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Exact Solutions for a Type of Electron Pairing Model with Spin-Orbit Interactions and Zeeman Coupling

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A type of electron pairing model with spin-orbit interactions or Zeeman coupling is solved exactly in the framework of the Richardson ansatz. Based on the exact solutions for the case with spin-orbit interactions, it is shown rigorously that the pairing symmetry is of the $p + ip$ wave and the ground state possesses time-reversal symmetry, regardless of the strength of the pairing interaction. Intriguingly, how Majorana fermions can emerge in the system is also elaborated. Exact results are illustrated for two systems, respectively, with spin-orbit interactions and Zeeman coupling.

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Recently, significant research attention has been paid to various physical systems with spin-orbit interactions, including quantum spin Hall effects [1–3], topological insulators [4], semiconductor heterostructures [5], and a number of artificial systems like ultracold atoms in optical lattices [6,7]. In particular, several important theoretical understandings have been obtained for pairing electrons in the presence of spin-orbit interactions [4,8]. Nevertheless, all of these theoretical investigations on pairing systems have been conducted in the framework of mean field theory (MFT), which is known to be a good approximation merely for weak pairing interactions. Therefore, more rigorous theoretical understanding or even exact solutions for these electron pairing systems are highly appreciated, particularly for strong pairing cases, even though it is extremely challenging to find exact solutions of models for interacting many-electron systems. This is a central motivation of this work.

It is noted that Richardson obtained exact solutions of some pairing models in the 1960s [9]. As is known, Richardson's exact solutions for pairing force models have played an important role in the research of interacting many-particle physics [10], including their connection with the well-known BCS model [11].

In this Letter, we first consider a type of electron pairing model with spin-orbit interactions and solve it exactly in the framework of the Richardson ansatz. Based on the exact solutions obtained, we show rigorously that the pairing order parameter always has $p + ip$-wave symmetry regardless of the strength of pairing interactions, which recovers an important conclusion deduced from the MFT. Notably, for a special example, we illustrate that our exact numerical results invalidate a ground state predicted by the MFT at the critical point. In addition, we address the same model with the Zeeman coupling [12]. Remarkably, we are also able to find an exact solution in the presence of a pure Zeeman term with the same scenario. Exact analytical results are presented for the special electron system. Moreover, we also elaborate how Majorana fermions can emerge in the system.

Let us consider a pairing electron Hamiltonian with spin-orbit interactions in a two-dimensional lattice, which may be written as

$$H = H_0 + H_{\text{int}}, \tag{1}$$

with

$$H_0 = \sum_{\mathbf{k}} (c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\downarrow})(\varepsilon_{\mathbf{k}} + \alpha \mathbf{k} \cdot \boldsymbol{\sigma})(c_{\mathbf{k}\uparrow}, c_{\mathbf{k}\downarrow})^T,$$

$$H_{\text{int}} = -\sum_{\mathbf{k},\mathbf{k}'} V_0(\mathbf{k}, \mathbf{k}') c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow},$$

where $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}}$ is the spin-independent single electron energy, $c^\dagger_{\mathbf{k}\uparrow(\downarrow)}$ and $c_{\mathbf{k}\uparrow(\downarrow)}$ are the creation and annihilation operators of electrons, $\mathbf{k} = (k_x, k_y)$ is the wave vector of the lattice [13], α is the effective strength of spin-orbit interaction, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ is the Pauli matrices. We here consider an s-wave pairing interaction, namely $V_0(\mathbf{k}, \mathbf{k}') = V_0 > 0$. Although the above Hamiltonian has been studied recently under various mean field approximations, to the best of our knowledge, it has not been solved exactly. Motivated by this, we here want to find an exact solution in the framework of Richardson ansatz. We first diagonalize the single-particle Hamiltonian by making the following unitary transformation:

