = \frac{x}{c}, |X_{\tau_X}| = 0\right). \tag{4.10}$$

Nonetheless, it is obvious that the representation (4.3) is simpler and more tractable than (4.10).

5 The effect of potential ruin of the reinsurer over the cedent

In the previous section, the distribution of the first ruin time of the two insurers is obtained via its Laplace transform. According to the decomposition in (4.1) of Proposition 3, we are able to keep track of which insurer ruins first. However, in the case of ruin, one natural question would be the consequence of one's ruin on the other. Once the bivariate process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ enters $\mathcal{A}^-$, then ruin of the process $\{Y_t^1\}_{t\geq 0}$ happens no later than the ruin event of the second insurer $\{Y_t^2\}_{t\geq 0}$. It would be interesting to know with what probability ruin of $\{Y_t^1\}_{t\geq 0}$ would also lead to ruin of $\{Y_t^2\}_{t\geq 0}$ (through claims process $\{L_t\}_{t\geq 0}$) and the amount of deficit in this case. Unfortunately this is still an open question for the moment that is hard to be solved due to the dependence between $\{Y_t^1\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ (see more comments at the end of the section). On the other hand, it is possible to get information on whether $\{Y_t^2\}_{t\geq 0}$ will get ruined before $\{Y_t^1\}_{t\geq 0}$ and the surplus level of $\{Y_t^1\}_{t\geq 0}$ when such an event occurs. This is particular important when $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ is interpreted as an insurer-reinsurer application, because the reinsurance company needs to be reliable visà vis its clients and needs to give some guarantee to its customers, whereas the insurer or the cedent would like to know its actual surplus if the reinsurer is ruined first, so that appropriate action can be taken.

Geometrically, it is easy to see that ruin of $\{Y_t^2\}_{t\geq 0}$ before $\{Y_t^1\}_{t\geq 0}$ is possible if and only if (y_1, y_2) belongs to $\mathcal{A}^+$, and in this case $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ remains in $\mathcal{A}^+$ when $\{Y_t^2\}_{t\geq 0}$ drops below zero. Assuming $(y_1, y_2) \in \mathcal{A}^+$, our goal here is to find the distribution of the surplus level of the insurer when the reinsurer ruins, jointly to the event $\{\tau_X > \tau_2\}$ (because in this scenario $\tau = \tau_2$). Mathematically, we are interested in the quantity $\mathrm{P}_{(y_1,y_2)}(Y_{\tau_2}^1 > u, \tau_X > \tau_2)$, where $\mathrm{P}_{(y_1,y_2)}$ is the conditional probability given the initial surplus levels $(Y_0^1, Y_0^2) = (y_1, y_2)$. Note that it is possible for $Y_{\tau_2}^1$ to be negative because a large claim from $\{L_t\}_{t\geq 0}$ could possibly ruin both $\{Y_t^1\}_{t\geq 0}$ and $\{Y_t^2\}_{t\geq 0}$ at the same time. Therefore, we shall evaluate $\mathrm{P}_{(y_1,y_2)}(Y_{\tau_2}^1 > u, \tau_X > \tau_2)$ for the domain $-\infty < u < \infty$. The result is summarized in the following Proposition.

Proposition 4 *For the bivariate risk process* $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$, *the survival function of the surplus level of*

$\{Y_t^1\}_{t\geq 0}$ when $\{Y_t^2\}_{t\geq 0}$ ruins jointly to the event $\{\tau_X > \tau_2\}$ is continuous in u and is given by

$$P_{(y_1,y_2)}(Y^1_{\tau_2} > u, \tau_X > \tau_2)$$
$$= \begin{cases} \sum_{n=0}^{\infty} \int_0^{\frac{x}{c}} \int_{-\frac{(1-a)u}{a}}^{\frac{x-(1-a)u-ct}{a}} e^{-\lambda_S t} \frac{(\lambda_S t)^n}{n!} F^{*n}_{a,S}(x-(1-a)u-ct-az)\eta(z,t|y_2)\, dz\, dt \\ \qquad + \sum_{n=0}^{\infty} \int_0^{\frac{x}{c}} \int_0^{-\frac{(1-a)u}{a}} e^{-\lambda_S t} \frac{(\lambda_S t)^n}{n!} F^{*n}_{a,S}(x-ct)\eta(z,t|y_2)\, dz\, dt, & u<0. \\ \sum_{n=0}^{\infty} \int_0^{\frac{x-(1-a)u}{c}} \int_0^{\frac{x-(1-a)u-ct}{a}} e^{-\lambda_S t} \frac{(\lambda_S t)^n}{n!} F^{*n}_{a,S}(x-(1-a)u-ct-az)\eta(z,t|y_2)\, dz\, dt, & 0 \leq u < \frac{x}{1-a}. \\ 0, & u \geq \frac{x}{1-a}. \end{cases} \tag{5.1}$$

