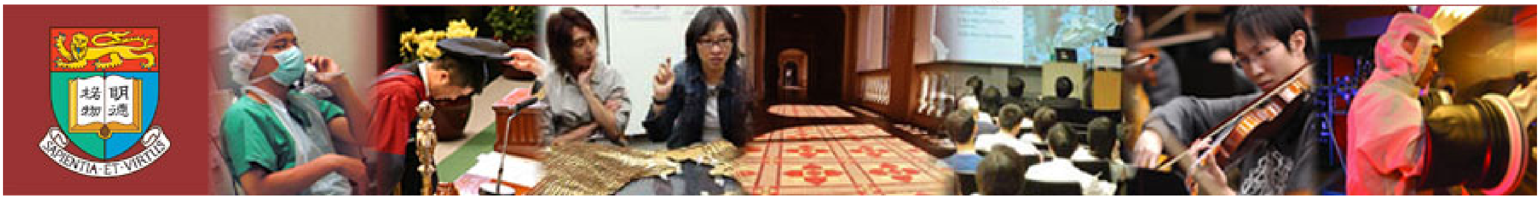

Pricing and Hedging American Options Analytically: A Perturbation Method

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Pricing and Hedging American Options Analytically: A Perturbation Method

Abstract

This paper studies the critical stock price of American options with continuous dividend yield. We solve the integral equation and derive a new analytical formula in a series form for the critical stock price. American options can be priced and hedged analytically with the help of our critical-stock-price formula. Numerical tests show that our formula gives very accurate prices. With the error well controlled, our formula is now ready for traders to use in pricing and hedging the S&P 100 index options and for the Chicago Board Options Exchange to use in computing the VXO volatility index.

1 Introduction

American options currently trade throughout the world. The most popular American option contracts in the United States are those on the S&P 100 Index (OEX), traded on the Chicago Board Options Exchange (CBOE). For example, the open interest of S&P 100 Index options on March 12, 2003 was 325,810 of which 167,768 contracts were calls and 157,042 contracts were puts. The trading volume was 72,713, including 37,703 calls and 35,010 puts. Most foreign-currency options traded on the Philadelphia Stock Exchange (PSE) are American style. For example, the total open interest of foreign-currency options on March 11, 2003 was 15,616 of which 12,702 contracts were American options.

Given the fact that American options are frequently traded on exchanges, pricing American options is very important. Due to the difficulty of dealing with the early-exercise feature, a closed-form formula[2] has not been found, and it seems unlikely that one will be found any time soon. In practice, the price of American options is often computed numerically by Cox, Ross and Rubinstein's (1979) binomial-tree method, by solving Black-Scholes (1973) partial differential equation with a moving boundary, or by solving an integral equation for the critical stock price[3] (see, e.g., Yu (1993), and Huang, Subrahmanyam and Yu (1996)). Even though these numerical methods are able to give accurate values, a good analytical approximate formula is still very useful and valuable for four reasons. First, the numerical computation could be time consuming. Second, an analytical formula can be used in case a computing engine is not available. Third, the detailed study of the critical stock price provides a methodology to study other moving boundary problems in finance, such as convertible bonds and real options with an early-exercise feature. Fourth, an analytical formula provides intuition of the relation between parameters. We focus on

[2]In this paper, we differentiate between two concepts: a *closed-form formula* and an *analytical formula*. By a *closed-form* formula, we mean that the formula is written in an easily computable function such as power, exponential or logarithmical functions, or a special function such as cumulative normal distribution, Bessel or confluent hypergeometric functions, etc. But the concept of an *analytical formula* has a wider scope. It covers that of a closed-form formula and a summation or multiplication series of some known functions.

[3]The critical stock price, the early-exercise boundary and the optimal exercise boundary have the same meaning. These terms are used interchangeably in this paper.

analytical approaches in this paper.