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= \frac{1}{\sqrt{2}}(a_{\mathbf{k},+} + e^{-i\theta(\mathbf{k})} a_{\mathbf{k},-}), \\ c_{\mathbf{k}\downarrow} &= \frac{1}{\sqrt{2}}(e^{i\theta(\mathbf{k})} a_{\mathbf{k},+} - a_{\mathbf{k},-}), \end{aligned} \tag{2}$$

with $e^{i\theta(\mathbf{k})} = (k_x + ik_y)/|\mathbf{k}|$ [14] for $\mathbf{k} \neq 0$ and $e^{i\theta(0)} = 1$ for $\mathbf{k} = 0$. Physically, this unitary transformation corresponds to a local spin-basis rotation to align the spin direction along the wave vector $\mathbf{k}$, which actually introduces

an effective local gauge field acting on electrons. The Hamiltonian is now rewritten as

$$H = \sum_{\mathbf{k},s} \varepsilon_{\mathbf{k},s} a^\dagger_{\mathbf{k},s} a_{\mathbf{k},s} - \frac{V_0}{4} \sum_{\mathbf{k}s,\mathbf{k}'s'} e^{-is\theta(\mathbf{k})+is'\theta(\mathbf{k}')} \times (A^\dagger_{\mathbf{k},s} - \delta_{\mathbf{k},0} A^\dagger_{0,0})(A_{\mathbf{k}',s'} - \delta_{\mathbf{k}',0} A_{0,0}), \quad (3)$$

where the dispersion $\varepsilon_{\mathbf{k},s} = \varepsilon_{\mathbf{k}} + s\alpha k$ with $s = \pm 1$ denoting the two branches of the diagonalized single-particle spectrum in the new basis. Here, the pairing operators are defined by

$$A^\dagger_{\mathbf{k},s} \equiv a^\dagger_{\mathbf{k},s} a^\dagger_{-\mathbf{k},s} (s = \pm), \qquad A^\dagger_{0,0} \equiv a^\dagger_{0,+} a^\dagger_{0,-}. \quad (4)$$

In derivation of the above Eq. (3), we have employed a useful relation $\theta(\mathbf{k}) - \theta(-\mathbf{k}) = \pm\pi$ for $\mathbf{k} \neq 0$. It is obviously seen from Eq. (4) that $A^\dagger_{\mathbf{k},s} = 0$ for $\mathbf{k} = 0$. The above operators satisfy the following commutation relations:

$$A^{\dagger 2}_{\mathbf{k},s} = 0, \qquad [A_{\mathbf{k},s}, A^\dagger_{\mathbf{k}',s'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}(1 - 2A^\dagger_{\mathbf{k},s}A_{\mathbf{k},s}),$$
$$[A^\dagger_{\mathbf{k},s}A_{\mathbf{k},s}, A^\dagger_{\mathbf{k}',s'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}A^\dagger_{\mathbf{k},s} \quad (5)$$

for $\mathbf{k} \neq -\mathbf{k}'$, and

$$[A_{0,0}, A^\dagger_{0,0}] = 1 - 2A^\dagger_{0,0}A_{0,0}, \quad (6)$$

for $\mathbf{k} = \mathbf{k}' = 0$. These relations play a crucial role in solving this model exactly. Although the pairing term in Eq. (3) is $\mathbf{k}$ dependent, we still make an ansatz in the same framework of Richardson's pioneering work on a pairing model [9]. In this framework, the eigenstates of Hamiltonian (3) should take the product form as

$$|n, S_+, S_-\rangle = \prod_{\mathbf{k}_i \in S_+} a^\dagger_{\mathbf{k}_i,+} \prod_{\mathbf{k}_j \in S_-} a^\dagger_{\mathbf{k}_j,-} \prod_{\nu=1}^{n} B^\dagger_\nu |0\rangle, \quad (7)$$

where

$$B^\dagger_\nu = \sum_{\substack{s,\mathbf{k}\in P_s \\ \mathbf{k}\neq 0}} \frac{e^{-is\theta(\mathbf{k})} A^\dagger_{\mathbf{k},s}}{2\varepsilon_{\mathbf{k},s} - E_\nu} + \frac{2A^\dagger_{0,0}}{2\varepsilon_0 - E_\nu}. \quad (8)$$

Here $S_\pm$ denotes the set of singly occupied levels (namely blocked levels) of the $\pm$ branch with cardinality $m_\pm$, while $P_\pm$ denotes the set of levels with the blocked ones excluded. The state vector defined in Eq. (7) describes an eigenstate of $N_e = m_+ + m_- + 2n$ electrons with n as the number of electron pairs. E_ν's in Eq. (8) are the parameters to be determined by n coupled algebraic equations to be given in the following.

