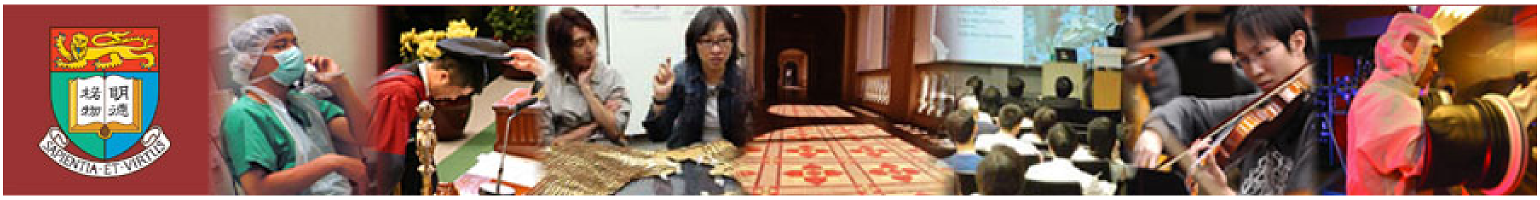

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RECENT PROGRESS ON THE DIRICHLET DIVISOR PROBLEM AND THE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION

Kai-Man Tsang[1]

Abstract

Let $\Delta(x)$ and $E(t)$ denote respectively the remainder terms in the Dirichlet divisor problem and the mean square formula for the Riemann zeta-function on the critical line. This article is a survey of recent developments on the research of these famous error terms in number theory. These include upper bounds, Ω-results, sign changes, moments and distribution, etc. A few open problems will also be discussed.

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1. Introduction

Let $d(n)$ denote the divisor function. The summatory function $D(x) = \sum_{n\le x} d(n)$ occurs in the study of many important problems in number theory, such as the asymptotic behaviour of the Riemann zeta-function. When written as $\sum_{uv\le x} 1$, $D(x)$ counts the number of lattice points (u, v) lying in the first quadrant under the hyperbola $uv = x$. By the so-called hyperbolic method, Dirichlet proved that, for any $x \ge 1$,

$$D(x) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Let

$$\Delta(x) = D(x) - x\log x - (2\gamma - 1)x$$

be the error term in the above asymptotic formula for $D(x)$. Dirichlet's divisor problem consists of determining the smallest α for which

$$\Delta(x) \ll_\varepsilon x^{\alpha+\varepsilon}$$

holds for any $\varepsilon > 0$. Thus, Dirichlet showed that $\alpha \le 1/2$, and throughout the past one and half century, there is a continual stream of improvement on this estimate. For instance, Voronoi [38] has proved by complicated analytic method that $\alpha \le 1/3$. The best estimate to-date is $\alpha \le 131/416$, due to Huxley [13], [14]. It is widely conjectured that $\alpha = 1/4$ is admissible, which is then the best possible (see §3 below).

Apart from getting improved upper bounds for $\Delta(x)$, there are, especially in the last three decades, many papers in the literature which study other interesting properties of $\Delta(x)$, including Ω-results, sign changes, moments, value distribution, etc. In this article, we will survey on the recent developments of these interesting research on $\Delta(x)$.

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[1]Dedicated to Professor Yuan Wang on the occasion of his eightieth birthday

The original publication is available at www.scichina.com and www.springerlink.com

2. Mean square of the Riemann zeta-function

Throughout this paper, the symbols c_j, c_j' etc. denote certain effective positive constants; c and ε denote respectively a generic positive constant and an arbitrarily small positive number, which may not be the same at each occurrence.

There is, so far, no simple closed expression for $\Delta(x)$ is known. At the beginning of the last century, Voronoi [38] proved the remarkable formula that

$$\Delta(x) = -\frac{2}{\pi}\sqrt{x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})\right), \tag{2.1}$$

where K_1, Y_1 are the Bessel functions, and the series on the right hand side is boundedly convergent for x lying in each fixed closed interval. Voronoi then showed that $\Delta(x) \ll x^{1/3+\varepsilon}$ holds for any $\varepsilon > 0$. Although (2.1) is an exact formula, in many applications the following truncated formula is more convenient (see [31] Chapter 12):

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\le N}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{nx} - \frac{\pi}{4}) + O(x^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}}) \quad \text{for} \quad 1 \le N \ll x. \tag{2.2}$$

For instance, by taking $N = x^{1/3}$ and using trivial bounds, this formula yields immediately the bound $\Delta(x) \ll x^{1/3+\varepsilon}$ for any $\varepsilon > 0$. This truncated formula forms the basis of many investigations on $\Delta(x)$.

A central problem in the theory of the Riemann zeta-function concerns the mean values

$$I_k(T) = \int_0^T \left|\zeta(\frac{1}{2} + it)\right|^{2k} dt, \quad k = 1, 2, 3, \ldots$$

of $\zeta(s)$ on the critical line. A long-standing conjecture states that for every integers $k \ge 1$,

$$I_k(T) \sim TP_k(\log T),$$

where P_k is a polynomial of degree k^2. So far asymptotic formula for $I_k(T)$ have been proved only for $k = 1$ and 2. For $k = 1$, Hardy and Littlewood [8] proved in 1918 that

$$I_1(T) = \int_0^T \left|\zeta(\frac{1}{2} + it)\right|^2 dt = T\log\frac{T}{2\pi} + (2\gamma - 1)T + E(T)$$

