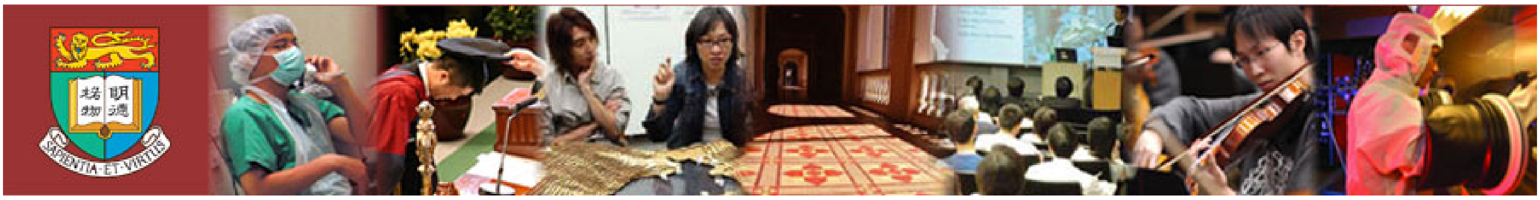

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On Maximizing the Throughput of Opportunistic Multicast in Wireless Cellular Networks with Erasure Codes

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***Abstract*—In this paper, we focus on designing opportunistic multicast scheduling (OMS) schemes that can maximize per-user throughput in a wireless network using erasure codes. We first design a maximal OMS (M-OMS) scheme for homogeneous networks where users experience i.i.d. channel conditions. We then build an analytical model to study the throughput performance of M-OMS. Given the channel statistical information of the users, we derive the upper bound on per-user throughput by solving a semidefinite optimization problem. For heterogeneous networks with non-i.i.d. channel conditions, we propose a modified M-OMS scheme which uses the average user channel condition as the weighting factor to improve the fairness among users. simulation results show that on average our proposed M-OMS schemes have about 15% throughput gain over the existing OMS scheme under the i.i.d. case; while for the non-i.i.d. case, the throughput gain can be as high as 100%.**

***Index Terms*—opportunistic multicast scheduling, multiuser diversity, multicast diversity, erasure coding**

I. INTRODUCTION

Stipulated by the increasing demand for multimedia contents like video streaming broadcast in wireless networks, mobile multicast services have received more and more research attentions [1]. Multicast is especially efficient in wireless networks due to the broadcast nature of the wireless medium, where one data packet can be received by multiple users through one transmission. To exploit such advantage, a naive approach is to broadcast to all the multicast users at each transmission. However, the broadcasting data rate is constrained by the user with the poorest channel condition, resulting in low throughput. Accordingly, to improve the multicast throughput, researchers proposed opportunistic multicast scheduling (OMS) schemes. Essentially, instead of broadcasting to all the multicast users, OMS selects a proper transmission rate that will guarantee successful reception by a *subset* of users with favorable channel conditions at each transmission. To design a proper OMS scheme, two major problems must be handled. The first problem is how to choose the subset of users in an intelligent way such that a high throughput is achieved without losing much wireless broadcast advantage. The other problem is the reliability issue. At each transmission, since OMS only targets at a subset of users, some receivers with worse channel conditions may not be able to receive the packet successfully. Conventional retransmission mechanism is not an efficient solution to this because each lost packet needs to be retransmitted to its corresponding receiver, leading to large retransmission overhead.

As a pioneer work, Gopala and Gamal proposed an OMS scheme in [2] [3]. Their idea was that at each transmission, the BS chooses a *fixed* fraction of the users with favorable channel conditions. In this paper, we call this type of multicast scheduling schemes as OMS with fixed selection ratio, or F-OMS. To address the reliability issue, the authors suggested that the BS should maintain a separate queue for each possible subset of users. However, in such scheme, the number of queues increases exponentially with the number of users, making it impossible for implementation when the number of users is large.

Recently, it is found that erasure codes [4] can effectively solve the reliability issue in opportunistic multicast. Specifically, instead of sending the original multicast packets, the BS encodes the original information using erasure codes and sends the coded packets. The adoption of erasure codes allows users to decode the source data once a minimum set of encoded packets is received, regardless of the specific receive sequence of the encoded packets. The number of encoded packets that can be generated from the source data is potentially limitless. As a result, regardless of the packet loss statistics of the users, the BS can send as many encoded packets as needed for the users to recover the source data without retransmission. Following such idea, in [5] the authors investigated the throughput performance by jointly utilizing the F-OMS proposed in [3] along with erasure codes. The authors also proposed an analytical model to find the optimal selection ratio for F-OMS.

