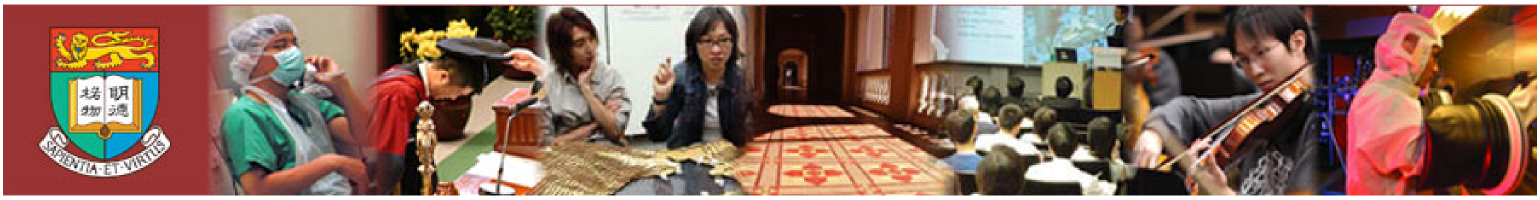

# Balanced Truncation for Time-Delay Systems Via Approximate Gramians

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**Abstract— In circuit simulation, when a large RLC network is connected with delay elements, such as transmission lines, the resulting system is a time-delay system (TDS). This paper presents a new model order reduction (MOR) scheme for TDSs with state time delays. It is the first time to reduce a TDS using balanced truncation. The Lyapunov-type equations for TDSs are derived, and an analysis of their computational complexity is presented. To reduce the computational cost, we approximate the controllability and observability Gramians in the frequency domain. The reduced-order models (ROMs) are then obtained by balancing and truncating the approximate Gramians. Numerical examples are presented to verify the accuracy and efficiency of the proposed algorithm.**

## I. INTRODUCTION

Time-delay phenomena frequently appear in high-speed circuits [1]. In circuit simulation, a time-delay system (TDS) may arise when a circuit network is connected with some delay elements (e.g., RLC interconnects connected with transmission lines [2]), which are usually from packages or PCBs. The propagation delays of transmission lines from packages and PCBs are the dominant effects, therefore, the attenuation effects of transmission lines can be ignored and they can be modeled as lossless transmission lines. Direct simulation of a large TDS is infeasible due to the prohibitive computational cost. Therefore, model order reduction (MOR) is desired to compact the model size for fast simulation.

Existing MORs can be classified into two main groups. The first group projects the original system onto a Krylov subspace to match a certain number of moments [3–5]. The second family, named truncated balanced realization (TBR) [6], is based on the concepts of controllability and observability. TBR reduces the order of systems by only preserving the dominant states of the original systems. Moment-matching methods are very efficient and numerically robust, but in general cannot provide an analytical error estimator, and therefore no optimal reduced-order models (ROMs) can be obtained. On the other hand, the Gramian-based methods often provide a higher global accuracy, and the resulting ROMs are nearly optimal under a given error bound. This motivates us to develop a TBR-type method to reduce the size of TDSs.

The major bottleneck of TBR lies in the calculation of the controllability and observability Gramians. For large linear time-invariant (LTI) systems, the computational cost of TBR is extremely high because two Lyapunov equations need to be solved at the cost of $O(n^3)$. Recently, an effective alternative, called the Poor Man's TBR, was proposed to approximate the Gramians in the frequency domain without solving the expensive Lyapunov equations [7]. This approach is very efficient because it uses frequency-weighted finite summation to approximate the infinite integration. The Poor Man's TBR can provide accurate ROMs, especially when the system has finite-bandwidth inputs.

The problem becomes more complicated when TBR is applied to reduce TDSs. Besides the huge model size, the complicated definitions involving the Lambert W function [8] prohibit the direct calculation of Gramians, as well as the Lyapunov-type equations for TDSs. The bottleneck lies in that, without an analytical convergence criterion, one can not determine how many branches of the Lambert W function are needed to approximate the TDS Gramians and the derived Lyapunov-type equations. Moreover, the calculation of Gramians via the Lambert W function involves unavoidable matrix inversion which further limits its practicality [9].