with $E(T) = o(T\log T)$. The error term $E(T)$ here is of immense interest and various authors have contributed to its estimation. In particular, the bounds $\sqrt{T}\log T$, $T^{5/12}\log^2 T$ and $T^{346/1067+\varepsilon}$ have been obtained by Ingham [15], Titchmarsh [30] and Balasubramanian [2] respectively, and the best bound to-date is $T^{131/416+\varepsilon}$, obtained by Huxley [13], [14]. By the following inequality of Heath-Brown [10]

$$\begin{aligned}
\zeta^2(\frac{1}{2} + it) &\ll (\log t)\int_{t-\log^2 t}^{t+\log^2 t}\left|\zeta(\frac{1}{2} + iu)\right|^2 du + \log t \\
&\ll (\log t)(I_1(t + \log^2 t) - I_1(t - \log^2 t)) + \log t \\
&\ll \log^4 t + (\log t)(E(t + \log^2 t) - E(t - \log^2 t)),
\end{aligned}$$

we see that a bound of the form $E(t) \ll t^{\beta}$ implies

$$\zeta(\frac{1}{2}+it) \ll t^{\beta/2}\sqrt{\log t}.$$

An important step forward on the research of $E(T)$ was made by Atkinson [1] in 1949, who proved for large T and $N \asymp T$ that

$$\begin{aligned} E(T) &= \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n\le N} (-1)^n \frac{d(n)}{n^{3/4}}\ e(T,n)\cos(f(T,n)) \\ &- 2\sum_{n\le N'} \frac{d(n)}{\sqrt{n}} \left(\log\frac{T}{2\pi n}\right)^{-1} \cos\left(T\log\frac{T}{2\pi n} - T + \frac{\pi}{4}\right) + O_\varepsilon(T^\varepsilon) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} e(T,n) &= \left(1+\frac{\pi n}{2T}\right)^{-1/4} \left\{\left(\frac{2T}{\pi n}\right)^{1/2} arsinh\left(\frac{\pi n}{2T}\right)^{1/2}\right\}^{-1}, \\ f(T,n) &= 2T arsinh\left(\frac{\pi n}{2T}\right)^{1/2} + (\pi^2 n^2 + 2\pi n T)^{1/2} - \frac{\pi}{4} \end{aligned}$$

and $N' = \frac{T}{2\pi} + \frac{N}{2} - \sqrt{(\frac{T}{2\pi}+\frac{N}{2})^2 - (\frac{T}{2\pi})^2}$. However this remarkable formula has been neglected for nearly thirty years until Heath-Brown [9] used it to prove the mean square formula

$$\int_0^T E(t)^2\, dt = c_2' T^{3/2} + O(T^{5/4}\log^2 T).$$

This brought to light the significance of Atkinson's formula, which then becomes the starting point of many of the investigations on $E(T)$.

By first order approximations, when $n = o(T)$ we have $e(T,n) \approx 1$ and $f(T,n) \approx 2\sqrt{2\pi nT} - \pi/4$. Thus the first sum for $E(T)$ on the right hand side of (2.3) is

$$\approx \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n\le N} (-1)^n \frac{d(n)}{n^{3/4}} \cos(2\sqrt{2\pi nT} - \frac{\pi}{4}).$$

Apart from the oscillating factor $(-1)^n$, this is the same as the Voronoi formula for $2\pi\Delta(\frac{T}{2\pi})$. Hence many of the methods used in the study of $\Delta(x)$ apply also to $E(T)$. As we shall see in the following sections, $\Delta(x)$ and $E(T)$ exhibit similar statistical behaviour.

The above similarity between the Voronoi series and the Atkinson formula led Jutila [21] to observe deeper connection between $\Delta(x)$ and $E(T)$. Jutila noted that, if

$$\Delta^*(x) = \sum_{n\le 4x} \frac{(-1)^n}{2} d(n) - x\log x - (2\gamma-1)x$$

then

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x), \tag{2.4}$$

and hence, by (2.2),

$$\Delta^*(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\le N} (-1)^n \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \frac{\pi}{4}) + O(x^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{2}}) \quad \text{for} \quad 1 \le N \ll x. \tag{2.5}$$

Thus $\Delta^*(x)$ instead of $\Delta(x)$ is the more appropriate analogue of $E(T)$.

It is clear from (2.4) that

$$\Delta(x) \ll x^\alpha \Rightarrow \Delta^*(x) \ll x^\alpha. \tag{2.6}$$

In [23], Lau and the author succeeded in inverting the recurrence (2.4), and hence showed the non-trivial result:

$$\Delta^*(x) \ll x^\beta \Rightarrow \Delta(x) \ll x^\beta. \tag{2.7}$$

Thus, $\Delta(x)$ and $\Delta^*(x)$ have the same upper bound order. From the very interesting Main Theorem proved in [21] and in view of (2.6), Jutila deduced that

$$\Delta(x) \ll x^\alpha \Rightarrow E(T) \ll T^{(1+2\alpha)/5} (\log T)^{12/5}. \tag{2.8}$$

In particular, the hypothetical best possible bound $\Delta(x) \ll x^{\frac{1}{4}+\varepsilon}$ implies $E(T) \ll T^{\frac{3}{10}+\varepsilon}$. The argument in the proof of (2.8) indeed shows also

$$E(T) \ll T^\alpha \Rightarrow \Delta^*(x) \ll x^{(1+2\alpha)/5} (\log x)^{12/5}.$$

Hence, in view of (2.7), we have

$$E(T) \ll T^{\frac{1}{4}+\varepsilon} \Rightarrow \Delta(x) \ll x^{\frac{3}{10}+\varepsilon}.$$

However, one expects that the best α satisfying $\Delta(x) \ll x^\alpha$ for all $x \ge 1$ should be the same as the best β for which $E(T) \ll T^\beta$ holds for all $T \ge 1$.