Solving the Schrödinger equation associated with Hamiltonian (3) and the eigenvector in Eq. (7) more tediously, we obtain the equations for the present two-branch electron system,

$$1 - \sum_{\substack{s,\mathbf{k}\in P_s \\ k\neq 0}} \frac{V_0/2}{2\varepsilon_{\mathbf{k},s} - E_\nu} - \frac{V_0}{2\varepsilon_0 - E_\nu} + \sum_{\mu\neq\nu}^{n} \frac{2V_0}{E_\mu - E_\nu} = 0, \quad (9)$$

where $\nu = 1, 2, \ldots, n$. The corresponding eigenenergy is given by

$$E(n, m_+, m_-) = \sum_{\mathbf{k}\in S_+} \varepsilon_{\mathbf{k},+} + \sum_{\mathbf{k}\in S_-} \varepsilon_{\mathbf{k},-} + \sum_{\nu=1}^{n} E_\nu. \quad (10)$$

Remarkably, here we have demonstrated the pairing model of Eq. (3) to be an integrable problem [15], making such a pairing model more promising and useful. The set of Eq. (9) is quite similar to Richardson's. In particular, when $\alpha = 0$, the two branches are degenerate and Eq. (9) recovers the usual Richardson's equation [9,16,17]. It has been shown by Gaudin that Eq. (9) has a continuum limit form in the thermodynamic limit [18,19].

Note that there are normally two kinds of spin-orbit interactions: one takes the form of $\mathbf{k}\cdot\boldsymbol{\sigma}$ as in Eq. (1) [4] with the exact solution being given above, while the other has the form $(\boldsymbol{\sigma}\times\mathbf{k})\cdot\hat{z}$ [5,20]. If the spin-orbit interaction in Eq. (1) is changed to the second form, one can replace $\theta(\mathbf{k})$ in Eqs. (2), (3), and (8) by $\theta'(\mathbf{k}) = \theta(\mathbf{k}) - \pi/2$ and accordingly the pairing model with $(\boldsymbol{\sigma}\times\mathbf{k})\cdot\hat{z}$-type spin-orbit interaction is exactly solvable as well.

Although it is still rather challenging to solve Eq. (9) even numerically, the computational loading is significantly reduced in comparison with the numerical exact diagonalization. In terms of this exact solution for the system described by the Hamiltonian (3), we are able to evaluate some quantities exactly and obtain relevant rigorous results, which are very helpful for validating or invalidating the related results based on the usual mean field framework.

As an important example, we use Eq. (7) to derive exactly the following dimensionless order parameter,

$$\triangle_{\mathbf{k},s} = \frac{\langle 0, 0, n-1 | a_{-\mathbf{k},s} a_{\mathbf{k},s} | n, 0, 0\rangle}{\sqrt{C_n C_{n-1}}} = e^{-is\theta(\mathbf{k})} \triangle^0_{\mathbf{k},s}, \quad (11)$$

where $C_n = \langle 0, 0, n | n, 0, 0\rangle$ and

$$\triangle^0_{\mathbf{k},s} = \frac{1}{\sqrt{C_n C_{n-1}}} \sum_{\nu=1}^{n} \frac{\sum_{\{j_i\}}^{\forall\ k_{j_i}\neq k} g^{(n)}_\nu g^{(n-1)*}_n}{2\varepsilon_k - E^{(n)}_\nu}, \quad (12)$$

with

$$g^{(n)}_\nu = \sum_P \prod_{\mu=1}^{\nu-1} \frac{1}{2\varepsilon_{k_{j_\mu}} - E^{(n)}_{P\mu}} \prod_{\mu=\nu+1}^{n} \frac{1}{2\varepsilon_{k_{j_{\mu-1}}} - E^{(n)}_{P\mu}}.$$

Here k denotes $(\mathbf{k}, s)$, the superscript (n) corresponds to the n-pair state, and P means the permutation of the corresponding terms. In the weak interaction limit ($V_0 \to 0$), E_ν's are all real, so that $\triangle^0_{\mathbf{k},s}$ is real as well. It is clearly seen that $\triangle_{\mathbf{k},s}$ has the $p_x + ip_y$ pairing symmetry. Notably, even when the solutions E_ν are complex numbers, we can show from Eq. (12) that $\triangle^0_{\mathbf{k},s}$ is still real and just the usual s-wave one because the complex solutions of Eq. (9) appear in the form of conjugate pairs. This finding for the pairing symmetry justifies a result expected by the MFT in the weak interaction limit [4,5]. Also remarkably, we indeed find from Eqs. (7)–(9) that the pairing ground state $|n, 0, 0\rangle$ is time-reversal symmetric despite the chiral p-wave pairing.