Johnson (1983) proposes an analytical approximation for the American put price based on a regression on Parkinson's (1977) numerical values. The formula is generated based on numbers through a statistical method instead of on a rigorous analysis of the intrinsic nature of the problem. Geske and Johnson (1984) give an analytical expression by treating an American put as a portfolio of an infinite number of compound options. Evaluating the multivariate cumulative normal distribution function is a practical problem for this method. They propose using a four-point extrapolation method to evaluate American options approximately. MacMillan (1986) and Barone-Adesi and Whaley (1987) use quadratic approximation for American option prices and find the critical stock price numerically by iteration. Bunch and Johnson (1992) propose a modified two-point Geske-Johnson method. Broadie and Detemple (1996) derive lower and upper bounds for the American option price. A comparison of the different methods is available in Broadie and Detemple (1996) and Ju (1998)[4].

It is by now well known that the price of American options can be written as the sum of the corresponding European option price and an integral in terms of its early-exercise boundary. The mathematical result appeared early in the literature by Kolodner (1956) and McKean (1965). It has been restudied by Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) to gain economic insights. The financial problem of pricing American options boils down to a mathematical problem of solving an integral equation with the critical stock price as an unknown function of time and other parameters. Ju (1998) approximates the early-exercise boundary as a piece-wise exponential function, obtains an analytical formula for the American option price, and then uses numerical iteration to determine a more accurate boundary. An analytical formula for the critical stock price has not yet been found. Even for the leading-order expansion near expiration, Kuske and Keller (1998), Stamicar, Ševčovič and Chadam (1999), and Bunch and Johnson (2000) give

[4] American option pricing has also been studied numerically by Brennan and Schwartz (1977) with a finite-difference method, by Carr (1998) with a randomization technique, and by Longstaff and Schwartz (2001) with a simulation-based least-squares approach.

different analytical expressions. In a recent development, Chen and Chadam (2007) provide a convincing mathematical justification to show that Ševčovič and Chadam (1999) give the correct asymptotic behavior near expiration, which is also confirmed by Evans, Kuske and Keller (2002). But Chen and Chadam (2007) only study the case when the underlying asset does not pay any dividend. We nontrivially extend Chen and Chadam's (2007) methodology to the case with dividend yield that models the price of American options on stock indices and currencies, which is one of the central concerns of academics and market participants. Recently Zhu (2006) develops a quasi-analytical expression for the critical stock price, however his complicated iteration procedure requires numerical integration in each step. An accurate and user-friendly approximate formula for the critical stock price is not available.

This paper makes three contributions to the literature. The first contribution is to entend the Chen and Chadam (2007) approach by taking into account a dividend yield. The second contribution is to derive approximate solutions for the critical stock price of the American option with continuous dividend yield at small $\sigma^2 T$, for three different regions where the difference between interest rate and dividend yield is positive, zero and negative. With the help of the critical-stock-price formula, one can price and hedge American options analytically. The third contribution is to test the validity of the formula. Compared with the highly accurate numerical values computed by solving the integral equation, our formulas up to the fourth-order term give very accurate prices with an accuracy up to 0.01 cent for the American options, with a one-month maturity and a strike price of 100 dollars, tested in this paper. With the error well controlled, our formula is now ready for traders to use in pricing OEX options, since most liquid OEX options have a maturity of about a month. The formula can also be used by the CBOE to compute the VXO volatility index since the index is defined as an implied volatility of the OEX options with one-month maturity.

2 The Model

For completeness, this section briefly reviews the Black-Merton-Scholes (1973) model of American option pricing. In a risk-neutral world, the price of an underlying stock, S_t, is modelled by a lognormal process

$$S_t = S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)t+\sigma w_t}, \tag{1}$$

where S_0 is the initial stock price, r is the risk-free rate, q is the continuous dividend yield, σ is the volatility of the underlying stock, w_t is a standard Wiener process (Brownian motion). The three parameters r, q and σ are assumed to be constant.

The owner of an American put has a right to claim the difference between the strike price, K, and the stock price, S_t, at any time, t, before maturity, T. Therefore, the American put price has a lower bound of $P_t \geq \max(K - S_t, 0)$. Pricing an American put option involves two steps. The first step is to determine the critical stock price, B_t^p, which is a function of time, t. If $S_t \leq B_t^p$, one should exercise the American put. Otherwise, one should hold for a possible later exercise. The second step is to determine the price of the American put when exercising the put is not optimal. A standard argument shows that the price of an American put option, $P(S,t)$, satisfies the following Black-Scholes partial differential equation, boundary conditions and final conditions:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP = 0, \qquad \text{for } \ t < T, \quad S > B_t^p, \tag{2}$$

$$P(S,t) = K - S, \qquad \frac{\partial P}{\partial S}(S,t) = -1, \qquad \text{for } \ t < T, \quad 0 < S \leq B_t^p, \tag{3}$$

$$P(S,T) = \max(K - S, 0), \tag{4}$$

$$B_T^p = \begin{cases} K & \text{if } \ r \geq q \\ \frac{r}{q}K & \text{if } \ r < q \end{cases}. \tag{5}$$