3. Ω-Results

The first Ω-results for $\Delta(x)$ were obtained by Hardy [7] in 1916, who showed that

$$\Delta(x) = \Omega_+ \left((x \log x)^{\frac{1}{4}} \log_2 x \right) \quad \text{and} \quad \Delta(x) = \Omega_-(x^{\frac{1}{4}}).$$

Here $\log_j = \log_{j-1}(\log)$ for $j = 2, 3, \ldots$. The Ω_--results was later improved by Corrádi and Kátai [3] to

$$\Delta(x) = \Omega_- \left(x^{\frac{1}{4}} \exp(c (\log_2 x)^{\frac{1}{4}} (\log_3 x)^{-\frac{3}{4}}) \right),$$

for some positive constant c. This is a consequence of a quantitative version of Kronecker's theorem. More than sixty years have lapsed before an improved Ω_+-result for $\Delta(x)$ appeared. Hafner [5] proved in 1981 that

$$\Delta(x) = \Omega_+ \left((x \log x)^{\frac{1}{4}} (\log_2 x)^\alpha \exp(-c\sqrt{\log_3 x}) \right),$$

where $\alpha = \frac{1}{4}(3 + 2\log 2) = 1.0965\ldots$ and $c > 0$ is a constant. In fact, this same result was also obtained independently (but unpublished) by A. Selberg. Their novel idea is as follows. First , let

$K(u) = (\pi u)^{-2}\sin^2(\pi u)$ be the Fejer kernel, and let $\Delta_1(x) = \pi\sqrt{\frac{2}{x}}\Delta(x^2)$. By using the truncated Voronoi formula (2.2), we easily show that for $1 \le \tau \le x^{1/7}$,

$$\int_{-\tau}^{\tau} \Delta_1(x + u\tau^{-1})K(u)\,du = \sum_{n\le(\tau/2)^2} \frac{d(n)}{n^{3/4}}\left(1 - \frac{2\sqrt{n}}{\tau}\right)\cos(4\pi\sqrt{n}x - \frac{\pi}{4}) + O(\log\tau).$$

Denote the sum on the right hand side by $S_\tau(x)$. Then clearly

$$\sup_{x-1\le y\le x+1} \Delta_1(y) \quad \ge S_\tau(x) + O(\log\tau),$$

and one obtains Ω_+-results for $\Delta_1(x)$ by exhibiting a large value of $S_\tau(x)$. By the Dirichlet theorem on simultaneous approximation, there exists $x \le 24^{(\tau/2)^2}$ such that $\|2\sqrt{n}x\| \le 1/24$ for each $n \le (\tau/2)^2$. Hence,

$$S_\tau(x) \ge \frac{1}{2}\sum_{n\le(\tau/2)^2} \frac{d(n)}{n^{3/4}}\left(1 - \frac{2\sqrt{n}}{\tau}\right) \gg \sqrt{\frac{\tau}{2}}\log\tau \gg (\log x)^{\frac{1}{4}}\log\log x,$$

which is the Ω_+-result of Hardy.

Hafner (and Selberg) noticed that the values of $d(n)$ distribute rather unevenly, and he took advantage of this by choosing a (thin) subset M of $[1,(\tau/2)^2]$, corresponding to those n for which $d(n)$ is large. Write $S_\tau(x) = S' + S''$, where S' is the subsum consisting of those terms for which $n \in M$. When Dirichlet's theorem is applied to S' as above we get the lower bound

$$S_\tau(x) \gg \sqrt{\frac{\tau}{2}}\log\tau + O(|S''|). \tag{3.1}$$

By estimating S'' trivially and then optimizing under the constraint that the O-term in (3.1) is dominated by the main term, Hafner obtained his improved Ω_+-result for $\Delta(x)$. Later Hafner and Ivić [6] were able to adapt this idea to $E(t)$ and obtained an Ω_+-result for $E(t)$ of the same quality.

Soundararajan [29] has a further idea. He saw that if the phase angle $\frac{\pi}{4}$ does not exist inside the cosine factors in $S_\tau(x)$, then instead of a trivial estimation, S'' can be made non-negative by summing over a set of values of x lying in some arithmetic progression. In this way, he achieved the even better Ω-result:

$$\Delta(x) = \Omega\left((x\log x)^{\frac{1}{4}}(\log_2 x)^{\frac{3}{4}(2^{4/3}-1)}(\log_3 x)^{-\frac{5}{8}}\right).$$

Here the exponent of $\log_2 x$ is $1.1398\ldots$. Soundararajan's argument does not give Ω_+ or Ω_--result because, in the course of removing the phase angle $\frac{\pi}{4}$, the sign of $\Delta(x)$ is lost. Moreover, his argument depends crucially on the non-negativity of the coefficients in the sum $S_\tau(x)$, and hence it does not apply to $E(t)$ where the oscillating factor $(-1)^n$ is attached to each of its terms. Soundararajan pointed out further that, by modelling the sequence $\{\sqrt{n}\}$, n square-free, as independent random variables, the above Ω-result should be the true maximal order of $\Delta(x)$ up to $(\log_2 x)^{o(1)}$.

