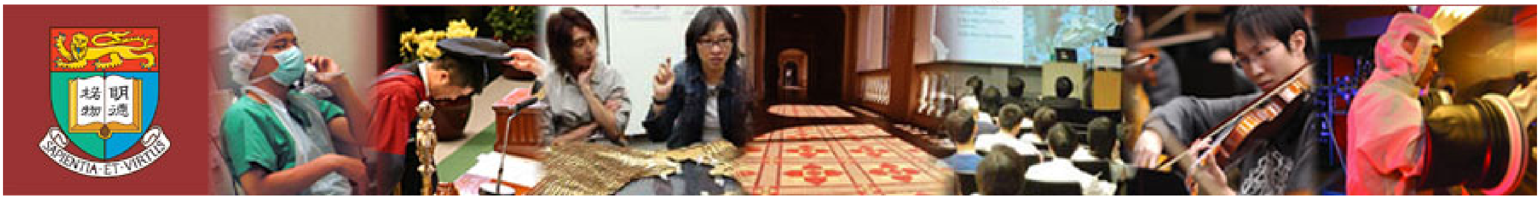

# A Simple and Optimum Geometric Decoding Algorithm for MIMO Systems

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## Abstract

*Geometric decoding (GD) is a newly proposed decoding technique for multiple-input multiple-output (MIMO) transmission over the fading channels. With a complete search on all symbol vectors in the lattice structure, GD requires about the same decoding complexity to achieve the same optimum block-error rates (BLERs) as that of ML decoding. In this paper, we propose a simple implementation of GD for optimum decoding of MIMO transmission. The GD decoder uses the channel matrix to construct a hyper paraboloid and the zero forcing solution to obtain a hyper ellipsoid projected from the hyper paraboloid. It then restricts the search among the symbol vectors within the hyper ellipsoid. Computer simulation studies on* $2\times2$, $3\times3$ *and* $4\times4$ *MIMO systems transmitting 8PAM and 16QAM show that the proposed GD algorithm can achieve the same BLERs as those of the ML decoders, yet having complexity reduction of more than 90%.*

## 1. Introduction

With the ever increasing demand on higher data rates for wireless communications systems, multiple-input multiple-output (MIMO) transmission has attracted much attention [1] due to its potential of significant increase of channel capacity without additional transmitted power or bandwidth. Theoretically, the channel capacity and error-rate performances of MIMO systems could be improved by increasing the numbers of transmit and receive antennas and sizes of constellation used for modulation [2, 3]. However, to realize these advantages, Maximum-likelihood (ML) decoding employing an exhaustive search algorithm is needed to be used at the receiver. The complexities of ML decoding increases exponentially with the numbers of transmit and receiver antennas and the sizes of constellation used in the MIMO systems [4]. For large numbers of antennas with large constellation sizes, the decoders become impractical to implement.

Different low-complexity decoding algorithms such as zero forcing (ZF), minimum-mean-square error (MMSE) and decision feedback equalization have been proposed for use in MIMO systems [5, 6]. These decoding algorithms are not optimum in terms of minimizing the block-error rate (BLER) and so have degradations in BLER performances. Sphere decoding (SD) has been introduced to significantly reduce the average decoding complexity [7], yet achieving the optimum performance as ML decoding. Both ML and SD decoding are search-based algorithms. In ML decoding, the search is done for the whole lattice structure. While in SD, the search is restricted to a hyper sphere inside the lattice structure, where the ML solution is likely to be inside the sphere.

Using a geometric approach to decode the signals for the MIMO channels was first introduced in [8] and the technique was called Geometric decoding (GD) in [9] and is also a search-based algorithm. By using the channel matrix, GD constructs an ellipsoid paraboloid where all the transmit symbol vectors lies on. Theoretically, GD could achieve the minimum BLER performance as ML decoding by searching all the symbol vectors, but then the complexity would be similar to that of ML decoding. In [9], a simplified GD algorithm was proposed for the cubic structure of pulse amplitude modulation (PAM) and quadrature amplitude modulation (QAM). The idea was to consider only the largest one or two eigenvalues with the corresponding eigenvectors to restrict the search among symbol vectors to a smaller subset. Although it significantly reduced the decoding complexity, since not all symbol vectors were considered, it was not an optimum GD.