Here $x=(1-a)y_1 - ay_2 > 0$, and $\eta(z,t|y_2)$ is the density of $(|Y^2_{\tau_2}|, \tau_2)$ at (z,t) given $Y^2_0 = y_2$, which is available from Corollary 3 of Landriault and Willmot (2009) (see also Dickson (2008)).

Proof. Because $Y_t^1 = \frac{a}{1-a}Y_t^2 + \frac{1}{1-a}X_t$ and $\{\tau_X > \tau_2\} = \{X_{\tau_2} > 0\}$ (recall that $\mathcal{A}^-$ is absorbing), and since $\{X_t\}_{t\geq 0}$ is independent of $\{Y_t^2\}_{t\geq 0}$ and hence $(|Y^2_{\tau_2}|, \tau_2)$, we get

$$\begin{aligned} \mathrm{P}_{(y_1,y_2)}(Y^1_{\tau_2} > u, \tau_X > \tau_2) &= \mathrm{P}_{(y_1,y_2)}\left(\frac{a}{1-a}Y^2_{\tau_2} + \frac{1}{1-a}X_{\tau_2} > u, X_{\tau_2} > 0\right) \\ &= \int_0^{\infty}\int_0^{\infty} \mathrm{P}_x\left(-\frac{a}{1-a}z + \frac{1}{1-a}X_t > u, X_t > 0\right)\eta(z,t|y_2)\, dz\, dt. \end{aligned}$$

Using the representation $X_t = x - ct - S_t^a$, the above equation can be rewritten as

$$\begin{aligned} \mathrm{P}_{(y_1,y_2)}(Y^1_{\tau_2} > u, \tau_X > \tau_2) &= \int_0^{\infty}\int_0^{\infty} \mathrm{P}\left(S_t^a < x-(1-a)u-ct-az, S_t^a < x-ct\right)\eta(z,t|y_2)\, dz\, dt \\ &= \int_0^{\infty}\int_0^{\infty} \mathrm{P}\left(S_t^a < \min(x-(1-a)u-ct-az, x-ct)\right)\eta(z,t|y_2)\, dz\, dt. \end{aligned} \tag{5.2}$$

Note that the probability term of the above integrand is zero for some combinations of values of z and t. Moreover, we need to distinguish between the cases $u \geq 0$ and $u < 0$ as follows.

1. First, for $u \geq 0$, one has $\min(x-(1-a)u-ct-az, x-ct) = x-(1-a)u-ct-az$ since z positive, and therefore

$$\begin{aligned}\mathrm{P}\left(S_t^a < \min(x-(1-a)u-ct-az, x-ct)\right) &= \mathrm{P}\left(S_t^a < x-(1-a)u-ct-az\right)\\ &= \sum_{n=0}^{\infty} e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}F_{a,S}^{*n}(x-(1-a)u-ct-az)\\ &\quad - e^{-\lambda_S t}\mathbf{1}_{\{x-(1-a)u-ct-az=0\}}. \end{aligned} \tag{5.3}$$

We remark that the last correction term at the end is to adjust for the fact that $\mathrm{P}(S_t^a < v) = \mathrm{P}(S_t^a \le v)$ for all $-\infty < v < \infty$ except for $v = 0$ due to the point mass arising from $S_t^a = 0$. Clearly, the above expression is zero if $x-(1-a)u-ct-az \le 0$. Since both z and t are positive, $x-(1-a)u-ct-az$ is less than zero if $x-(1-a)u \le 0$. This explains the final equation in (5.1) for $u \ge \frac{x}{1-a}$. The integration domain in (5.2) for which (5.3) is generally non-zero is given by $\{(z,t) : ct+az < x-(1-a)u\}$ when $x-(1-a)u > 0$. By noting that $\{(z,t) : ct+az = x-(1-a)u\}$ is a set of measure zero, one can ignore the correction term in (5.3). Combining these observations leads to the second equation in (5.1) for $0 \le u < \frac{x}{1-a}$.