Recently, Lau and the author [23] found a way to adapt Soundararajan's idea to obtain large values of $E(t)$. (2.7) shows that any Ω-result for $\Delta(x)$ will automatically be an Ω-result for $\Delta^*(x)$. As mentioned in §2 (see (2.5)), $\Delta^*(x)$ and $E(t)$ are approximated by similar finite sums. Hence, via this indirect route, Ω-result for $\Delta(x)$ is passed onto $E(t)$. Thus it was proved in [23] that

$$E(T) = \Omega\left((T\log T)^{\frac{1}{4}}(\log_2 T)^{\frac{3}{4}(2^{4/3}-1)}(\log_3 T)^{-\frac{5}{8}}\right).$$

4. Sign Changes

The Ω_+ and Ω_--results described in the last section imply $\Delta(x)$ changes signs infinitely often. In particular, $\Delta(x)$ has infinitely many zeros, since $\Delta(x)$ increases (jumps) only at the positive integers and it decreases continuously like $-logx$ elsewhere. One natural way to study the oscillatory behaviour of $\Delta(x)$ is to consider the gaps between its zeros.

In 1955, Tong [32] proved that:

There exist positive constants c and c' such that , for any $X \geq 1$ and any $v \in [-cX^{1/4}, cX^{1/4}]$, there is $x \in [X,\ X+c'\sqrt{X}]$ for which $\Delta(x) = v$. In particular, $\Delta(x)$ changes signs in $[X,\ X+c'\sqrt{X}]$ for all $X \geq 1$, that is, the gap between the zeros of $\Delta(x)$ is $O(\sqrt{x})$.

Indeed, Heath-Brown and the author [12] showed that the above length of gaps are essentially best possible:

Let δ be any sufficiently small positive number. Then for $X \geq X_0(\delta)$, there are at least $\delta\sqrt{X}\log^5 X$ disjoint subintervals of length $\delta\sqrt{X}\log^{-5} X$ in $[X, 2X]$ such that $|\Delta(x)| > \delta x^{\frac{1}{4}}$ whenever x lies in any of these subintervals. In particular $\Delta(x)$ does not change sign in any of these subintervals.

As was pointed out in [36], $|\Delta(x)|$ in the above can be replaced by $\Delta^+(x)$ and $\Delta^-(x)$ respectively (see also [37]). Here for any real-valued function g,

$$g^+ = \frac{1}{2}(|g| + g), \quad g^- = \frac{1}{2}(|g| - g)$$

denote the positive and negative parts of g respectively. As we shall see in the next section, this refinement has implications on the asymptotic formulas for the moments of $\Delta(x)$. We also point out that the results about the gaps between zeros in this section hold verbatim for the error term $E(T)$.

5. Moments of $\Delta(x)$ and $E(T)$

The results described in the last two sections show that $\Delta(x)$ and $E(T)$ exhibit considerable fluctuations. However, these error terms behave rather well on the average, especially over intervals of length $\gg \sqrt{X}$. In particular, their moments have quite simple asymptotic formulas.

It is already contained in Voronoi's work [38] that

$$\int_0^X \Delta(x)\,dx = \frac{X}{4} + \frac{X^{3/4}}{2\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{5/4}}\sin(4\pi\sqrt{nX} - \frac{\pi}{4}) + O(1) = \frac{X}{4} + O(X^{3/4}). \tag{5.1}$$

Cramér [4] later took a step further and proved the mean square formula

$$\int_0^X \Delta(x)^2\,dx = c_2 X^{3/2} + O_\varepsilon(X^{5/4+\varepsilon}). \tag{5.2}$$

An immediate consequence of this is that $\Delta(x) = \Omega(x^{1/4})$. However, it is easy to see, by constructing a suitable example, that (5.1) and (5.2) together is still not enough to deduce that $\Delta(x) = \Omega_+(x^{1/4})$ or $\Delta(x) = \Omega_-(x^{1/4})$.

As we have already mentioned in §2, the corresponding mean square formular for $E(t)$ appeared much later. Heath-Brown [9] applied Atkinson's formula to prove

$$\int_0^T E(t)^2\,dt = c_2' T^{3/2} + O_\varepsilon(T^{5/4+\varepsilon}).$$

It is not difficult to see that the following two statements are equivalent;

(i) $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$,

(ii) $\int_0^X |\Delta(x)|^A\,dx \ll_{A,\varepsilon} X^{1+\frac{A}{4}+\varepsilon}$ for all $A \geq 1$.

Not much was known about higher power moments of these error terms until Ivić [16] used the method of large values to prove, for $g = \Delta$ or E, that

$$\int_0^X |g(x)|^A\,dx \ll_\varepsilon X^{1+\frac{A}{4}+\varepsilon} \quad \text{for} \quad 1 \leq A \leq 35/4. \tag{5.3}$$

The range of A was later extended to 28/3 by Heath-Brown [11].

By (5.2) and Hölder's inequality, we find that

$$\int_0^X |\Delta(x)|^A\,dx \gg X^{1+\frac{A}{4}} \quad \text{for all} \quad A \geq 2.$$

Hence apart from the ε power of X, the upper bound (5.3) is best possible.

The author [34] obtained in 1990 the asymptotic formula

$$\int_0^X \Delta(x)^k\,dx = c_k X^{1+\frac{k}{4}} + O(X^{1+\frac{k}{4}-\delta_k}) \tag{5.4}$$

for $k = 3, 4$, with $\delta_3 = 1/14$, $\delta_4 = 1/23$. The asymptotic formula for $k = 3$ is particularly interesting. It shows that the values of $\Delta(x)$ have a substantial bias towards the positive side. If $\Delta(x)$ has a distribution function, it must be asymmetric.