The first two equations, (2) and (3), can be combined to give

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP = \begin{cases} 0 & \text{if } S > B_t^p \\ qS - rK & \text{if } S \leq B_t^p \end{cases}, \quad \text{for } \ t < T. \tag{6}$$

The backward inhomogeneous linear diffusion equation with the final condition (4) has the following solution:

$$P(S,t) = p_E(S,t) + \int_t^T [rKe^{-rs}N(-d_2(S, B_s^p, s)) - qSe^{-qs}N(-d_1(S, B_s^p, s))]ds, \tag{7}$$

where p_E is the price of the corresponding European put option given by the Black-Scholes (1973) formula,

$$p_E(S,t) = Ke^{-r(T-t)}N(-d_2(S,K,T-t)) - Se^{-q(T-t)}N(-d_1(S,K,T-t)), \tag{8}$$

where $N(x)$, defined by

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy, \tag{9}$$

is the cumulative normal distribution function and

$$d_1(x,y,t) = \frac{\ln(x/y) + (r - q + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, \qquad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}. \tag{10}$$

Equation (7) expresses the value of an American put as the sum of the value of a European put and the early-exercise premium. The early-exercise premium can be viewed as the value of a contingent claim that allows dividends paid by the stock, $qSdt$, to be exchanged for interest earned on the exercise price, $rKdt$, whenever the stock price is below the optimal exercise boundary. The expression for B_t^p is crucial when we evaluate the integration in equation (7). We are unable to price an American put without a formula for the critical stock price. Applying equation (7) at the boundary, $S = B_t^P$, gives us a single integral equation

$$\begin{aligned} K - B_t^p \;=\; & Ke^{-r(T-t)}N(-d_2(B_t^p,K,T-t)) - B_t^p e^{-q(T-t)}N(-d_1(B_t^p,K,T-t)) \\ & + \int_t^T [rKe^{-rs}N(-d_2(B_t^p,B_s^p,s)) - qB_t^p e^{-qs}N(-d_1(B_t^p,B_s^p,s))]ds. \end{aligned} \tag{11}$$

One may notice that $B_t^p = 0$ if $r = 0$. One should never exercise an American put if the interest rate is zero.

The owner of an American call has a right to claim the difference between the stock price, S_t, and the strike price, K, at any time, t, before maturity, T. Therefore the American call price has a lower bound, $C_t \geq \max(S_t - K, 0)$. Pricing an American call option also involves two steps. The first step is to determine the critical stock price, B_t^c. If $S_t \geq B_t^c$, one should exercise the American call. Otherwise, one should hold for a possible later exercise. The second step is to determine the price of the American call when exercising the call is not optimal. The price of an American call option, $C(S,t)$, satisfies the following Black-Scholes partial differential equation, boundary conditions and final conditions:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0, \qquad \text{for } \ t < T, \ \ 0 < S < B_t^c, \tag{12}$$

$$C(S,t) = S - K, \qquad \frac{\partial C}{\partial S}(S,t) = 1, \qquad \text{for } \ t < T, \ \ S \geq B_t^c, \tag{13}$$

$$C(0,t) = 0, \tag{14}$$

$$C(S,T) = \max(S-K, 0), \tag{15}$$

$$B_T^c = \begin{cases} K & \text{if } \ r \leq q \\ \frac{r}{q}K & \text{if } \ r > q \end{cases}. \tag{16}$$