In this paper, we also consider the opportunistic multicast problem using erasure codes. Unlike [5], our approach is not restricted to using a single selection ratio. We first propose a maximal OMS (M-OMS) scheme that can maximize the per-user throughput for homogeneous networks. To evaluate the performance of M-OMS, we establish an analytical model and derive the upper bound on the per-user throughput achieved by M-OMS. We further provide a possible extension of M-OMS scheme for heterogeneous networks. Numerical results demonstrate that our proposed schemes can significantly improve throughput performance over existing F-OMS schemes.

The rest of the paper is organized as follows. In Section II, we describe our system model and assumptions. In Section III, we introduce our M-OMS scheduler for homogeneous networks. We then extend the proposed M-OMS scheme to heterogeneous networks in Section IV. Numerical results are presented in Section V. Finally we conclude our paper in Section VI.

II. System Model and Problem Formulation

We consider a time slotted wireless network with a single BS and N users. All the N users belong to a common multicast group and subscribe to a single information stream. The users in the network each experiences time-varying channel condition. We assume quasi-stationary channel conditions, and the length of each time slot is comparable with the channel coherence time. Therefore, any user's channel condition remains constant during a given time slot and varies independently from one time slot to another. We assume that the channel stochastic process is stationary and ergodic.

Define random variable $r_i(k)$ as the throughput capacity, or the "*channel rate*" of user i at time slot k, and r_i as the channel rate of user i at a generic time slot. At the beginning of each time slot, we assume that each user can feedback its channel state information (CSI) to the BS in an error-free manner such that the BS knows the particular realization of the channel rates of the users at that time slot.

Let $r(k)$ denote the transmission rate chosen by the BS at the kth time slot. It is shown in [6] that $r(k)$ is always equal to the channel rate of some user in the network at that time slot. We also assume that when the BS transmits at data rate $r(k)$, then any user whose channel rate is greater than or equal to $r(k)$ is able to receive the data successfully, while others experience channel outage. Accordingly, the average throughput of user i until the end of the kth time slot can be expressed as:

$$\Psi_i(k) = \frac{1}{k}\sum_{j=0}^{k} I_{r_i(k)\geq r(j)} r(j)$$

where I_A is the indication function that equals to one when event A is true and zero otherwise. Since in multicast, all the users within the same multicast group want the same data, the multicast throughput is thus constrained by the user with the worst average throughput. As a result, our objective is to solve the following max-min fairness optimization problem:

$$\max\left\{\min_{i\in\{1,\ldots,N\}} \lim_{k\to\infty} \Psi_i(k)\right\}$$

III. M-OMS for Homogeneous Networks

In this section, we consider opportunistic multicast in homogeneous wireless networks where all users have i.i.d. channel statistics. Due to the symmetry property of a homogeneous network, we expect that the users will have the same average per-user throughput. Therefore, the worst user throughput becomes the average per-user throughput of all the users. Such expectations are well validated by our simulation results shown in Section V.

Let $F(r)$ and $f(r)$ denote the cdf and pdf of the user channel rate, respectively. At each time slot, we rank the users according to their channel rate in ascending order as follows:

$$r_{1:N}(k) \leq r_{2:N}(k) \leq \ldots \leq r_{N:N}(k)$$

Since we assumed that each user's channel condition follows a stationary random process, we can drop the time index and rewrite $r_{i:N}(k)$ as $r_{i:N}$, which is a stationary random process representing the ith order statistics in a sample of N random variables with a common cdf $F(r)$.

A. F-OMS

In F-OMS, the BS always chooses a fixed fraction of users for transmission. In order words, $r(k) = r_{i:N}(k)$, where $(N-i+1)/N$ represents the selection ratio. The average per-user throughput achieved under such scheme can be obtained as [5]:

$$R_{F-OMS} = \frac{N-i+1}{N} E[r_{i:N}]$$

From [7], $E[r_{i:N}]$, the expected value of $r_{i:N}$, can be derived as:

$$E[r_{i:N}] = N\binom{N-1}{i-1}\int_0^\infty rf(r)[F(r)]^{i-1}[1-F(r)]^{N-i}dr \tag{1}$$

The selection ratio can be optimized according to the channel rate distribution as follows [5]:

$$R^*_{F-OMS} = \underset{i\in\{1,\ldots,N\}}{\operatorname{argmax}} \frac{N-i+1}{N} E[r_{i:N}] \tag{2}$$

Since i only takes on discrete values, the optimal i value in (2) can be found by exhaustive search once the channel rate distribution is given.