By Hölder's inequality, we further deduce from (5.2) and (5.4) that

$$X^{3/2} \ll \int_0^X \Delta(x)^2\,dx \leq \left(\int_0^X |\Delta(x)|\,dx\right)^{2/3} \left(\int_0^X \Delta(x)^4\,dx\right)^{1/3} \ll \left(\int_0^X |\Delta(x)|\,dx\right)^{2/3} X^{2/3},$$

and hence

$$\int_0^X |\Delta(x)|\,dx \gg X^{5/4}.$$

This together with (5.1) shows that

$$\int_0^X \Delta^{\pm}(x)\,dx \gg X^{5/4}.$$

In particular, we have $\Delta(x) = \Omega_\pm(x^{1/4})$. Same results hold for $E(t)$.

These investigations sparked off a series of study on the higher-power moments of $\Delta(x)$ and $E(T)$. Among other things, Heath-Brown proved in [11] that for any integer $k \le 9$, the limit

$$c_k = \lim_{X\to\infty} X^{-1-\frac{k}{4}} \int_0^X \Delta(x)^k \, dx$$

exists. In fact, such limit exists for any $k < K$ where K satisfies

$$\int_0^X |\Delta(x)|^K \, dx \ll X^{1+\frac{K}{4}+\varepsilon}.$$

Same holds with $E(T)$ in place of $\Delta(x)$.

It should be pointed out that the above result of Heath-Brown does not give the asymptotic formula for the k-th moment with an explicit error term, and also the above limit is not necessarily non-zero. In fact, from (5.1) we have $c_1 = 0$.

Initiated by the work of the author in [34] and Heath-Brown's paper [11], there followed a series of investigation on explicit asymptotic formula of the type (5.4) for larger values of k. At the present moment, our available techniques only allow us to handle k up to 9 (see Zhai [39] and Ivić and Sargos [20]).

There are also considerable interests on the size of the error term $F_k(X)$ in the asymptotic formula for the higher power moments:

$$\int_0^X \Delta(x)^k \, dx = c_k X^{1+\frac{k}{4}} + F_k(X).$$

Ivić (see [18], Theorem 3.8) observed that

$$F_2(X) \ll V(X) \Rightarrow \Delta(x) \ll (V(x)\log x)^{1/3}.$$

Thus, from the Ω_+-result of $\Delta(x)$, Ivić and Ouellet [19] deduced that

$$F_2(X) = \Omega(X^{\frac{3}{4}}(\log X)^{-\frac{1}{4}}(\log_2 X)^{\frac{3}{4}(3+\log 4)}\exp(-c\sqrt{\log_3 X})).$$

In view of this, they conjectured that for every $k \ge 2$,

$$F_k(X) \ll_\varepsilon X^{\frac{1}{4}(k+1)+\varepsilon} \quad \textit{for all} \quad \varepsilon > 0.$$

This is a very strong conjecture, whose truth would imply the main conjecture $\Delta(x) \ll x^{1/4+\varepsilon}$. But, at least for odd k, this conjecture is false. To see this, we apply our theorem in §4 that for some interval $J = [X,\ X+U]$ with $U = \delta\sqrt{X}\log^{-5} X$, we have $\Delta(x) \le -cX^{1/4}$ for all $x \in J$. Then

$$\int_J \Delta(x)^k \, dx \le (-cX^{1/4})^k U,$$

and hence

$$F_k(X+U) - F_k(X) \le -cUX^{\frac{k}{4}}.$$

Thus, $F_k(X) = \Omega(X^{\frac{k}{4}+\frac{1}{2}}\log^{-5} X)$.

The above argument does not work for even k. But for $k = 2$, Lau and the author [22] showed that

$$\int_0^Z F_2(X)\, dX = -(8\pi^2)^{-1} Z^2 \log^2 Z + cZ^2 \log Z + O(Z^2), \tag{5.5}$$

and whence

$$F_2(X) = \Omega_-(X \log^2 X). \tag{5.6}$$

Clearly (5.5) can be reformulated as

$$\int_0^Z (F_2(X) + (4\pi^2)^{-1} X \log^2 X - cX \log X)\, dX \ll Z^2.$$

In [35], the following stronger result was proved:

For any $r \geq 1$, the estimate

$$\int_0^Z |F_2(X) + (4\pi^2)^{-1} X \log^2 X - cX \log X|^r \, dX \ll (cr)^{4r} Z^{r+1}$$

holds. Consequently, if $G(X)$ is any increasing function satisfying $2 \leq G(X) \leq \log^4 X$, we have

$$|F_2(X) + (4\pi^2)^{-1} X \log^2 X - cX \log X| \leq XG(X)$$

for all but $O(Ze^{-cG(Z)^{1/4}})$ values of X in $[2, Z]$.

In general, we conjecture that asymptotic formula of the form (5.4) exists, and the best possible O-term there is $O(X^{1+\frac{k}{4}-\frac{1}{2}+\varepsilon})$. Same is conjectured for $E(t)$.