The solution to the problem can be written in an integral form:

$$C(S,t) = c_E(S,t) + \int_t^T [qSe^{-qs}N(d_1(S, B_s^c, s)) - rKe^{-rs}N(d_2(S, B_s^c, s))]ds, \tag{17}$$

where c_E is the price of the corresponding European call option given by the Black-Scholes (1973) formula. The critical stock price, B_t^c, for the American call satisfies the following integral equation:

$$\begin{aligned} B_t^c - K \;\; &= \;\; B_t^c e^{-q(T-t)}N(d_1(B_t^c, K, T-t)) - Ke^{-r(T-t)}N(d_2(B_t^c, K, T-t)) \\ &\quad + \int_t^T [qB_t^c e^{-qs}N(d_1(B_t^c, B_s^c, s)) - rKe^{-rs}N(d_2(B_t^c, B_s^c, s))]ds. \end{aligned} \tag{18}$$

This equation has the property: if $q \to 0$, $B_t^c \to +\infty$, which leads to the well-known result that one should never exercise an American call if the underlying stock does not pay any dividend.

The analytical formula, (7) or (17), of the American option price was first introduced to the mathematics literature by Kolodner (1956) to study change of phase, and to the economics literature by McKean (1965). It has been restudied by Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) to gain financial insights into the context of American option pricing. The American option pricing problem boils down to a mathematical problem of solving the integral equation, (11) or (18), for the critical stock price, B_t^p or B_t^c. Huang, Subrahmanyam and Yu (1996) solve the optimal exercise boundary with a recursive numerical integration approach. Ju (1998) solves the problem by approximating the early-exercise boundary as a multi-piece exponential function. We solve the problem with a perturbation method. Our target is to obtain an analytical formula for the critical stock price.

3 The Main Theorems

Our main results are summarized in the following two theorems.

Theorem 1. *The price of an American put option is given by the analytical formula (7) where the critical stock price, $B_t^p(r, q, \sigma, K, T)$, is given as follows:*

If $r > q \geq 0$,

$$B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)u(\xi)}}, \tag{19}$$
$$u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3} + O\left(\frac{1}{\xi^4}\right),$$
$$\xi = \ln\sqrt{8\pi(r-q)^2(T-t)/\sigma^2}.$$

If $r = q$,

$$B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)v(\eta)}}, \tag{20}$$
$$v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1 - \frac{5}{4\sqrt{2\pi}}}{\eta} + o\left(\frac{1}{\eta}\right),$$
$$\eta = \ln[4\sqrt{\pi}r(T-t)].$$

If $r < q$,

$$B_t^p = \frac{r}{q}Ke^{-2\sqrt{\tau^*}w(\sqrt{\tau^*})}, \qquad \tau^* = \frac{1}{2}\sigma^2(T-t), \qquad r^* = \frac{r}{\frac{1}{2}\sigma^2}, \qquad q^* = \frac{q}{\frac{1}{2}\sigma^2}, \tag{21}$$
$$w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\tau^{*3/2} + O(\tau^{*2}),$$
$$\beta_0 = 0.451723, \qquad \beta_1 = 0.144914\ (r^* - q^*),$$
$$\beta_2 = -0.009801 - 0.041764\ (r^* + q^*) + 0.014829\ (r^* - q^*)^2,$$
$$\beta_3 = -0.000618 - 0.002087\ (r^* - q^*) - 0.015670\ (r^{*2} - q^{*2}) - 0.001052\ (r^* - q^*)^3.$$

The proof of Theorem 1 is given in Appendix A.

Remark 1. The present formulas of $u(\xi)$, $v(\eta)$ and $w(\sqrt{\tau^*})$ are derived with a perturbation method under the assumption of small $\sigma^2 T$. It covers almost all the existing approximate formulas in the same family as a special case. For example, Chen and Chadam (2007) provide a formula for $u(\xi)$ with $q = 0$. Evans, Kuske and Keller (2002) provide the first terms of the present formulas for $u(\xi)$, $v(\eta)$ and $w(\sqrt{\tau^*})$.