B. Maximal OMS

To see how the throughput performance of F-OMS can further be improved, let us consider the following example. Suppose we have a multicast network with 10 users, and the optimal i value of F-OMS is 6, i.e., the BS transmits to 5 users at each transmission. At time slot 1, the channel rate of users 1 to 5 is 1000(bps), while the channel rate of users 6-10 is 900(bps); at time slot 2, the channel rate of user 1 is 1000(bps), while the channel rate of user 2-10 is 10(bps). Using F-OMS, the transmission rates $r(k)$ chosen by the BS at each time slot will be 1000(bps) and 10(bps), and the corresponding per-user throughput obtained will be 500(bps) and 10(bps), respectively. However, if we adopt another strategy that: set $r(k)$ as 900(bps) in time slot 1, and 1000(bps) in time slot 2, then the resulting per-user throughput at each time slot changes to 900(bps) and 100(bps), respectively. We can see that the alternative strategy largely boosts the throughput performance for both cases. Above example shows that, to compensate the inefficiency of F-OMS, we should dynamically adjust the selection ratio according to the channel rates of the users. Accordingly, we propose the Maximal OMS, or M-OMS scheme. At each time slot, the M-OMS selects a transmission rate that can maximize the *summed* throughput of the users, which is defined as the transmission rate chosen by the BS multiplied by the number of users that are capable of receive the packet under such transmission rate. In other words, the transmission rate $r(k)$ of the BS is determined as follows:

$$r(k) = \underset{r_i(k), i \in 1,\ldots,N}{\operatorname{argmax}} \sum_{j=1}^{N} r_i(k) I_{r_j(k) \geq r_i(k)}$$

To derive the per-user throughput of M-OMS, note that the transmission rate of the BS $r(k)$ can only take on N possible values, i.e., $r_{1:N}, \ldots, r_{N:N}$. If the BS chooses $r(k) = r_{i:N}$, then $N-i+1$ users can receive the packet, and the summed throughput will be $(N-i+1)r_{i:N}$. Since the BS chooses the i value with the maximal summed throughput, the average per-user throughput achieved by M-OMS can then be expressed as:

$$R_{M-OMS} = E\left[\max_{i \in 1,\ldots,N} \frac{(N-i+1)}{N} r_{i:N}\right]$$

To analyze R_{M-OMS}, first notice that R_{M-OMS} is the extreme value of a series of random variables Y_i where:

$$Y_i \triangleq \frac{(N-i+1)}{N} r_{i:N} \tag{3}$$

Interestingly, we observe that R_{M-OMS} is the extreme order statistics of a series of random variables defined by the order statistics of the users' channel conditions. However, the general results in order statistics theory like (1) can not be directly applied because $\{r_{i:N}\}, i \in 1,\ldots,N$ are correlated. As a result, $\{Y_i\}, i \in 1,\ldots,N$ are also neither identical nor independent. For such case, close form solution may not be possible to find. As a result, we resort to find an upper bound for R_{M-OMS} using the results from [8].

For convenience, we first define some notations to be used in the following description. Let $\mu_{i:N}$ denote $E[r_{i:N}]$ and $\sigma^2_{i:N}$ denote $Var[r_{i:N}]$, we have:

$$\begin{aligned} E\left[Y_i\right] &= \frac{(N-i+1)}{N} \mu_{i:N} \\ \text{Var}\left[Y_i\right] &= \left[\frac{(N-i+1)}{N}\right]^2 \sigma^2_{i:N} \end{aligned} \tag{4}$$

Again, $\mu_{i:N}$ and $\sigma^2_{i:N}$, the mean and the variance of $r_{i:N}$ can be derived from results in [7].