2. Next, for $u < 0$, note that $x-(1-a)u$ is always positive. Moreover, one has

$$\min(x-(1-a)u-ct-az, x-ct) = \begin{cases} x-(1-a)u-ct-az, & z \ge -\frac{(1-a)u}{a}.\\ x-ct, & z < -\frac{(1-a)u}{a}.\end{cases}$$

Hence, for $z \ge -\frac{(1-a)u}{a}$, the representation (5.3) still holds true and the relevant domain of integration in (5.2) is $\{(z,t) : ct+az < x-(1-a)u, z \ge -\frac{(1-a)u}{a}\}$. On the other hand, for $z < -\frac{(1-a)u}{a}$,

$$\begin{aligned}\mathrm{P}\left(S_t^a < \min(x-(1-a)u-ct-az, x-ct)\right) &= \mathrm{P}\left(S_t^a < x-ct\right)\\ &= \sum_{n=0}^{\infty} e^{-\lambda_S t}\frac{(\lambda_S t)^n}{n!}F_{a,S}^{*n}(x-ct) - e^{-\lambda_S t}\mathbf{1}_{\{x-ct=0\}},\end{aligned}$$

which is generally non-zero for $\{(z,t) : x-ct > 0, z < -\frac{(1-a)u}{a}\}$. Taking into account the above two contributions in the integrand of (5.2) (and again ignoring the terms on sets with measure zero), one arrives at the first equation in (5.1) for $u < 0$. □

Note that Proposition 4 is concerned with the surplus level of $\{Y_t^1\}_{t\ge0}$ when $\{Y_t^2\}_{t\ge0}$ is ruined within the set $\mathcal{A}^+$ (i.e. $\tau = \tau_2$). An open question would be to find the **distribution of the surplus level of $\{Y_t^2\}_{t\geq 0}$ when $\{Y_t^1\}_{t\geq 0}$ is ruined (for the first time) within the set $\mathcal{A}^-$ (i.e. $\tau = \tau_1$)**. We remark that regardless of whether ruin of the bivariate process $\{(Y_t^1, Y_t^2)\}_{t\geq 0}$ occurs from $\mathcal{A}^+$ or $\mathcal{A}^-$, it is possible that $\tau = \tau_1 = \tau_2$ due a large claim from $\{L_t\}_{t\geq 0}$. If ruin of $\{Y_t^2\}_{t\geq 0}$ occurs within $\mathcal{A}^+$, the probability of the event $\{\tau = \tau_1 = \tau_2\}$ is given by $\mathrm{P}_{(y_1,y_2)}(Y_{\tau_2}^1 < 0, \tau_X > \tau_2) = \mathrm{P}_{(y_1,y_2)}(\tau_X > \tau_2) - \mathrm{P}_{(y_1,y_2)}(Y_{\tau_2}^1 > 0, \tau_X > \tau_2)$, where $\mathrm{P}_{(y_1,y_2)}(\tau_X > \tau_2)$ can be retrieved from (4.2) at $\beta = 0$ and $\mathrm{P}_{(y_1,y_2)}(Y_{\tau_2}^1 > 0, \tau_X > \tau_2)$ from (5.1) at $u = 0$. In a similar manner, if the afore-mentioned open question is solved, one could also obtain the probability of the event $\{\tau = \tau_1 = \tau_2\}$ for ruin of $\{Y_t^1\}_{t\geq 0}$ within the set $\mathcal{A}^-$.