In this paper, we propose a simple but optimum GD algorithm implementation in terms of minimizing the BLER for use in the MIMO systems over fading channels. The decoder constructs an elliptic paraboloid using the channel matrix and then restricts the search among the symbol vectors to a small region inside a hyper ellipsoid projected from the elliptic paraboloid which is obtained using the zero forcing solution. A hyper rectangle enclosing the hyper ellipsoid is then used to search the exact symbol vectors within the

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hyper ellipsoid. Monte Carlo computer simulation has been used to assess the complexity reductions of our proposed GD and BLERs of $2\times2$, $3\times3$ and $4\times4$ MIMO systems transmitting 8PAM and 16QAM over a block fading channel. Results show that our proposed GD is an optimum decoder and can achieve complexity reductions of more than 90% over ML decoding.

## 2. MIMO System Model

Consider a symbol synchronized and uncoded MIMO system with $N$ transmit and $N$ receive antennas over a fading channel. The received signal matrix is given by

$$\mathbf{r} = \mathbf{Hx} + \mathbf{w} \tag{1}$$

where $\mathbf{r} = [r_1, r_2, ..., r_N]^T$ is the $N$-dimensional received signal vector, $\mathbf{x} = [x_1, x_2, ..., x_N]^T$ a symbol vector in the transmit lattice, with $[]^T$ denoting vector or matrix transposition, $\mathbf{H}$ is the channel matrix, with elements $h_{ij}$ representing the transfer function from the $j$-th transmit antenna to $i$-th receive antenna, and $\mathbf{w} = [w_1, w_2, ..., w_N]^T$ is an independently and identically distributed (i.i.d.) zero-mean Gaussian noise vector with elements having a fixed variance. For real multi-level PAM (MPAM) transmission, the elements in the matrices of (1) are all real values and (1) is a real matrix equation. While for multi-level QAM (MQAM) transmission, the elements in the matrices of (1) are all complex, but (1) can also be transformed into a real matrix equation:

$$\begin{bmatrix}\mathbf{Re}(\mathbf{r})\\ \mathbf{Im}(\mathbf{r})\end{bmatrix} = \begin{bmatrix}\mathbf{Re}(\mathbf{H}) & \mathbf{-Im}(\mathbf{H})\\ \mathbf{Im}(\mathbf{H}) & \mathbf{Re}(\mathbf{H})\end{bmatrix}\begin{bmatrix}\mathbf{Re}(\mathbf{x})\\ \mathbf{Im}(\mathbf{x})\end{bmatrix} + \begin{bmatrix}\mathbf{Re}(\mathbf{w})\\ \mathbf{Im}(\mathbf{w})\end{bmatrix} \tag{2}$$

with $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ being the real and imaginary parts, respectively, of $(\cdot)$. Since it is much easier and more convenient to explain GD in the real dimensional space, therefore, without lost of generality, real value vectors and matrices are assumed in (1) for discussions in the rest of the paper.

Assuming the channel matrix $\mathbf{H}$ in (1) is known at the receiver, a ML decoder selects the vector with the minimum Euclidean distance as the transmitted symbol vector, i.e.,

$$\mathbf{x}_{\mathbf{ML}} = \arg\min_{\mathbf{x}\in\Omega} \left\|\mathbf{r} - \mathbf{Hx}\right\|^2 \tag{3}$$

where $\mathbf{r}, \mathbf{x} \in \Re^N$, $\mathbf{H} \in \Re^{N\times N}$ and $\Omega$ is the set of all possible symbols in the transmitted lattice. For signals with a large number of symbols, the computation requirement for ML decoding is too complicated to implement in practice. GD has been proposed to reduce the complexity.