Theorem 2. *The price of an American call option is given by the analytical formula (17) where the critical stock price, $B_t^c(r, q, \sigma, K, T)$, is given by the following duality relation:*

$$B_t^c(r, q, \sigma, K, T) = \frac{K^2}{B_t^p(q, r, \sigma, K, T)} \tag{22}$$

or explicitly as follows:

If $r > q \geq 0$,

$$B_t^c = \frac{r}{q} K e^{2\sqrt{\tau^*} w(\sqrt{\tau^*})}, \qquad \tau^* = \frac{1}{2}\sigma^2(T-t), \qquad r^* = \frac{r}{\frac{1}{2}\sigma^2}, \qquad q^* = \frac{q}{\frac{1}{2}\sigma^2},$$
$$w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\tau^{*3/2} + O(\tau^{*2}),$$
$$\beta_0 = 0.451723, \qquad \beta_1 = 0.144914\ (q^* - r^*),$$
$$\beta_2 = -0.009801 - 0.041764\ (q^* + r^*) + 0.014829\ (q^* - r^*)^2,$$
$$\beta_3 = -0.000618 - 0.002087\ (q^* - r^*) - 0.015670\ (q^{*2} - r^{*2}) - 0.001052\ (q^* - r^*)^3.$$

If $r = q$,

$$B_t^c = K e^{\sqrt{2\sigma^2(T-t)v(\eta)}},$$
$$v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1 - \frac{5}{4\sqrt{2\pi}}}{\eta} + o\left(\frac{1}{\eta}\right),$$
$$\eta = \ln[4\sqrt{\pi} r(T-t)].$$

If $r < q$,

$$B_t^c = K e^{\sqrt{2\sigma^2(T-t)u(\xi)}},$$
$$u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3} + O\left(\frac{1}{\xi^4}\right),$$
$$\xi = \ln\sqrt{8\pi(r-q)^2(T-t)/\sigma^2}.$$

The proof of Theorem 2 is given in Appendix B.

Remark 1. It seems to us that the duality relation (22) between the critical stock price of an American call and that of an American put with the same strike is not well known. Evans, Kuske and Keller (2002) study the problem of the American call with asymptotic analysis and provide the first terms of our formulas $u(\xi)$, $v(\eta)$ and $w(\sqrt{\tau^*})$.

Remark 2. In principle, one is able to push the series of $u(\xi)$, $v(\eta)$ and $w(\sqrt{\tau^*})$ to any higher order, but it involves much more algebra. We choose to stop at the level of the fourth-order term and go on to test the accuracy of the formulas. The formulas are derived theoretically with an assumption of small $\sigma^2 T$. How small is it practically? This important question has to be answered before academics and market participants can use the formula. A numerical test on the accuracy of the formulas must be performed in order to answer the question.

4 The Computation of Highly Accurate Numerical Values of the Critical Stock Price and the American Option Price

In order to test the accuracy of the present formulas, we need highly accurate numerical values as a benchmark. There are many different ways to compute the critical stock price and the American option price numerically. The most popular ones include the binomial-tree method, the PDE method and the integral-equation approach. We adopt the integral-equation approach in this paper because it computes the critical stock price directly.

Highly accurate numerical values of the critical stock price of an American put can be computed by solving the integral equation (11) numerically. For a given set of parameters, (r, q, σ, K, T), in order to solve the equation for B^p at a particular time, t, we need B^p for the time interval between t and T. Therefore, we need to solve B^p backward from T, where B^p_T is known to be K if $r \geq q$ or $(r/q)K$ if $r < q$. For example, at time $T - \Delta t$, $B^p_{T-\Delta t}$ solves the following equation:

$$K - B^p_{T-\Delta t} = p_E(B^p_{T-\Delta t}, T - \Delta t) + \frac{1}{2}\left[f(B^p_{T-\Delta t}, B^p_{T-\Delta t}, T - \Delta t) + f(B^p_{T-\Delta t}, B^p_T, T)\right]\Delta t,$$

where

$$f(B_t^p, B_s^p, s) = rKe^{-rs}N(-d_2(B_t^p, B_s^p, s)) - qB_t^p e^{-qs}N(-d_1(B_t^p, B_s^p, s))$$

and a trapezoidal rule is used in the numerical integration. The equation can be solved numerically with a standard root-finding algorithm. Once $B_{T-\Delta t}^p$ is known, $B_{T-2\Delta t}^p$ can be found by solving the following equation

$$K - B_{T-2\Delta t}^p = p_E(B_{T-2\Delta t}^p, T-2\Delta t) + \frac{1}{2}[f(B_{T-2\Delta t}^p, B_{T-2\Delta t}^p, T-2\Delta t) + 2f(B_{T-2\Delta t}^p, B_{T-\Delta t}^p, T-\Delta t) + f(B_{T-2\Delta t}^p, B_T^p, T)]\Delta t.$$