As an illustration, we address a special case where all N_e electrons are on the Fermi surface $k = k_F$, which is an

approximation for the considered system as many physical phenomena are only closely related to the electrons near the Fermi surface. The degeneracy of the Fermi level is supposed to be Ω. In the presence of the SO interaction, the energy level is split into two branches $\varepsilon_{\mathbf{k},\pm} = \varepsilon_F \pm \alpha k_F$. The symbols in Fig. 1 denote the exact condensation energy $\Delta E(n)$ calculated from Eqs. (9) and (10) numerically. The condensation energies obtained from the MFT, denoted by solid lines, are always higher than the exact results; in particular, they vanish at the quantum critical point $n_c/\Omega = 0.5$ (half filling) predicted by the MFT, implying that the system would be in a normal state. However, in sharp contrast, the exact condensation energy is always finite at half-filling even in the thermodynamic limit, giving rise to a superconducting state. We here wish to pinpoint that the differences between the exact solution and MFT are more significant for small n cases, such as those for cold-atom systems.

Next we turn to consider the Zeeman term [12] induced by an external magnetic field $\mathbf{B} = (B_x, B_y, B_z)$, which reads $\hat{H}_Z = \sum_{\mathbf{k}} (c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\downarrow}) \mathbf{B} \cdot \boldsymbol{\sigma} (c_{\mathbf{k}\uparrow}, c_{\mathbf{k}\downarrow})^T$ and is added to Hamiltonian (1). We now make another transformation as

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= \cos\varphi_{\mathbf{k}} a_{\mathbf{k},+} + \sin\varphi_{\mathbf{k}} e^{-i\tilde{\theta}(\mathbf{k})} a_{\mathbf{k},-}, \\ c_{\mathbf{k}\downarrow} &= \sin\varphi_{\mathbf{k}} e^{i\tilde{\theta}(\mathbf{k})} a_{\mathbf{k},+} - \cos\varphi_{\mathbf{k}} a_{\mathbf{k},-}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tan(2\varphi_{\mathbf{k}}) &= \eta_{\mathbf{k}}/B_z, \\ e^{i\tilde{\theta}(\mathbf{k})} &= [(B_x + \alpha k_x) + i(B_y + \alpha k_y)]/\eta_{\mathbf{k}}, \\ \eta_{\mathbf{k}} &= \sqrt{(B_x + \alpha k_x)^2 + (B_y + \alpha k_y)^2}. \end{aligned} \tag{14}$$

The single-particle spectrum still has two branches with $\varepsilon_{\mathbf{k},s} = \varepsilon_{\mathbf{k}} + s\sqrt{\eta^2_{\mathbf{k}} + B^2_z}$. In addition to the operators in Eq. (4), we also need new operators defined as

$$A^\dagger_{\mathbf{k},0} \equiv a^\dagger_{\mathbf{k},+} a^\dagger_{-\mathbf{k},-}, \qquad A_{\mathbf{k},0} \equiv a_{-\mathbf{k},-} a_{\mathbf{k},+}. \tag{15}$$

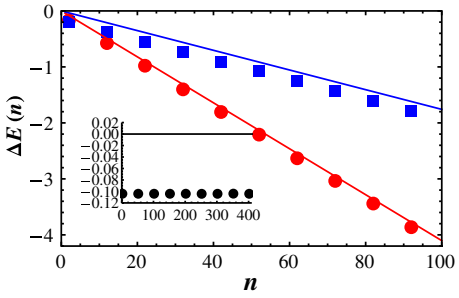

FIG. 1 (color online). The condensation energy $\Delta E(n)$ in units of αk_F as a function of pair number n with $V_0\Omega = 0.2$ and $n/\Omega = 0.3$ (red line and dots) and 0.7 (blue line and squares). To emphasize, we show in the inset the results for the half-filled case, namely $n/\Omega = 0.5$. Here, solid lines are results from the MFT while symbols represent those from exact solutions.