## 3. Geometric Decoding (GD)

### 3.1. Basic Concept of GD

The Euclidean distance in (3) can be written as [9]:

$$f(\mathbf{x}) = \left\|\mathbf{r} - \mathbf{Hx}\right\|^2 \tag{4}$$

$$= (\mathbf{x} - \mathbf{x_c})^{\mathbf{T}} \mathbf{H}^{\mathbf{T}}\mathbf{H}(\mathbf{x} - \mathbf{x_c})$$

$$= (\mathbf{x} - \mathbf{x_c})^{\mathbf{T}} \mathbf{M}^{-1}(\mathbf{x} - \mathbf{x_c}) \tag{5}$$

which is an elliptic paraboloid in an $N$+1 dimensional space with the axis perpendicular to a subspace spanned by the symbol vectors in $\Omega$ and has a global minimum point given by $\mathbf{x_c} = \mathbf{H}^{-1}\mathbf{r}$ as shown in Fig. 1 [10]. Using eigenvalue decomposition, the matrix $\mathbf{M}$ in (5) can be written as:

$$\mathbf{M} = \left(\mathbf{H}^{\mathbf{T}}\mathbf{H}\right)^{-1} = \left(\mathbf{V}\Lambda\mathbf{V}^T\right)^{\mathbf{T}} \tag{6}$$

where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_N)$ are the eigenvalues of $\mathbf{M}$ arranged in descending order, and $\mathbf{V}$ is the corresponding eigenvector matrix given by:

$$\mathbf{V} = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \cdots, \mathbf{v_N}] \tag{7}$$

$$= \begin{bmatrix} v_{11} & v_{21} & v_{31} & \cdots & v_{N1}\\ v_{12} & v_{22} & v_{32} & \cdots & v_{N2}\\ v_{13} & v_{23} & v_{33} & \cdots & v_{N3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ v_{1n} & v_{2n} & v_{3n} & \cdots & v_{NN}\end{bmatrix} \tag{8}$$

The horizontal cross section of the elliptic paraboloid (5) is given by:

$$f(\mathbf{x}) = a^2 \tag{9}$$

which is a hyper ellipsoid with its center at a distance $a^2$, where $a \in \Re$, from $\mathbf{x_c}$. This hyper ellipsoid, with lengths and directions of the semiaxes given by $a\sqrt{\lambda_i}$

and $\mathbf{v_i}$ , respectively [9], can be projected onto the subspace spanned by the vectors in $\Omega$ as shown in Figure 1. Since all transmitted symbol vectors must be lying on the elliptic paraboloid (5), a set of hyper ellipsoids containing all the transmit symbol vectors can be obtained and projected onto the subspace. It can be seen from (3), (4) and (9) that an optimum GD is to obtain the solution of:

$$\mathbf{x_{GD}} = \arg\min_{\mathbf{x}\in\Omega} f(\mathbf{x}) \tag{10}$$

The symbol vector lying on the smallest hyper ellipsoid is decoded as the transmitted symbol vector. If all possible symbol vectors in $\Omega$ are considered by (10) for obtaining the decoding solution, it will have the same implementation complexity as that of the ML decoder. Here we propose a simple and novel GD searching algorithm which requires less than 10% complexity of the ML decoder to achieve the minimum BLER performance.

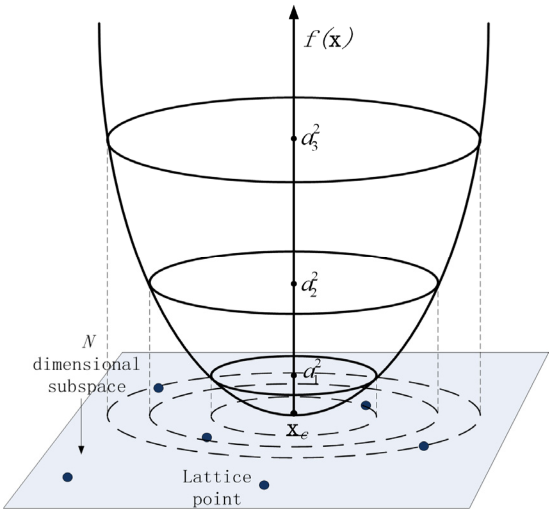


Figure 1. Elliptic paraboloid in an $N$+1 dimensional space with axis perpendicular to a subspace spanned by the vectors in $\Omega$ .