Once $B_{T-i\Delta t}^p$, $i = 1,\ 2,\ \cdots,\ n-1$, are known, $B_{T-n\Delta t}^p$ can be found by solving

$$K - B_{T-n\Delta t}^p = p_E(B_{T-n\Delta t}^p, T-n\Delta t) + \frac{1}{2}[f(B_{T-n\Delta t}^p, B_{T-n\Delta t}^p, T-n\Delta t) + 2\sum_{i=1}^{n-1} f(B_{T-n\Delta t}^p, B_{T-(n-i)\Delta t}^p, T-(n-i)\Delta t) + f(B_{T-n\Delta t}^p, B_T^p, T)]\Delta t.$$

We have solved $B_{T-i\Delta t}^p$, $i = 1,\ 2,\ \cdots,\ n$, successively. With all the information on the boundary, we can price the American put at time $t = T - n\Delta t$ by computing the integration formula (7) numerically, i.e.,

$$P(S, T-n\Delta t) = p_E(S, T-n\Delta t) + \frac{1}{2}[f(S, B_{T-n\Delta t}^p, T-n\Delta t) + 2\sum_{i=1}^{n-1} f(S, B_{T-(n-i)\Delta t}^p, T-(n-i)\Delta t) + f(S, B_T^p, T)]\Delta t.$$

The numerical results are presented in Tables 1 and 2. Table 1 is the critical stock price of an American put option as a function of the time to maturity from a week up to 17 weeks. As the grid size reduces from a week to half of a week, to a quarter of a week, 1/8 of a week, $\cdots$, 1/64 of a week, the computed numerical values converge to the true value with an error of 0.01 cent for an option with up to three month maturity and a strike price of 100 dollars. Table 2 shows the American put prices for a range of moneyness from -10% to 10% and volatility from 0.2 to 0.5. The computed prices of a put option with one-month maturity[5] are also convergent with an error of 0.01 cent as the time grid size decreases from 1/20 of a month to 1/320 of a month.

[5] Most of the time, our test is restricted to a one-month maturity in this paper, partly because most liquidly traded American-style OEX options have a maturity of about a month.

With the highly accurate numerical values computed, we are now able to test the accuracy of the present formulas with certain truncated terms.

5 Numerical Test on the Accuracy of the Present Formulas

Since the American call and put have a duality relationship, we only need to test the formulas for the American put. The conclusion on the accuracy of the American put price automatically applies to that of the American call price.

We have done many numerical tests for different values of parameters (r, q). We find out that the American option price errors are quite stable in each region of $r > q \geq 0$, $r = q$ and $r < q$. We now present one typical case in each region to illustrate the errors.

The results for $r > q \geq 0$ are presented in Tables 3 and 4. Table 3 is the critical stock price as a function of the time to maturity from one week up to four months. The first column shows the highly accurate numerical values computed from the integral equation. The other four columns are computed by using the present analytical formulas with one, two, three and four truncated terms. The convergency of the series becomes an issue. Intuitively the series for $u(\xi)$ converges for large $|\xi|$. Based on the numerical values in Table 3, the series converges for small $T - t$, e.g., $T - t = 1/12$, which corresponds to $\xi = -1.42213$. It does not converge for large $T - t$, e.g., $T - t = 1/3$, which corresponds to $\xi = -0.72898$. The condition of convergency is under investigation. Table 4 is the price of the American options with a one-month maturity and a strike price of 100 dollars for a range of moneyness from -10% to 10% and volatility from 0.2 to 0.5. Compared with the highly accurate numerical values computed with the numerical-integration method described in §4, the present formulas give very accurate prices. For example, the formula with the first two terms gives a price with an error of 0.07 cent. This corresponds to OEX options with an error of 0.3 cent, since OEX options have a strike price of about 400 dollars (the S&P 100 index level on March 12, 2003 was $408.92). The accuracy is good enough for practical application. The price given by the present formula can be treated as the true model price if it is used to price OEX options.