Under this transformation, the total Hamiltonian with Zeeman term can be rewritten as

$$\begin{aligned} H = &\sum_{\mathbf{k},s=\pm} \varepsilon_{\mathbf{k},s} a^\dagger_{\mathbf{k},s} a_{\mathbf{k},s} - V_0 \sum_{\mathbf{k}s,\mathbf{k}'s'} e^{-is\tilde{\theta}(-s\mathbf{k}) + is'\tilde{\theta}(-s'\mathbf{k}')} \\ &\times \lambda^*_s(\mathbf{k}) \lambda_{s'}(\mathbf{k}') A^\dagger_{\mathbf{k},s} A_{\mathbf{k}',s'}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \lambda_s(\mathbf{k}) &\equiv s\cos\varphi_{s\mathbf{k}} \sin\varphi_{s(-\mathbf{k})} \lambda_0(\mathbf{k}) \\ &\equiv -(\cos\varphi_{\mathbf{k}} \cos\varphi_{-\mathbf{k}} + \sin\varphi_{\mathbf{k}} \sin\varphi_{-\mathbf{k}} e^{i\tilde{\theta}(\mathbf{k}) - i\tilde{\theta}(-\mathbf{k})}) \end{aligned}$$

and $s, s' = 0, \pm 1$ in the second summation of Eq. (16).

Generally, because $[A^\dagger_{\mathbf{k},\pm}, A_{\mathbf{k}',0}] \neq 0$ for $\mathbf{k} \neq 0$ and $\lambda_s(\mathbf{k})$ is $\mathbf{k}$ dependent, it is hard to find an exact solution of Hamiltonian (16) by adopting a similar ansatz used above. Nevertheless, Hamiltonian (16) can still be solved exactly for some special but relevant cases. When the external magnetic field $\mathbf{B} = 0$, we have $\varphi_{\mathbf{k}} = \varphi_{-\mathbf{k}} = \pi/4$ and $\tilde{\theta}(\mathbf{k}) = \theta(\mathbf{k})$, so that $\lambda_s(\mathbf{k}) = s/2$ ($s = \pm$), $\lambda_0(\mathbf{k}) = 0$ ($\mathbf{k} \neq 0$), and $\lambda_0(0) = -1$. In this case, Hamiltonian (16) reduces to Eq. (3).

On the other hand, when $\alpha = 0$ and $\mathbf{B} \neq 0$, $\lambda_0(\mathbf{k}) = -1$ and $\lambda_s(\mathbf{k}) = \lambda_s(-\mathbf{k})$ ($s = \pm$) because $\varphi_{\mathbf{k}} = \varphi_{-\mathbf{k}}$ and $\tilde{\theta}(\mathbf{k}) = 0, \pi$. In addition, since $A_{\mathbf{k},s} = -A_{-\mathbf{k},s}$ for $s = \pm$, the second summation for $s = \pm$ in Eq. (16) vanishes, respectively. Considering the relation

$$\left[\sum_{s=\pm} \varepsilon_{\mathbf{k},s} a^\dagger_{\mathbf{k},s} a_{\mathbf{k},s}, A^\dagger_{\mathbf{k},0}\right] = (\varepsilon_{\mathbf{k},+} + \varepsilon_{\mathbf{k},-}) A^\dagger_{\mathbf{k},0}, \tag{17}$$

we can take another ansatz

$$C^\dagger_\nu = \sum_{\mathbf{k}} \frac{A^\dagger_{\mathbf{k},0}}{\varepsilon_{\mathbf{k},+} + \varepsilon_{\mathbf{k},-} - E_\nu} \tag{18}$$

to replace $B^\dagger_\nu$ in Eq. (7). Solving the Schrödinger equation with Hamiltonian (16) and the corresponding eigenvector, we obtain the equations that the parameters E_ν's satisfy,

$$1 - \sum_{\mathbf{k}} \frac{V_0}{\varepsilon_{\mathbf{k},+} + \varepsilon_{\mathbf{k},-} - E_\nu} + \sum_{\mu \neq \nu}^{n} \frac{2V_0}{E_\mu - E_\nu} = 0, \tag{19}$$

where $\nu = 1, 2, \ldots, n$. The expression of the eigenenergy of the whole system is the same as Eq. (10). Equation (19) implies that even when the single-particle energies of electrons are spin dependent, the Hamiltonian is still exactly solvable. With the solutions of Eq. (19), we can similarly evaluate the dimensionless order parameter for this system with Eq. (11). Now the order parameter $\triangle_{\mathbf{k},0} = \triangle^0_{\mathbf{k},0}$, which has the usual s-wave symmetry.