To derive the covariance matrix of $\{Y_i\}$, we will need the joint distribution information of two order statistics. Let $f_{ij}(x,y)$ $(1 \leq i < j \leq N)$ denote the joint pdf of $r_{i:N}$ and $r_{j:N}$, we have [7]:

$$\begin{aligned} f_{ij}(x,y) = &\frac{N!}{(i-1)!(j-i-1)!(N-j)!} F^{i-1}(x) f(x) \times \\ &\left[F(y) - F(x)\right]^{j-i-1} f(y) \left[1 - F(y)\right]^{N-j} \end{aligned} \tag{5}$$

And then $\mu_{ij:N}$ can be obtained as:

$$\mu_{ij:N} = E[r_{i:N} r_{j:N}] = \int_0^\infty \int_0^\infty xy f_{ij}(x,y) dx dy \tag{6}$$

Let $Q_{ij} \triangleq \text{Cov}[Y_i, Y_j]$ denote the covariance between Y_i and Y_j, it can then be expressed as:

$$Q_{ij} = \begin{cases} \frac{(N-i+1)(N-j+1)}{N^2} \mu_{ij:N} & \\ -\mu_{i:N}\mu_{j:N}, & i \neq j \\ \left[\frac{N-i+1}{N}\right]^2 \sigma^2_{i:N}, & i = j \end{cases} \tag{7}$$

Once the mean and covariance values are obtained, the upper bound for R_{M-OMS} can be obtained by solving the semidefinite optimization problem below:

$$\begin{aligned} R^*_{M-OMS} = &\min(\boldsymbol{x'\mu} + \boldsymbol{X} \cdot (\boldsymbol{Q} + \boldsymbol{\mu\mu'}) + x_0) \\ &\text{s.t.} \begin{pmatrix} \boldsymbol{X} & (\boldsymbol{x} - \boldsymbol{e_i})/2 \\ (\boldsymbol{x} - \boldsymbol{e_i})/2 & y_0 \end{pmatrix} \succcurlyeq 0, i = 1,\ldots,N. \end{aligned} \tag{8}$$

In (8), $\boldsymbol{\mu} = (E[Y_1], \ldots, E[Y_N])$, $\boldsymbol{e_i}$ is a unit vector with the ith component equal to 1 and zero otherwise. $\boldsymbol{Q}$ is the covariance matrix with its components $Q_{ij} = \text{Cov}[Y_i, Y_j]$. $\boldsymbol{x}$,$\boldsymbol{X}$, and x_0 are constraint variables where $\boldsymbol{x}$ is a N length vector, $\boldsymbol{X}$ is an $N \times N$ matrix, and x_0 is a scalar. The last inequality $A \succcurlyeq 0$ denotes the constraint that matrix A is positive semidefinite. Formulation (8) that can be solved within $\varepsilon > 0$ of the optimal solution in polynomial time in the problem data and $\log(1/\varepsilon)$ [9]. In practice, standard semidefinite optimization codes such as SeDuMi [10] can be used to find the solution.

C. Throughput Analysis for I.I.D. Rayleigh Channels

In this section, we analyze the average per-user throughput of M-OMS under i.i.d. Rayleigh channel conditions using the analytical model established in the previous subsection. We assume a linear relationship between the channel rate and the channel fade state (squared magnitude). Such assumption holds true for Rayleigh fading channel in the low SNR regime. Given such assumption, the channel rate under Rayleigh fading is exponentially distributed. Hence the density function of r_i can be expressed as $f(r_i) = 1/\mu e^{r_i/\mu}, r_i \geq 0$, where μ is the average channel rate of a user. We begin with a discussion of the stochastic structure of and distributional representations for the vector of order statistics $\{r_{1:N}, \ldots, r_{N:N}\}$. When r_i are i.i.d. with common mean μ, it is known that [7]:

$$r_{i:N} \triangleq \mu \left(\sum_{j=1}^{i} \frac{W_j}{N-j+1} \right)$$

where the W_i are i.i.d. standard exponential random variables with unit mean. This formation is known as Rényi's representation [11].