In this paper, we use the Poor Man's TBR for fast evaluation of the approximate TDS Gramians, then a TBR procedure can be applied to obtain the ROMs. The proposed algorithm takes advantage of the facts that typical TDSs from circuit simulation usually have a finite bandwidth and the delay effects occur only in the high frequency range. We use lossless multiconductor transmission lines (MTLs) to model the delay elements. When the operation frequency increases, the MTLs can no longer be regarded as a pure resistive network: the coupling capacitance and inductance will delay the signal propagation. The typical time-delay value in lossless MTLs is in the magnitude of nanoseconds, and the time-delay effects will occur only when the input signal is of high frequency. Therefore, it is reasonable to evaluate the TDS Gramians in the specific frequency bandwidth by finite summation rather than by infinite integration. The accuracy and efficiency of the proposed algorithm are demonstrated by numerical experiments.

This paper is organized as follows. Section II reviews TDSs and the Poor Man's TBR. In Section III we present the existing definitions of the Gramians for TDSs and propose a TBR method for TDSs. The proposed method is then verified by numerical examples in Section IV. Finally, Section V concludes the paper.

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## II. BACKGROUND

### A. Time-delay Systems

In this paper, a TDS is modeled from a large interconnect network connected with a set of lossless MTLs, resulting in a TDS with constant delays in the states. A TDS is formulated as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t), \;\; t > 0 \\ &y(t) = Cx(t) \\ &x(0) = x_0 \\ &x(t) = g(t), \;\; t \in [-h, 0) \end{aligned} \quad , \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A, A_d \in \mathbb{R}^{n\times n}$ are the state matrices, $h > 0$ is a constant time delay in the states, $B \in \mathbb{R}^{n\times p}$ is the input matrix, $C^T \in \mathbb{R}^{n\times q}$ is the output matrix, $y(t) \in \mathbb{R}^q$ is the output vector and $u(t) \in \mathbb{R}^p$ is the input vector. For simplicity, we assume the system has a single constant delay, and the generalization to systems with multiple delays will be presented in Section III-D. For a TDS, initial conditions including the state values in the delay interval $[-h, 0)$, i.e., $x_0$ and the preshape function $g(t)$, are needed to determine the state solution after origin.

In [10], the time-domain state solution of (1) is formulated as

$$x(t, 0, g, u) = x(t, 0, g, 0) + \int_0^t K(t,\tau)Bu(\tau)d\tau, \tag{2}$$

where $x(t, 0, g, u)$ denotes a solution at time $t$ corresponding to the initial time 0, initial delay-interval function $g(t)$ and input signal $u(t)$. $x(t, 0, g, 0)$ denotes the homogeneous solution of the system with $u(t) = 0$. $K(t,\tau)$ denotes the fundamental matrix function of (1) and it satisfies the following equations

$$\begin{aligned} &\partial K(t,\tau)/\partial\tau = -K(t,\tau)A - K(t,\tau+h)A_d, \; 0 \le \tau \le t-h \\ &K(t,t) = I \\ &K(t,\tau) = 0, \; \text{for } \tau > t. \end{aligned} \quad . \tag{3}$$

To analyze the controllability and observability Gramians, we need to derive an analytical solution of the fundamental matrix function $K(t,\tau)$, which will be discussed in details in Section III-A.

### B. Review of the Poor Man's TBR

For a minimally realized, stable LTI standard state-space model, the controllability and observability Gramians $P$ and $Q$ are defined respectively as

$$P = \int_0^\infty e^{At}BB^Te^{A^Tt}dt, \qquad Q = \int_0^\infty e^{A^Tt}C^TCe^{At}dt, \tag{4}$$

which are the unique positive definite solutions to the Lyapunov equations

$$AP + PA^T + BB^T = 0, \tag{5a}$$

$$A^TQ + QA + C^TC = 0. \tag{5b}$$

After computing the Gramians from the Lyapunov equations, a standard TBR is then performed to obtain the ROM. The main advantage of TBR over moment-matching methods is that the $H^\infty$ norm of the transfer function approximation has a bounded error.