### 3.2. Proposed GD Algorithm Implementation

We first use zero-forcing decoding [11] to obtain the initial solution $\mathbf{x}_{zf}$, then compute $a_{zf}^2$ using (5) and obtain a hyper ellipsoid with:

$$f(\mathbf{x}) = a_{zf}^{\ 2} \tag{11}$$

on the subspace spanned by the symbol vectors in $\Omega$ . Then we construct a smallest hyper rectangle which encloses the hyper ellipsoid. We use a new $N$-dimensional rectangular coordinate system, with $x_i'$ -axes, for $i = 1,2,3,\cdots,N$ , coinciding with the $i^{th}$-semiaxis of the hyper ellipsoid and the origin coinciding with $\mathbf{x_c}$ . The superscript prime is used to denote the variables in the new coordinate system. The hyper rectangle in an $N$-dimensional space has $2^N$ apexes. The coordinates of these apexes are composed of $N$ variables in these $x_i'$ - axes directions and can be denoted in the new coordinate system as:

$$\mathbf{k_p}' = [x'_{p1}, x'_{p2}, x'_{p3}, \cdots, x'_{pN}] \tag{12}$$

where $p = 1,2,\cdots,2^N$ and $x'_{pi} = \pm a_{zf}\sqrt{\lambda_i}$ , for $i$ = 1, 2, .., $N$. By using coordinate transformation, the coordinates of these $2^N$ apexes can be transformed to the lattice point space as:

$$\mathbf{k_p}^{\mathbf{T}} = \mathbf{V}\cdot\left(\mathbf{k_p}'\right)^{\mathbf{T}} + \mathbf{x_c} \tag{13}$$

where $\mathbf{V}$ is the eigenvector matrix (7). Thus the $i^{th}$ -component in $\mathbf{k_p}$ can be obtained as:

$$x_{pi} = \sum_{q=1}^{N}\left(v_{qi}x'_{pq}\right) + x_{ci} \tag{14}$$

where $x_{ci}$ is the $i^{th}$-component in $\mathbf{x_c}$ in the $x_i$ - axis direction. The perimeters of this smallest hyper rectangle are not in parallel with the axes of the lattice point space, so it is not convenient to use. We need to find another hyper rectangle which encloses this smallest hyper rectangle and has the perimeters in parallel with the axes of the lattice point space.

Since $x'_{pq} = \pm a_{zf}\sqrt{\lambda_i}$ , the maximum and minimum values of the element in $\mathbf{k_p}$ in the $x_i$ - axis direction of the lattice space is:

$$\begin{aligned} x_{i\_\max} &= x_{ci} + \sum_{q=1}^{N}\left|v_{qi}a_{zf}\sqrt{\lambda_q}\right| \\ x_{i\_\min} &= x_{ci} - \sum_{q=1}^{N}\left|v_{qi}a_{zf}\sqrt{\lambda_q}\right| \end{aligned} \tag{15}$$

The new hyper rectangle enclosing the smallest hyper rectangle has the apexes with coordinates given by:

$$\hat{\mathbf{k}}_{\mathbf{p}} = [\breve{x}_{p1}, \breve{x}_{p2}, \breve{x}_{p3}, \cdots, \breve{x}_{pN}] \tag{16}$$

where $p = 1, 2, \cdots, 2^N$ and $\breve{x}_{pi} = x_{i\_max}$ or $x_{i\_min}$, for $i$ = 1, 2, .., $N$. Assume that, within the boundaries of $x_{i\_max}$ and $x_{i\_min}$ given by (15), there are $iNum$ symbol vectors, with coordinates in the $x_i$ - axis direction given by:

$$\{\hat{x}_i\} = \{\hat{x}_{i1}, \hat{x}_{i2}, \hat{x}_{i3}, \cdots, \hat{x}_{iiNum}\} \tag{17}$$

If $iNum$ = 0, there is no symbol vector inside the hyper rectangle and the GD needs to do no searching. The zero-forcing point is considered to be the solution. If $iNum$ is not zero for $i$ = 1, .., $N$, the total number of symbol vectors inside the hyper rectangle is $\prod_{i=1}^{N} iNum$ which must include all the symbol vectors inside the enclosed hyper ellipsoid. Then we narrow the search region to be exactly inside the hyper ellipsoid as described below.