Using Rényi's representation, $\mu_{i:N}$ can be easily obtained as:

$$\mu_{i:N} = \mu \sum_{j=1}^{i} \frac{1}{N-j+1} \tag{9}$$

Note that W_i are i.i.d standard exponential random variables, the variance of $r_{i:N}$ can be similarly derived as:

$$\sigma_{i:N}^2 = \mu^2 \sum_{j=1}^{i} \frac{1}{N-j+1} \tag{10}$$

And the product mean value of $r_{i:N}$ and $r_{j:N}$ also follows:

$$\mu_{ij:N} = E\left[\mu\left(\sum_{k_1=1}^{i} \frac{W_{k_1}}{N-k_1+1}\right)\mu\left(\sum_{k_2=1}^{j} \frac{W_{k_2}}{N-k_2+1}\right)\right] \tag{11}$$

Again using the fact that W_i are i.i.d. standard exponential random variables, we can establish the result that:

$$E\left[W_{k_1}W_{k_2}\right] \begin{cases} 1, & k_1 \neq k_2 \\ 2, & k_1 = k_2 \end{cases} \tag{12}$$

We can substitute (12) into (11) to evaluate $\mu_{ij:N}$.

Now $Q_{i,j}$ can be obtained by substituting (9) — (11) into (4) and (7). Accordingly the semidefinite optimization problem can be established and solved using existing software packages.

IV. M-OMS for Heterogenous Networks

When the channel conditions of the users vary independently but non-identically, simply using the proposed M-OMS scheme may cause fairness issues. This situation typically arises from the geographical spread of users. Users who are closer to the BS have typically better channel conditions than the ones further away. As a result, the remote users may suffer from low throughput and become the bottleneck of the multicast group. In [5], the authors proposed a modified F-OMS algorithm that uses the users' average channel rates as the weighting factor in the selection process.

In this paper, we also modify our M-OMS scheme to cope with heterogeneous networks. First let μ_i denote the average channel rate of user i. We revise (3) to become:

$$r(k) = \underset{r_i(k), i \in 1,\ldots,N}{\operatorname{argmax}} \sum_{j=1}^{N} \frac{r_i(k)}{\mu_i^{\beta}} I_{r_j(k) \geq r_i(k)} \tag{13}$$

In (13), $\beta \geq 0$ is a weighting exponent which controls the scaling of the weighting factor. A larger β implies that the BS prefers users with low average channel rates since the denominator increases exponentially with β. As two extreme cases, when $\beta = 0$, (13) becomes the original M-OMS algorithm; when $\beta = \infty$, the user with the lowest average channel rate dominates in the sense that the BS always chooses its channel rate as the transmission rate. Note that finding an optimal β value is non-trivil since the non-i.i.d property complicates the optimization problem. Intuitively, when the differences among users' average channel rates are large, a larger β is preferred; otherwise the BS should use a smaller β.

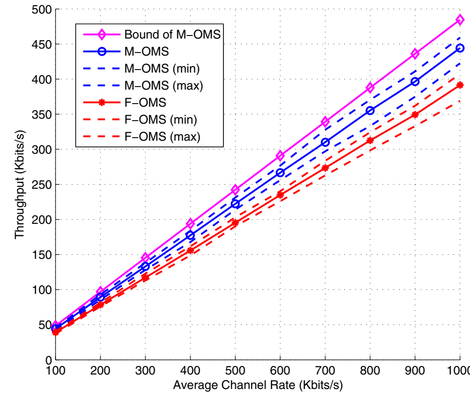

Fig. 1. Throughput comparison between M-OMS and F-OMS with the number of users $N = 30$.

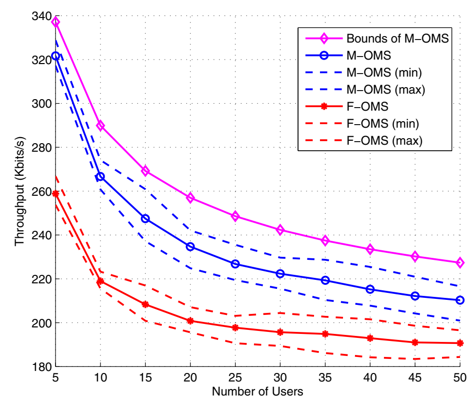

Fig. 2. Throughput comparison between M-OMS and F-OMS with the average channel rate $\mu = 500$ (kbits/s).

V. Performance Evaluation

In this section, we use Monte Carlo simulation to evaluate the performance of proposed M-OMS schemes. Each simulation run consists of 50000 time slots (i.e. 50 seconds), where the length of each slot is 1ms. Each average value is the mean of 30 simulation runs. M-OMS and F-OMS are implemented, and in all the figures shown below, we always plot F-OMS with the optimal i value. To derive the upper bound for M-OMS, we use `CVX` optimization package [12] [13] to solve the optimization problem defined in (8).