In the opposite direction, Tong [33] has improved the mean square formula (5.2) to

$$\int_0^X \Delta(x)^2 \, dx = c_2 X^{3/2} + O(X \log^5 X) \tag{5.7}$$

Thus, in view of (5.6) the error term here is just larger than its true order of magnitude by a small power of $\log X$. It was therefore quite an interesting development when Preissmann [28] was able to further reduce $F_2(X)$ to $O(X \log^4 X)$. In fact, Preissmann treated in detail the corresponding problem for the circle problem, that is, the error term $P(x) := \sum_{n \leq x} r(n) - \pi x$, where $r(n)$ denotes the number of representations of n as the sum of two integer squares. But since $P(x)$ has representation similar to the Voronoi formula for $\Delta(x)$ and the Atkinson formula for $E(T)$, his method applies to $\Delta(x)$ and $E(T)$ as well.

To obtain upper bound for $F_2(X)$ of such precision, the truncated Voronoi formula (2.2) is not sufficient and we need the following refined Voronoi formula proved by Meurman [27]:

For x large and $N \gg x$,

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \frac{\pi}{4}) + R_N(x), \tag{5.8}$$

where $R_N(x) \ll x^{-1/4}$ if $\|x\| \gg x^{5/2} N^{-1/2}$ and $R_N(x) \ll x^{\varepsilon}$ otherwise. The merit of this formula is that the remainder term $R_N(x)$ is $\ll x^{-1/4}$ for most of the time, and this is just enough for our purpose.

Take $N = X^7$ in (5.8). On squaring the sum on the right hand side and then integrating term by term, the diagonal terms together yield the main term in (5.7). A little calculation shows that

$$F_2(X) \ll X \left| \sum_{\substack{m,n \leq X^7 \\ m \neq n}} \frac{d(m) m^{-\frac{3}{4}} d(n) n^{-\frac{3}{4}}}{\sqrt{n} - \sqrt{m}} \, e^{4\pi i (\sqrt{n} - \sqrt{m}) X} \right|. \tag{5.9}$$

To bound the double sum on the right hand side, Preissmann applied Hilbert's inequality in the following form: *Let* $\{a_n\}$, $\{b_n\}$ *be two sequences of complex numbers. Suppose* λ_n *is a strictly increasing sequence of positive real numbers. Then*

$$\sum_{m\neq n} \frac{a_n\overline{b_m}}{\lambda_n-\lambda_m} \ll \left(\sum_n |a_n|^2\delta_n^{-1}\right)^{\frac{1}{2}} \left(\sum_n |b_n|^2\delta_n^{-1}\right)^{\frac{1}{2}} \tag{5.10}$$

where $\delta_n = \min_{m\neq n}|\lambda_n-\lambda_m|$. In our case, $a_n = b_n = d(n)n^{-\frac{3}{4}}\, e^{4\pi i\sqrt{n}X}$ and $\lambda_n = \sqrt{n}$, and the best lower bound for δ_n is $\gg 1/\sqrt{n}$. Thus, Preissmann deduced that

$$F_2(X) \ll X(\sum_n d(n)^2 n^{-1}).$$

The last sum, by the well-known estimate: $\sum_{n\leq y} d(n)^2 \ll y\log^3 y$, is easily seen to be $\ll \log^4 X$ and this leads to the result of Preissmann. Recently in [24], the bound for $F_2(X)$ is further reduced to

$$F_2(X) \ll X\log^3 X\log\log X. \tag{5.11}$$

This and (5.6) together shows that we are now within only one power of $\log X$ from the true order. Lau and the author had earlier obtained the improved bound $F_2(X) \ll X\log^{7/2} X(\log\log X)^{5/2}$ by a different idea. The anonymous referee of our paper, however, saw a better way of applying Hilbert's inequality and hence resulted in the above better bound. His interesting idea is contained in the following.

Lemma *Suppose* a_n *are complex numbers for* $N < n \leq 2N$. *Let* $K \leq N$ *be a positive integer. Then*

$$\sum_{N<m\neq n\leq 2N} \frac{a_m\overline{a_n}}{\sqrt{m}-\sqrt{n}} \ll S_1+S_2+S_3$$

$$S_1 = N^{3/2}K^{-1}\sum_{h\geq N/K} h^{-2}\sum_{N<n\leq 2N-h} |a_na_{n+h}|,$$

$$S_2 = N^{1/2}\sum_{h\leq N/K} h^{-1}\sum_{N<n\leq 2N-h} |a_na_{n+h}|$$

and

$$S_3 = KN^{-1/2}\sum_{N<n\leq 2N} |a_n|^2.$$

To prove this lemma we divide $(N, 2N]$ into K disjoint intervals

$$I = \left(\frac{k-1}{K}N, \frac{k}{K}N\right], \quad K < k \leq 2K.$$

We write $g(n)$ for the lower end-point of the interval I for which $n \in I$, so that $g(n) < n \le g(n) + N/K$. We claim that

$$\sum_{N<m\ne n\le 2N} \frac{a_m\overline{a_n}}{\sqrt{m}-\sqrt{n}} = \sum_{\substack{N<m,n\le 2N \\ g(m)\ne g(n)}} \frac{a_m\overline{a_n}}{\sqrt{g(m)}-\sqrt{g(n)}} + O(S_1) + O(S_2). \tag{5.12}$$