Since the event $\{\tau = \tau_1 = \tau_2\}$ is caused by a large claim from $\{L_t\}_{t\geq 0}$, a related problem would be to determine the **distribution of the claim causing ruin** in such case. This is of particular importance because such a claim has adverse effect on both insurers at the same time and the associated risk has to be assessed properly. On the other hand, one might also be interested in finding **joint distribution of (Y_τ^1, Y_τ^2), which represents the surplus levels of the two lines at the time of ruin τ**. We remark that a similar problem has been considered in the form of a Gerber-Shiu function (Gerber and Shiu (1998)) by Gong et al. (2010) in a different two-dimensional risk model using recursive methods. We leave these as open questions in the current model.

Acknowledgment: Andrei Badescu gratefully acknowledges financial support received from the Natural Sciences and Engineering Research Council of Canada (NSERC). Eric Cheung also acknowledges a start-up fund provided by the Faculty of Science and the Department of Statistics and Actuarial Science in the University of Hong Kong.

References

[1] Avram, F., Palmowski, Z. and Pistorius, M. 2008a. A two-dimensional ruin problem on the positive quadrant. *Insurance: Mathematics and Economics* 42(1): 227-234.

[2] Avram, F., Palmowski, Z. and Pistorius, M. 2008b. Exit problem of a two-dimensional risk process from the quadrant: Exact and asymptotic results. *Annals of Applied Probability* 18(6): 2421-2449.

[3] Bertoin, J. 1997. Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Annals of Applied Probability* 7(1): 156-169.

[4] Cai, J. and Li, H. 2005. Multivariate risk model of phase type. *Insurance: Mathematics and Economics* 36(2): 137-152.

[5] Cai, J. and Li, H. 2007. Dependence properties and bounds for ruin probabilities in multivariate compound risk models. *Journal of Multivariate Analysis* 98(4): 757-773.

[6] Chan, W.-S., Yang, H. and Zhang, L. 2003. Some results on the ruin probability in a two-dimensional risk model. *Insurance: Mathematics and Economics* 32(3): 345-358.

[7] Cheung E.C.K. and Landriault D. 2010. A generalized penalty function with the maximum surplus prior to ruin in a MAP risk model. *Insurance: Mathematics and Economics* 46(1): 127-134.

[8] Dang, L., Zhu, N. and Zhang, H. 2009. Survival probability for a two-dimensional risk model. *Insurance: Mathematics and Economics* 44(3): 491-496.

[9] Dickson, D.C.M. 2008. Some explicit solutions for the joint density of the time of ruin and the deficit at ruin. *ASTIN Bulletin* 38(1): 259-276.

[10] Dickson, D.C.M. and Willmot, G.E. 2005. The density of the time to ruin in the classical Poisson risk model. *ASTIN Bulletin* 35(1): 45-60.

[11] Gerber, H.U. and Shiu E.S.W. 1998. On the time value of ruin. *North American Actuarial Journal* 2(1): 48-72.

[12] Gong, L., Badescu, A.L. and Cheung, E.C.K. 2010. Recursive methods for a two-dimensional risk process with common shocks. Submitted for publication.

[13] Landriault, D. and Willmot, G.E. 2009. On the joint distributions of the time to ruin, the surplus prior to ruin, and the deficit at ruin in the classical risk model. *North American Actuarial Journal* 13(2): 252-270.

[14] Li, J., Liu, Z. and Tang, Q. 2007. On the ruin probabilities of a bidimensional perturbed risk model. *Insurance: Mathematics and Economics* 41(1): 185-195.

[15] Lin, X.S. and Willmot, G.E. 1999. Analysis of a defective renewal equation arising in ruin theory. *Insurance: Mathematics and Economics* 25(1): 63-84.

[16] Suprun, V.N. 1976. Problem of destruction and resolvent of a terminating process with independent increments. *Ukrainian Mathematical Journal* 28(1): 39-45.

[17] Rabehasaina, L. 2009. Risk processes with interest force in Markovian environment. *Stochastic Models* 25(4): 580-613.

[18] Tijms, H. 1994. *Stochastic Models.* Chichester: John Wiley.

[19] Willmot, G.E. and Woo, J.-K. 2007. On the class of Erlang mixtures with risk theoretic applications. *North American Actuarial Journal* 11(2): 99-115.

[20] Yuen, K.C., Guo, J. and Wu, X. 2006. On the first time of ruin in the bivariate compound Poisson model. *Insurance: Mathematics and Economics* 38(2): 298-308.