However, direct solution of the Lyapunov equations (5) needs $O(n^3)$ cost, which limits the application of TBR. To reduce the complexity, Poor Man's TBR [7] was proposed to approximate the Gramians in the frequency domain. Poor Man's TBR reformulates the system Gramians (4) in the Laplace domain as

$$P = \int_{-\infty}^{\infty} (sI - A)^{-1}BB^T(sI - A)^{-H}ds, \tag{6a}$$

$$Q = \int_{-\infty}^{\infty} (sI - A^T)^{-1}C^TC(sI - A^T)^{-H}ds, \tag{6b}$$

where the superscript $H$ denotes the Hermitian transpose. A finite summation is then used to approximate the infinite integration for the sake of efficiency. Defining

$$z_{ck} = (j\omega_kI - A)^{-1}B, \quad z_{ok} = (j\omega_kI - A^T)^{-1}C^T, \tag{7}$$

the approximate Gramians can be calculated as

$$\widetilde{P} = \sum_k z_{ck}z_{ck}^H, \qquad \widetilde{Q} = \sum_k z_{ok}z_{ok}^H \;. \tag{8}$$

With the approximate Gramians, standard TBR can then be applied to perform MOR.

## III. GRAMIAN-BASED MOR OF TDSS

### A. Controllability and Observability Gramians of TDSs

There are several definitions of controllability for a TDS [10–12], such as $M_2$ controllability, absolute controllability, point-wise controllability, and some hybrid types. Here we use the most popular definition, the point-wise controllability.

***Definition 1*** (Controllability of a TDS) [13]: The system (1) is point-wise controllable if, for any given initial conditions $g(t)$ and $x_0$, there exists $0 < t_1 < \infty$, and an admissible input $u(t)$ for $t \in [0,\ t_1]$ such that $x(t_1,\ 0,\ g(t),\ u(t)) = 0$.

The lack of a well-established analytical criterion of the controllability imposes a significant limitation to its applicability. An algebraic criterion was proposed by Weiss [10] to check the point-wise controllability of linear time-varying systems. The most recent result was presented in [13], where the Gramians matrices are defined via the Lambert W function. The derivation of algebraic Gramians stems from the time-domain solution to (1) via the matrix Lambert W function [13]

$$x(t) = \sum_{k=-\infty}^{\infty} e^{S_kt}C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\tau)}C_k^NBu(\tau)d\tau, \tag{9}$$

with

$$S_k = \frac{1}{h}W_k(A_dhQ_k) + A, \tag{10}$$

where $W_k(H_k)$ is the matrix Lambert W function of $H_k$ defined in [13]. For more details of the convergence conditions, definitions and calculations of $C_k^I$, $C_k^N$ and $Q_k$, we refer the readers to [9, 13, 14].

As discussed in Section II-A, the fundamental matrix function $K(t,\tau)$ now has an analytical form [13]

$$K(t,\tau) = \sum_{k=-\infty}^{\infty} e^{S_k(t-\tau)}C_k^N. \tag{11}$$

Analogous to the LTI systems, the controllability Gramian can be defined as [13]

$$P(0,\ t_1) = \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{S_k(t_1-\tau)} C_k^N B B^T \{ \sum_{k=-\infty}^{\infty} e^{S_k(t_1-\tau)} C_k^N \}^T d\tau. \tag{12}$$

The observability condition and its Gramian are defined similarly in the following.

***Definition 2*** (Observability of a TDS) [13]: The system (1) is point-wise observable in $[0, t_1]$ if the initial point $x_0$ can be uniquely determined from the knowledge of $u(t)$, $g(t)$ and $y(t)$.

The observability Gramian is defined using the fundamental matrix function (11) as [13]

$$Q(0, t_1) = \int_0^{t_1} \{ \sum_{k=-\infty}^{\infty} e^{S_k(\tau-0)} C_k^N \}^T C^T C \sum_{k=-\infty}^{\infty} e^{S_k(\tau-0)} C_k^N d\tau. \tag{13}$$

Note that if $A_d = 0$, the TDS reduces to a LTI system, and the controllability and observability Gramians are the same as in the linear case without delays. Therefore the TBR procedure for a TDS is consistent to the LTI case when $A_d \to 0$.

These analytical Gramians and their time-domain evaluations were proposed in [13] for the first time, where the Gramians are approximated in the time domain using the first several branches of the Lambert W function. However, as discussed in [9], such an evaluation is too expensive for practical implementation due to the unavoidable matrix inversion in computing the coefficients $S_k$ and $C_k^N$. Moreover, without a convergence criterion of the infinite Lambert W function branches, one cannot determine how many branches should be used to approximate the Gramians. When the original TDSs are of high order, which is common in VLSI circuit simulation, the cost becomes extremely expensive, making this approach infeasible.