We first consider terms with $m \ge n + N/K$, so that $g(m) \ne g(n)$. Then $0 < n - g(n) \le N/K$, whence $0 < \sqrt{n} - \sqrt{g(n)} \ll \sqrt{N}/K$, and similarly for m. We also have $\sqrt{m}-\sqrt{n} \gg (m-n)/\sqrt{N}$. Moreover if $m - n \ge 2N/K$ then

$$g(m) - g(n) \ge m - N/K - n \ge (m-n)/2.$$

On the other hand, if $N/K \le m - n < 2N/K$ we have

$$g(m) - g(n) \ge N/K > (m-n)/2,$$

since $g(m) \ne g(n)$, as remarked above. In either case we conclude that

$$\sqrt{g(m)} - \sqrt{g(n)} \gg (g(m)-g(n))/\sqrt{N} \ge (m-n)/\sqrt{N}.$$

It follows that

$$\frac{1}{\sqrt{m}-\sqrt{n}} = \frac{1}{\sqrt{g(m)}-\sqrt{g(n)}} + O\left(\frac{N^{3/2}}{K(m-n)^2}\right)$$

when $m \ge n + N/K$. Thus these O-terms contribute $O(S_1)$ to (5.12).

Secondly we consider terms in which $n < m < n+N/K$. Here we have $\sqrt{m}-\sqrt{n} \gg (m-n)/\sqrt{N}$. Moreover if $g(m) > g(n)$ we have $g(m) - g(n) \ge N/K$, whence $\sqrt{g(m)} - \sqrt{g(n)} \gg \sqrt{N}/K \gg (m-n)/\sqrt{N}$. It follows that the terms under consideration contribute $O(S_2)$ to (5.12), which establishes the claim.

We now group terms n from a particular interval I, all of which have the same lower end-point $g(n)$. Thus, letting $I_1, \cdots, I_J$ be the sequence of intervals I we set

$$A(j) = \sum_{n\in I_j} a_n$$

and we write g_j for the lower end-point of I_j. Thus

$$\sum_{\substack{N<m,n\le 2N \\ g(m)\ne g(n)}} \frac{a_m\overline{a_n}}{\sqrt{g(m)}-\sqrt{g(n)}} = \sum_{j\ne k} \frac{A(j)\overline{A(k)}}{\sqrt{g_j}-\sqrt{g_k}}.$$

Moreover for $j > k$ we have $\sqrt{g_j} - \sqrt{g_k} \gg (g_j - g_k)/\sqrt{N} \gg \sqrt{N}/K$. We can now apply Hilbert's inequality as in (5.10) with $a_n = b_n$ to give a bound

$$\ll KN^{-1/2} \sum_{j\le J} |A(j)|^2.$$

When we expand $|A(j)|^2$ we get terms $a_n\overline{a_{n+h}}$ with $n, n+h \in I_j$. Thus $|h| \le N/K$. When $h \ne 0$ we have $KN^{-1/2} \le N^{1/2}|h|^{-1}$, so that the overall contribution is $O(S_2)$. Finally, terms with $h = 0$ produce a contribution $O(S_3)$, which completes the proof of the Lemma.

We proceed now to prove the bound in (5.11) by proving that the double sum on the right hand side of (5.9) is $\ll \log^3 X \log\log X$, that is,

$$\left| \sum_{\substack{m,n \le X^7 \\ m \ne n}} \frac{d(m)m^{-\frac{3}{4}}\, d(n)n^{-\frac{3}{4}}}{\sqrt{n}-\sqrt{m}}\, e^{4\pi i(\sqrt{n}-\sqrt{m})X} \right| \ll \log^3 X \log\log X. \tag{5.13}$$

Let $N_j = 2^{j/2}$ and decompose the above sum into $O((\log X)^2)$ subsums

$$S_{i,j} = \sum_{\substack{N_i < m \le \sqrt{2}N_i \\ N_j < n \le \sqrt{2}N_j}} \frac{a_m \overline{a_n}}{\sqrt{m}-\sqrt{n}},$$

where $a_n = d(n)\, n^{-3/4} e^{4\pi i\sqrt{n}X}$ and it is understood that $m \ne n$ if $i = j$. When $i \ge j+2$ we have $\sqrt{m}-\sqrt{n} \gg \sqrt{N_i}$, and a trivial bound yields $S_{i,j} \ll N_{i-j}^{-1/4}(\log X)^2$. Thus the total contribution from all such sums $S_{i,j}$ is $O((\log X)^3)$. Sums with $j \ge i+2$ may be handled similarly. We also have

$$S_{i,i+1} + S_{i+1,i} = -S_{i,i} - S_{i+1,i+1} + \sum_{N_i < m \ne n \le 2N_i} \frac{a_m \overline{a_n}}{\sqrt{m}-\sqrt{n}}.$$

The sum on the right may be bounded by our Lemma, as can $S_{i,i}$ and $S_{i+1,i+1}$. We take $K = [N_i/(1+\log N_i)]$ and find that

$$\sum_{N < n \le 2N-h} |a_n a_{n+h}| \ll \sum_{N < n \le 2N-h} d(n)\, d(n+h) n^{-3/2} \ll N^{-1/2}(\log X)^2 \sum_{d|h} d^{-1},$$

whence $S_1 \ll (\log X)^2$, $S_2 \ll (\log X)^2 \log\log X$ and $S_3 \ll (\log X)^2$. It follows that

$$S_{i,i} \ll (\log X)^2 \log\log X \quad \text{and} \quad S_{i,i+1} + S_{i+1,i} \ll (\log X)^2 \log\log X.$$

Thus on summing over $i \ll \log X$ we obtain (5.13).

Closer scrutiny of the proof reveals that the main saving comes from bounding sums of the form $\sum_{n \le N} d(n)\, d(n+h)$ rather than $\sum_{n \le N} d^2(n)$.