As an interesting example, we also look into the special case addressed above with $\alpha = 0$, where all N_e ($\leq \Omega$) electrons are on the Fermi surface $k = k_F$ in Eq. (19). In the presence of the Zeeman field, unpaired electrons prefer to occupy the spin-down states. Therefore, the eigenenergy

of the partially polarized pairing state $|n, 0, S_-\rangle$ $(2n + S_- = N_e)$ is found to be

$$E = (N_e - 2n)(\varepsilon_{\mathbf{k}_F} - |\mathbf{B}|) + 2n\varepsilon_{\mathbf{k}_F} - V_0 n(\Omega' - n + 1),$$

where $\Omega' = \Omega - (N_e - 2n)$ considering that $S_- = N_e - 2n$ states are blocked by unpaired electrons. Thus the condensation energy $\triangle E = E - E_0$, with E_0 the energy of the fully polarized ferromagnetic state, is given by

$$\triangle E = -V_0 n^2 + n[2|\mathbf{B}| - V_0(\Omega - N_e + 1)]. \quad (20)$$

From the above equation, we can readily find a critical value $B_c = V_0(\Omega - N_e/2 + 1)/2$. When $|\mathbf{B}| < B_c$, the system is in the pairing state, otherwise the ferromagnetic state.

We now attempt to elaborate how Majorana fermions (MF) can emerge in the system described by Hamiltonian (1) based on our exact solution. To capture the essential physics but without the loss of generality, we consider that $2n$ electrons occupy the states in a narrow ribbon around $\varepsilon(\mathbf{k}_F)$ and confined in an annular region $r_0 < r < R_0$. In the continuum limit and from H_0, in addition to the bulk states $\varepsilon_{\mathbf{k},s}$ one can also find the inner and outer edge states with the energies $E_{\text{in}} = \varepsilon(\mathbf{k}_F) + \alpha L_z/r_0$ and $E_{\text{out}} = \varepsilon(\mathbf{k}_F) - \alpha L_z/R_0$, where L_z is the angular momentum. In the presence of the pairing interaction, the zero modes with $L_z = 0$ survive due to the topological protection [12] and they could be occupied by pairs of MFs: $a_{\text{MF}}(m\mathbf{k}_F) = (\gamma_{1m} - i\gamma_{2m})/2$ $(m = \pm)$ with $\gamma_{im} = \gamma_{im}^\dagger$ the MF operators, while the occupied bulk states are in the pairing states described by Eq. (7) with a lower energy. If the nth occupied pairing state in the branch $\varepsilon_{\mathbf{k},+}$ is lowered by the pairing energy to touch the edge state energy level $\varepsilon(\mathbf{k}_F) = E_{\text{MF}}$ with E_{MF} the occupation energy for one pair of m-MFs, i.e., $E(n, 0, 0) = E(n - 1, 2_{\text{edge}}, 0) = E(n - 1, 0, 0) + 2E_{\text{MF}}$ in terms of Eqs. (7) and (10), the MFs may emerge as gapless excitations. This condition also shows the degeneracy of occupation and vacuum of MF states, which the non-Abelian statistics originates from. Note that the probability amplitude for the emergence of a pair of m-MFs is proportional to $\langle 0, 2_{\text{edge}}, n - 1|\gamma_{1m}^\dagger \gamma_{2m}^\dagger|n - 1, 2_{\text{edge}}, 0\rangle \neq 0$ for the present system. The above analysis asserts some important results of the MFT [21,22].

In summary, by making a spin-rotation unitary transformation and in the framework of Richardson ansatz, we have found that a class of the electron pairing model with two kinds of spin-orbit interactions is exactly solvable, which is closely relevant to recent research hot spots on topological superconductors and Dirac fermions. More importantly, based on the exact solution, we have rigorously shown that the pairing symmetry is of the $p + ip$ wave regardless of the strength of pairing interactions and the ground state possesses the time-reversal symmetry. Intriguingly, we have also elaborated how Majorana fermions can emerge in the system. Finally, we wish to pinpoint that the present exact solutions for the mentioned pairing systems may shed light on profound understanding of topological superfluids.

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