### *B. Complexity of Lyapunov-Type Equations of TDSs*

Here we briefly analyze the complexity of Lyapunov-type equations for a TDS, and show that it is reasonable to approximate the Gramians by the Poor Man's TBR.

By defining

$$w_{ij} = e^{S_i t} C_i^N B B^T (e^{S_j t} C_j^N)^T, \tag{14}$$

$$P_{ij} = \int_0^{\infty} w_{ij} dt, \tag{15}$$

the controllability Gramians (12) can be rewritten as

$$P = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_0^{\infty} w_{ij} dt = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} P_{ij}. \tag{16}$$

For a stable TDS we can obtain a set of Lyapunov-type equations for the controllability Gramian components $P_{ij}$

$$S_i P_{ij} + P_{ij} S_j^T + C_i^N B B^T (C_j^N)^T = 0. \tag{17}$$

Then the controllability Gramian (16) can be approximated by its first several branches of solutions to (17). The observability Gramian can be approximated similarly.

However, calculating the Gramians from time-domain Lyapunov-type equations are as difficult as calculating the Gramians from their time-domain definitions. In our experiments, the delay elements are modeled from MTLs which have strong coupling effects in the high frequency range, and this motivates us to calculate the Gramians in the frequency domain for computational efficiency.

### *C. Proposed TBR of TDSs via Approximate Gramians*

By Laplace transform, the fundamental matrix function of a TDS becomes [13]

$$\mathcal{L}\{ \sum_{k=-\infty}^{\infty} e^{S_k(t-0)} C_k^N B \} = (sI - A - A_d e^{-sh})^{-1} B, \tag{18a}$$

$$\mathcal{L}\{\{ \sum_{k=-\infty}^{\infty} e^{S_k(\tau-0)} C_k^N \}^T C^T \} = (sI - A^T - A_d^T e^{-sh})^{-1} C^T, \tag{18b}$$

where $\mathcal{L}$ denotes the Laplace transform operation. Then the controllability and observability Gramians can be calculated in the frequency domain as

$$P = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sh})^{-1} B B^T (sI - A - A_d e^{-sh})^{-H} ds, \tag{19a}$$

$$Q = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sh})^{-H} C^T C (sI - A - A_d e^{-sh})^{-1} ds. \tag{19b}$$

Evaluating the Gramians over the whole frequency band is impractical. In practical VLSI circuits, the input signals usually have a finite bandwidth, therefore the infinite integration can be approximated by a finite summation at a set of frequency sampling points. In this paper, we focus on the TDSs consisting of a linear RLC interconnect network connected with a set of lossless MTLs. The time-delay values in such models are in the magnitude of nanoseconds, therefore the delay effects occur only when the input signal is of high frequency. This fact makes the proposed approximation viable. The scheme of frequency selection is elaborated in [7], and in this paper we select the frequency-sampling points uniformly within the frequency band of interest. By defining

$$z_{ck} = (j\omega_k I - A - A_d e^{-j\omega_k h})^{-1} B, \tag{20a}$$

$$z_{ok} = (j\omega_k I - A - A_d e^{-j\omega_k h})^{-H} C^T, \tag{20b}$$

the approximate Gramians can be calculated as

$$\widetilde{P} = \sum_k z_{ck} z_{ck}^H, \qquad \widetilde{Q} = \sum_k z_{ok} z_{ok}^H \ . \tag{21}$$

Denoting

$$Z_c = [z_{c1},\ z_{c2}, \ldots,\ z_{cN}], \tag{22a}$$

$$Z_o = [z_{o1},\ z_{o2}, \ldots,\ z_{oN}], \tag{22b}$$

the approximate Gramians are expressed as

$$\widetilde{P} = Z_c Z_c^H, \qquad \widetilde{Q} = Z_o Z_o^H \ . \tag{23}$$

After calculating the Gramians, a standard TBR is then performed to obtain the ROM. The proposed algorithm is summarized in ***Algorithm 1***. Note that MOR via the projection technique can preserve TDS structure, and the delay effect remains the same as in the original systems. Similar to classical TBR, the proposed Gramian-based algorithm for TDSs may not preserve system passivity. However, some passivity test and enforcement approaches [15, 16] can be incorporated to obtain passive ROMs.