The same method works for $E(t)$ as well, except that we need to use the refined Atkinson formula as given by Meurman in [27]. However, it is interesting to note that, Lee and the author [26] have proved very recently that if

$$H(T) = \int_0^T E(t)^2\, dt - c_2' T^{3/2}$$

denotes the error term in the mean square formula for $E(t)$, then

$$\int_0^Z H(T)\, dT = \frac{-3}{\pi^2} Z^2 \log^2 Z \log\log Z + O(Z^2 \log^2 Z).$$

In particular $H(T) = \Omega_-(T \log^2 T \log\log T)$. This is different from (5.5), and is the first result in the literature that exhibits a fundamental difference between $\Delta(x)$ and $E(t)$. A related result that reveals the discrepancy between $\Delta(x)$ and $E(t)$ is the mean square

$$\int_0^T \left(E(t) - 2\pi\Delta^*(\frac{t}{2\pi}) \right)^2 dt = T^{\frac{4}{3}} P(\log T) + O_\varepsilon(T^{\frac{7}{6}+\varepsilon})$$

proved by Ivić [17]. Here P is a certain cubic polynomial.

6. Distributions

It is well-known that asymptotic estimates for higher power moments can give information on the value distribution. It was therefore of great interest when Heath-Brown [11] proved the existence of distribution function for $x^{-1/4}\Delta(x)$ (and also for $t^{-1/4}E(t)$) without using its higher power moments. More precisely, he proved the following ([11], Theorem 1)

There exists a function $f(\alpha)$ such that for any interval I, we have

$$X^{-1} meas\{x \in [1,\ X] : x^{-1/4}\Delta(x) \in I\} \to \int_I f(\alpha)\, d\alpha$$

as $X \to \infty$. The function $f(\alpha)$ and its derivatives satisfy the growth condition

$$\frac{d^k}{d\alpha^k} f(\alpha) \ll_{A,k} (1+|\alpha|)^{-A} \quad \text{for} \quad k = 0, 1, 2, \ldots \quad \text{and any} \quad A > 0.$$

Moreover, $f(\alpha)$ extends to an entire function on $\mathbb{C}$.

The main idea of his proof is based on the fact that $x^{-1/4}\Delta(x)$ can be closely approximated in the mean by a very short initial section of its Voronoi series. The whole Voronoi series is too long for high power moments to be handled satisfactorily, but a very short initial section can be. In general, his method applies to any function $F(x)$ that can be approximated by an oscillating series in the following sense:

$$\lim_{N\to\infty} \limsup_{X\to\infty} \frac{1}{X}\int_0^X \min\{1, |F(x) - \sum_{n\le N} a_n(\gamma_n x)|\}\, dx = 0,$$

where $a_1(x), a_2(x), \ldots$ etc. are continuous real-valued functions of period 1 and $\gamma_1, \gamma_2, \ldots$ are non-zero constants.

From our earlier result (5.4) that

$$c_3 = \lim_{X\to\infty} X^{-7/4} \int_0^X \Delta(x)^3\, dx > 0,$$

we know the above distribution function $f(\alpha)$ for $x^{-1/4}\Delta(x)$ is biased towards the positive. The same phenomenon holds for $t^{-1/4}E(t)$.

7. Some Open Problems

An intuitive explanation for the phenomenon that the distribution function of $x^{-1/4}\Delta(x)$ skews towards the positive side is that, while $\Delta(x)$ decreases steadily like $-\log x$, it jumps at each positive integer with a magnitude which can be as large as $x^{c/\log\log x}$. Therefore, an aggregation of large jumps over a short interval may result in exceptionally large value of $\Delta(x)$. This is also consistent with the much larger Ω_+-result we have obtained so far for $\Delta(x)$. However this explanation does not seem to apply to $E(t)$.

At the present moment, we have asymptotic formula for the k-th moments of $\Delta(x)$ and $E(t)$ for k up to 9, and as shown by Zhai [39], the odd moments (namely for $k = 3, 5, 7, 9$) all have positive leading terms. Zhai further pointed out that the existence of all the moments is equivalent to the main conjecture $\Delta(x) \ll x^{1/4+\varepsilon}$. Thus , it might be very difficult to obtain asymptotic formulas for all the high power moments in the near future.

Problem 1 Is it true that all the odd moments of $\Delta(x)$ are positive? One may also try to investigate this under the hypothetical bound $\Delta(x) \ll x^{1/4+\varepsilon}$.

It may be of interest to note that the positivity of the k-th moments for $k = 3, 5, 7, 9$ cannot imply the same for higher k (see [25] for a discussion on this).

So far we have a number of statistical results for $\Delta(x)$ and $E(t)$, all of which show that these error terms have "normal" sizes around $x^{1/4}$ and $t^{1/4}$ respectively. It is therefore of interest to look for exceptional values. These functions do have zeros, and since they change rather slowly, they will remain small in size over short intervals around the zeros. An interesting question that worths investigation is the following.

Problem 2 Do there exist intervals $I = [X,\ X + X^{\beta}]$ with $\beta > 1/4$ such that

$$\int_I |\Delta(x)|\, dx \ll X^{\beta+\frac{1}{4}-\delta}$$

for some small positive δ?

In conclusion, these classical error terms in number theory not only provide many interesting questions for further investigations, it is also a good testing field for novel ideas and techniques researchers have developed in connection with other problems in number theory.

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