We first compare F-OMS and M-OMS with a fixed number of users $N = 30$ and varying average channel rates. We assume the users experience i.i.d. Rayleigh fading and their channel rates are exponentially distributed, with average channel rate μ varying from 100(kbits/s) to 1000(kbits/s). In Fig. 1, solid lines with circle and dot shaped markers represent average per-user throughput of M-OMS and F-OMS, respectively. Note that in simulations for i.i.d. case, the average per-user throughput coincides well with the worst user throughput, confirming that our conjecture is correct. The dashed lines around the solid lines are the minimal and maximal per-user throughput values during all the simulation runs. The solid line with diamond markers is the upper bound derived by solving (8), and the gap between the upper bound and the simulation results of M-OMS is about 9% . In Fig. 1, it can be observed that M-OMS consistently performs better than F-OMS, with a steady 13.5% performance improvement. Also we notice that when the number of users is fixed, the throughput scales almost linearly with the increasing of average channel rate. With another fact that even the minimal throughput of M-OMS surpasses the maximal throughput achieved by F-OMS, the effectiveness of M-OMS scheme is obviously validated in this simulation.

Next, we conduct another set of simulations to see how the throughput performance is affected by the number of users. Here we fix the average channel rate to $\mu = 500$ (kbits/s), and vary the number of users from 5 to 50. The simulation results are plotted in Fig. 2. In this simulation, the gap between the upper bound and the simulation results of M-OMS is about 8% . In Fig. 2, we notice that the both the throughput of the different schemes decrease and the performance gain of M-OMS over F-OMS decrease with the number of users. As can be seen from Fig. 2, when $N \in [5, 15]$, M-OMS performs 20% or more better than F-OMS. When $N = 50$, the performance gain decays to about 10%.

Next, we compare the performance of the modified M-OMS and the modified F-OMS scheme for heterogeneous networks. In the simulation, the number of users N varies from 5 to 50, and their average channel rates are chosen uniformly from the range of $[100, 1000]$ (kbits/s). Each average value is the mean of 30 simulation runs with different random generated scenarios. In Fig. 3, we show the average throughput of the worst user of the two schemes, where the modified M-OMS is indicated as WM-OMS (weighted M-OMS) and the modified F-OMS is similarly specified as WF-OMS. Note that the plot is not as smooth as the homogeneous case since the scenarios are randomly generated. We plot the simulation results of WM-OMS with three different weighting exponents, $\beta = 1, 2, 3$. We can see that within such range, higher β values provide better performance. For the best case with $\beta = 3$, WM-OMS almost achieves a 100% performance gain over WF-OMS. However, further increasing the β value is not able to increase the throughput in our simulation. In fact, when we use $\beta = 4$ in the simulation, the result is actually worse than the case with $\beta = 3$. Therefore in practice, the BS can adaptively adjust the weighting exponent value according to the real situation to achieve a better performance.

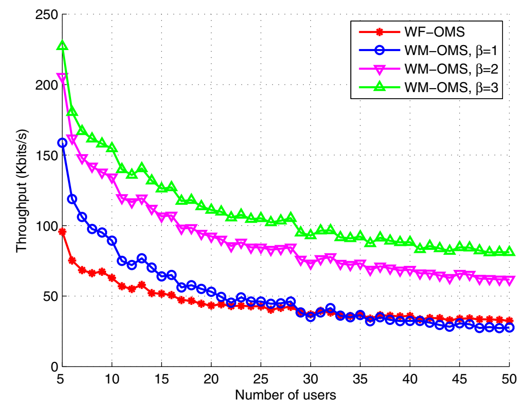

Fig. 3. The average throughput of the worst user of WM-OMS and WF-OMS.

VI. Conclusion

In this paper we investigated how to design an opportunistic multicast scheduling scheme to improve per-user throughput performance in a wireless network using erasure codes. As one of the main results, we showed that for homogeneous networks, maximizing the summed throughput can equivalently maximize the per-user throughput performance and we accordingly proposed the M-OMS scheme. Noticing that the throughput performance of M-OMS can not be derived directly, we resort to numerical methods to find its upper bound instead. We further presented a generalization of M-OMS to enhance the fairness among users in a heterogeneous network. In the simulation part, we used Rayleigh fading channel model to evaluate the performance of proposed schemes. Results demonstrated a significant performance gain of the proposed M-OMS over existing F-OMS.

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