**Algorithm 1** : TBR of TDSs via Approximate Gramians

1: Input: $A, A_d, B, C, h$
2: Select frequency-sampling points: $\omega_k \ (k = 1 \dots N)$
3: Compute $Z_c = [z_{c1},\ z_{c2}, \dots,\ z_{cN}]$
where $z_{ck} = (j\omega_k I - A - A_d e^{-j\omega_k h})^{-1} B$
4: Compute $Z_o = [z_{o1},\ z_{o2}, \dots,\ z_{oN}]$
where $z_{ok} = (j\omega_k I - A - A_d e^{-j\omega_k h})^{-H} C^T$
5: Compute the Gramians $P = Z_c Z_c^H$ and $Q = Z_o Z_o^H$
6: Compute Cholesky factors $P = XX^H$ and $P = YY^H$
7: Compute SVD of cross factors and partition for truncation
$X^H Y = U \Sigma V^T = U \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} V^T$
8: Compute the balanced transformation matrices
$T_L = \Sigma^{-1/2} V^H Y^H$ and $T_R = XU\Sigma^{-1/2}$
9: Partition the balanced realization according to Step 7
$A_b = T_L A T_R = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{db} = T_L A_d T_R = \begin{bmatrix} A_{d_{11}} & A_{d_{12}} \\ A_{d_{21}} & A_{d_{22}} \end{bmatrix}$, $B_b = T_L B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C_b = C T_R = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$
10: Truncate the balanced realization to obtain the ROM
$A_r = A_{11}, A_{dr} = A_{d_{11}}, B_r = B_1, C_r = C_1$
11: Output: $A_r, A_{dr}, B_r, C_r, h$

### D. Practical Implementation

In practical implementations, TDSs constructed from an interconnect network connected with MTLs usually have the state-evolution equations in the form of descriptor systems (DSs), and the TDSs can have multiple delay terms resulting from different signal propagation paths. Therefore, the state-evolution equation (1) becomes

$$E\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_{d_i} x(t - h_i) + Bu(t), \qquad (24)$$

where $m$ is the number of delay terms. In this situation, the formulation (20) now changes to

$$z_{ck} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k h_i})^{-1} B, \qquad (25a)$$

$$z_{ok} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k h_i})^{-H} C^T. \qquad (25b)$$

By replacing Step 3 and 4 in ***Algorithm 1*** with (25), we can perform TBR of TDSs with multiple delays. The accuracy and efficiency of our proposed algorithm are presented in the experimental section.

## IV. Experimental Results

### A. Modeling of TDSs

In this paper we focus on time-delay effects resulting from MTLs when excited in the high frequency range. The whole system consists of a linear interconnect network connected with a set of lossless MTLs. Figure 1 shows the model structure of TDSs. When the input signal is of high frequency, the MTLs will have strong coupling effects, i.e., coupled capacitance and inductance between each pair of lossless transmission lines. These coupled parameters along the MTLs defer the signal propagation, resulting in the time-delay effects. The detailed procedures to construct the TDSs from a linear interconnect network and a set of lossless MTLs are elaborated in [2], and the per-unit-length (PUL) parameter matrices of the MTLs are extracted utilizing the approaches in [17].

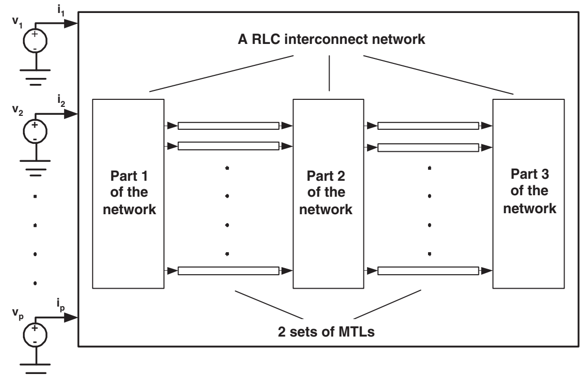


Fig. 1. A TDS consisting of a RLC interconnect network connected with two sets of lossless MTLs

The time-delay values in our examples are in the magnitude of nanoseconds, so the delay effects only occur when the input signal is beyond GHz. In order to capture such high-frequency delay effects, we select the sampling points uniformly within the high-frequency range of interest. The experimental results are compared with the existing TDS moment-matching approach [2].