Assume that among all possible value sets $\{\hat{x}_i\}$ of (17), for $i$ = 1, 2, ..., $N$, the set $\{\hat{x}_l\}$ in the $l^{th}$-axis direction, where $1 \le l \le N$, has the largest number of elements. The rest of the value sets can be used to form $\prod_{i=1, i \ne l}^{N} iNum$ vectors with different combinations given by:

$$\begin{aligned}\mathbf{c_k} = [&\hat{x}_1(k_1), \hat{x}_2(k_2), \cdots \hat{x}_{l-1}(k_{l-1}),\\ &\hat{x}_{l+1}(k_{l+1}), \cdots, \hat{x}_N(k_N)]\end{aligned} \tag{18}$$

where $k = 1, 2, ..., \prod_{i=1, i \ne l}^{N} iNum$ and the $i^{th}$-component in $\mathbf{c_k}$ is taken from the $k_j^{th}$-component in the value set $\{\hat{x}_i\}$ given by (17), with $k_j = 1, 2, ..., jNum$ (that is $k_1 = 1, 2, ..., 1Num$, $k_2 = 1, 2, ..., 2Num$, $k_3 = 1, 2, ..., 3Num$ and so on). The elements in $\mathbf{c_k}$ are substituted into (11) to find the $l^{th}$ coordinates of the symbol vectors within the hyper ellipsoid, i.e.:

$$\{\bar{x}_l\} = \{\bar{x}_{l1}, \bar{x}_{l2}, \bar{x}_{l3}, \cdots \bar{x}_{llNum}\} \tag{19}$$

where $lNum$ is the number of symbol vectors. Each of the elements in (19) is used in (18) to form a complete set of coordinates for a symbol vector:

$$\begin{aligned}\mathbf{x_k} = [&\hat{x}_1(k_1), \hat{x}_2(k_2), \cdots \hat{x}_{l-1}(k_{l-1}),\\ &\bar{x}_l(k_l), \hat{x}_{l+1}(k_{l+1}), \cdots, \hat{x}_N(k_N)]^T\end{aligned} \tag{20}$$

inside the hyper ellipsoid. Finally, the value of $a^2$ for each symbol vector is computed using (9) and the one with minimum $a^2$ is decoded as the transmitted symbol vector.

## 4. Simulation Results and Discussions

To assess the complexity reduction of our proposed GD algorithm, different MIMO systems have been studied by using computer simulation. They are the $2 \times 2$, $3 \times 3$ and $4 \times 4$ MIMO systems transmitting 8PAM and 16QAM over a fading channel. Complexity is measured by the number of symbol vectors used in the decoding process. Table 1 shows the total numbers of symbol vectors used in ML decoding and the average number of symbol vectors used in our GD algorithm at a BLER of about $10^{-2}$ for these MIMO systems. For ML decoding, all symbol vectors are used, so the implementation complexity of the ML decoder increases exponentially with the number of antennas and level of modulation used. With our proposed GD algorithm, the symbol vectors used are limited to those inside the hyper ellipsoid obtained using zero forcing. Table 1 shows that, for the $2 \times 2$ MIMO system transmitting 8PAM and the $4 \times 4$ MIMO system transmitting 16QAM, the numbers of symbol vectors used in the ML decoder are 64 and 65536, respectively. However, with the use of our GD algorithm, the numbers of symbol vectors used in these two MIMO systems are substantially reduced to only 5 and 1273, leading to the complexity reduction of 92.2% and 98.1% respectively.

Table 1 Number of symbol vectors visited in ML decoding/proposed GD (reduction is indicated as percentage)

| Modulation | 2×2 | 3×3 | 4×4 |
|---|---|---|---|
| **8PAM** | 64/5 | 512/21 | 4096/65 |
| | 92.2% | 95.8% | 98.4% |
| **16QAM** | 256/11 | 4096/86 | 65536/1273 |
| | 95.7% | 97.9% | 98.1% |

Monte Carlo computer simulations using Matlab have also been used to assess the BLER performances of these MIMO systems in a fading channel. Results have shown that, in all these MIMO systems studied, the BLER using the ML decoder and our proposed GD are the same, indicating that our proposed GD is optimum. This is as expected because both ML decoder and our proposed GD attempt to find the same symbol vector with the minimum distance and decode it as the transmitted symbol vector. Our proposed GD simply speeds up the decoding process by narrowing the search region and this does not affect the search result. The BLER performances of using the ML decoder and our proposed GD in different MIMO systems transmitting 8PAM and 16QAM are shown in Figs. 2 and 3, respectively.