### B. Proposed TBR Compared with Moment-Matching Approach

In the first example, we use a 3-port linear interconnect network connected with 70 lossless three-conductor transmission lines to construct an order-1098 TDS. A ROM of order 220 is constructed utilizing the proposed TBR approach. The TDS Gramians are approximated by using 50 sampling points distributed uniformly in logarithm scale within the frequency range of interest, i.e., from $1e6$ to $1e12$ Hz in our experiments. We also utilize the moment-matching approach to construct an order-231 ROM matching the first 77 moments of the original model, and this ROM uses 2nd-order Taylor expansion to approximate the delay terms. Figure 2 compares the frequency response of various ROMs with the original TDS. It is shown that the time-delay effects occur in the high frequency range, and the proposed TBR approach provides a much higher accuracy than the moment-matching method. Because the sizes of ROMs from these two approaches are similar, the simulation speedup is also comparable (about 10x). The time-domain response with respect to the input signal $u(t) = \sin(2\pi \times 10^9 t)$ $(mV)$ is further compared in Figure 3 which also shows that the TBR approach provides a much higher accuracy than the moment-matching method.

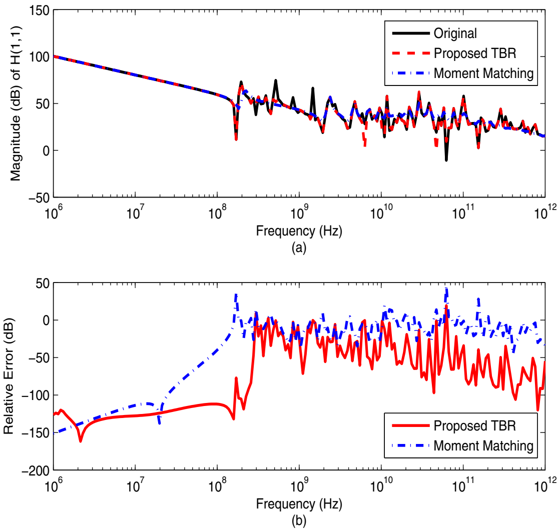


Fig. 2. Frequency response of Example 1 and comparison between the proposed TBR and the moment-matching approach. (a) The magnitude of transfer function $H(1,1)$. (b) Relative error comparison.

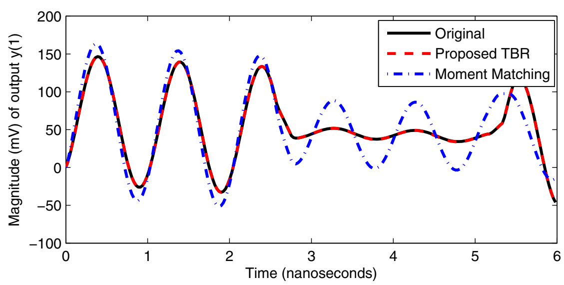


Fig. 3. Time-domain response of output $y(1)$ in Example 1 and comparison between the proposed TBR and the moment-matching approach.

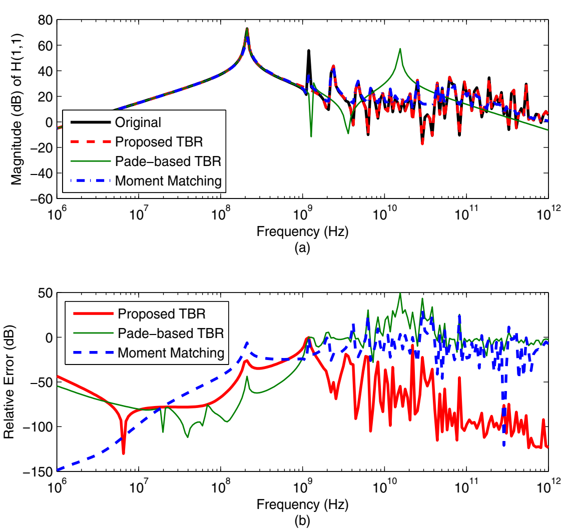


Fig. 4. Frequency response of Example 2 and comparison with various ROMs. (a) The magnitude of transfer function $H(1,1)$. (b) Relative error comparison.

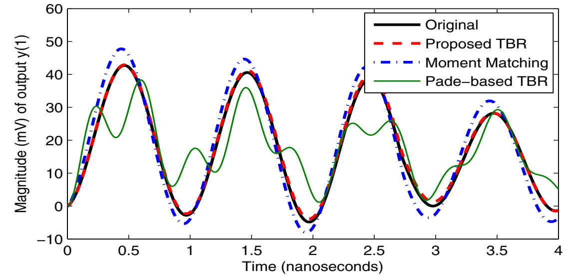


Fig. 5. Time-domain response of output $y(1)$ in Example 2 and comparison with various ROMs.

### *C. Proposed TBR Compared with Padé-Approximation Approach*

The proposed TBR approach is a direct TBR of the original model. However, one can still perform indirect TBR based on an equivalent DS constructed via Padé approximation of the exponential delay terms from the original TDS [18]. The drawback of the indirect TBR approach lies in that the resulting DS is of much higher order than the original TDS (the size of DS will increase in proportion to the product of the original order, the number of delay terms, and the Padé-approximation order), and the nature of Padé approximation further limits its accuracy performance. The comparison between our proposed direct TBR and the indirect TBR is shown in the second example. In this example, we use a 6-port interconnect network connected with 25 lossless two-conductor transmission lines to construct a TDS of order 193. An order-22 ROM is constructed utilizing the proposed TBR approach. The TDS Gramians are approximated by using 65 sampling points distributed uniformly in logarithm scale within the frequency range of interest, i.e., from $1e6$ to $1e12$ Hz in our experiments. An order-579 DS is constructed using 2nd-order Padé approximation of the delay term, and then an order-188 ROM is obtained via TBR based on the DS. We also utilize the moment-matching approach to construct an order-30 ROM matching the first 5 moments of the original model, and this ROM uses 4th-order Taylor expansion to approximate the delay term. Figure 4 (a) compares the frequency response of various ROMs with the original TDS. From the relative-error results plotted in Figure 4 (b), it is seen that the proposed TBR method provides the best accuracy performance among the various MOR approaches. The time-domain response with respect to the input signal $u(t) = \sin(2\pi \times 10^9 t)$ $(mV)$ is also compared in Figure 5 showing that the proposed TBR provides a higher accuracy than the moment-matching method and the indirect TBR based on Padé approximation.

### *D. Comparison of MOR Times*

Table I compares the MOR times of the proposed TBR scheme versus the moment-matching approach and the indirect TBR based on Padé approximation. The tests are performed on a platform of Intel Core 2 Q8400 with 2.66GHz CPU and 3.25GB RAM. The CPU times show that the efficiency of various MOR schemes is case-dependent. For Example 1, though moment-matching approach takes less time to construct the ROM, the proposed TBR can provide a much higher accuracy than the moment-matching method, as shown in Figure 2 and Figure 3, and the ability to capture the delay effects is of high importance in TDS modeling. However, the proposed TBR approach is faster for Example 2, where the moment-matching approach requires 4th-order Taylor expansion of the delay terms to capture the delay effects well. The equivalent model before moment matching for Example 2 is 4 times larger than the original TDS [2], making it less efficient than the proposed TBR approach. The MOR times of Padé-based TBR are also shown in Table I. The test for Example 1 uses 2nd-order Padé approximation of the delay terms, resulting in an order-5490 equivalent DS which is of much higher order than the original TDS. From the results we can see that the Padé-based method is inaccurate and inefficient, making it infeasible for TDS MOR. Therefore, the proposed TBR provides the best accuracy performance among the various MORs, and it can even be more efficient than the moment-matching approach when higher accuracy is required.

TABLE I
COMPARISON OF THE MOR TIMES OF THE PROPOSED TBR, PADÉ-BASED TBR, AND THE MOMENT-MATCHING APPROACH (SECONDS)

| | Proposed TBR | Moment Matching | Padé TBR |
|---|---|---|---|
| Example 1 (Order 1098) | 71.07 (Order 220) | 43.51 (Order 231) | out of memory |
| Example 2 (Order 193) | 0.85 (Order 22) | 1.46 (Order 30) | 7.59 (Order 188) |

## V. CONCLUSIONS

In VLSI circuit simulation, it is important to facilitate the simulation of TDSs when large RLC networks are connected with delay elements from packages or PCBs. In this paper, we have presented a TBR-type MOR for TDSs, based on approximate system Gramians. To the best of our knowledge, it is the first time to perform model reduction on TDSs via a Gramian-based approach. The experimental results have shown that the proposed TBR-type algorithm provides higher accuracy and comparable efficiency than the conventional moment-matching methods, making the proposed algorithm suitable for fast simulation of TDSs.

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