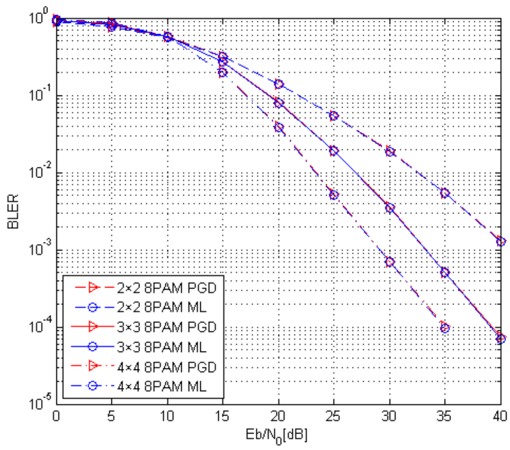


Figure 2. BLERs using proposed GD and ML decoding in different MIMO systems transmitting 8PAM

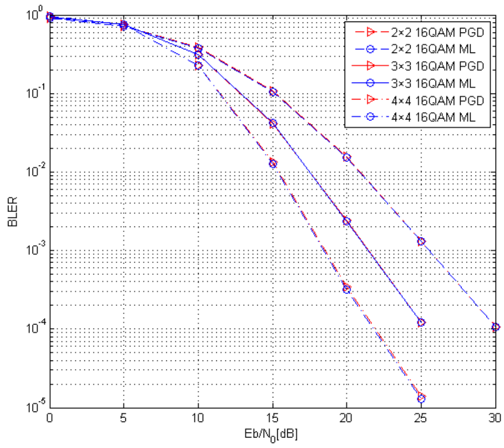


Figure 3. BLERs using proposed GD and ML decoding in different MIMO systems 16QAM

## 5. Conclusions

We have presented the simple implementation of a new GD algorithm for optimum decoding of signals for MIMO systems. The decoder uses the channel matrix to construct a hyper paraboloid containing all the transmit symbol vectors. A hyper ellipsoid enclosing the ML solution is projected from the hyper paraboloid using the zero forcing solution. The symbol vectors to be searched are then restricted to those inside the hyper ellipsoid. Monte Carlo computer simulation results for different MIMO systems transmitting 8PAM and 16QAM show that the proposed GD can achieve complexity reductions on decoding of more than 90% over the ML decoder, yet having the same minimum BLER performances as those of using ML decoder.

## 6. References


[1] A. Goldsmith, S. A. Jafar, N. Jindal and S. Vishwanath, "Capacity limits of MIMO channels," IEEE Journal on Selected areas in Communications, vol. 21, no. 5, pp. 684–702, Jun. 2003.

[2] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Commun.*, vol. 6, pp. 311-335, Mar. 1998.

[3] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecommun.*, vol. 10, no. 6, pp. 585-595, 1999.

[4] L. Azzam, E. Ayanoglu, "Reduction of ML decoding complexity for MIMO sphere decoding, QOSTBC, and OSTBC," *Information theory and application workshop*, pp:18 – 25, Jan. 2008.

[5] P. Wolniansky, C. Foschini, G. Golden, and R. Valenzuela, "V-BLAST: an architecture for realizing very high data rates over the rich-scattering wireless channel," in *International Symposium on Signals, Systems and Electronics ISSSE98*, pp. 295–300.

[6] G. J. Foschini, "Layered space-time architecture for wireless communications in a fading environment when using multi-element antennas," *Bell labs Tech. j.,* vol. 2, pp. 41–59, 1996.

[7] E. Viterbo and J. Boutros, "A Universal Lattice Code Decoder for Fading Channels," *IEEE Trans. Information Theory*, vol. 45, no. 5, pp. 1639–1642, July 1999.

[8] H. Artes, D. Seethaler, and F. Hlawatsch, "Efficient detection algorithms for mimo channels: A geometrical approach to approximate ml detection," *IEEE Trans. Signal Processing*, vol. 51, no. 11, pp. 2808–2820, Nov. 2003.

[9] M. Samuel and M. P. Fitz, "Geometric Decoding Of PAM and QAM Lattices," in *Proc. IEEE Global Telecommunications Conf.*, pp. 4247–4252, Nov.2007.

[10] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[11] A. Paulraj, R. Nabar and D. Gore, *Introduction to Space-Time Wireless Communications*, Cambridge University Press, May 2003.