The results for $r = q$ are presented in Tables 5 and 6. The series of the critical stock price in Table 5 converges for small $T - t$, but not for large $T - t$. But the convergency is slightly better than that for the case of $r > q$ in Table 3, because of the fact that the singularity near expiration in this case is weaker than that in the last case. The accuracy of the American option price in Table 6 is also better than that for the case $r > q$ in Table 4. For example, the formula with the first four terms gives a price with an error of 0.03 cent. This corresponds to OEX options with an error of 0.1 cent.

The results for $r < q$ are presented in Tables 7 and 8. The critical stock price in Table 7 computed with the present formulas converges to a value that is different from the highly accurate numerical value. We are still investigating why there is such a difference. The American option prices in Table 8 are almost identical to the highly accurate numerical values. The error, even with only the first term, is only 0.01 cent, which corresponds to 0.04 cent for OEX options. The accuracy of the formula in this case is the best of the three cases due to its weak singularity near expiration. Our results suggest that the formula in this case has the potential to price American options with longer maturities. A more comprehensive test will be reported in a subsequent study on developing an analytical formula for the critical stock price of a long-term American option.

The solutions obtained for the regions $r > q \geq 0$, $r = q$ and $r < q$ are totally different. To further test the continuity of three formulas (19), (20) and (21) in Theorem 1 near the neighborhood of $|r - q| \ll 1$, we present the results of $(r, q) = (0.05, 0.0499)$, $(r, q) = (0.05, 0.05)$ and $(r, q) = (0.05, 0.0501)$ in Table 9. The computed American option prices are indeed very close with three different critical-stock-price formulas. The relative difference is smaller than 0.3%. The accuracy is good enough for the application of pricing OEX options.

Our study shows that further research is required to enhance the convergency and accuracy of the series for the critical stock price. The key is to find some other ways to expand the two functions $u(\xi)$ and $v(\eta)$, so that the series converge for large value of $T - t$.

This is a problem for further research.

6 Conclusion

Pricing American options has been an outstanding issue in finance for thirty years, since the no-arbitrage option pricing model was established by Black and Scholes (1973) and Merton (1973). The difficulty comes from the early-exercise feature in the contract. Analytically describing the critical stock price is challenging.

This paper solves the problem near expiration by using a perturbation method with an assumption of small $\sigma^2 T$. We have obtained an analytical formula in a series form for the critical stock price of American option. We also present a duality relationship between the critical stock price of an American call and that of an American put with the same strike price. With the present analytical formula, one is able to price and hedge American options by using the analytical integration formula. We have also performed comprehensive numerical tests on the accuracy of the early-exercise boundary and the option price computed by using the present formulas with some truncated terms. The numerical tests show that our formula with up to four terms, one or two terms in some cases, is sufficient in pricing short-term American options with maturities of one or two months. The error is under half a cent if the formula is used to price liquidly traded OEX options with a maturity of about one month, a strike price of about 400 dollars, moneyness under 10% and underlying volatility from 0.2 to 0.5. The present formula is now ready for traders to use in pricing OEX options and for the CBOE to use in computing VXO, since the volatility index is defined as the implied volatility of one-month at-the-money OEX options.

The perturbation method presented in this paper can be used to study the price of other derivatives[6] and convertible bonds with some embedded American options. The extension of the present method to price a long-term American option, such as long-term equity anticipation securities (LEAPS) with maturities up to three years, is left for further

[6]For example, Zhang (2003) studies the price of continuously sampled Asian options by using the perturbation method.

research.

Appendix

A Proof of Theorem 1

Applying a standard transformation $S \leftarrow x$, $t \leftarrow t^*$, $P(S,t) \leftarrow p(x,t^*)$ and $B_t^p \leftarrow s(t^*)$ as

$$S = Ke^x, \qquad t = T - \frac{2}{\sigma^2}t^*, \qquad P(S,t) = Kp(x,t^*), \qquad B_t^p = Ke^{s(t^*)},$$

to equations (2, 3, 4, 5) yields

$$\begin{cases} p_{t^*} - p_{xx} - (r^* - q^* - 1)p_x + r^*p = 0, & \text{for } t^* > 0, \quad x > s(t^*), \\ p(x,t^*) = 1 - e^x, \qquad p_x(x,t^*) = -e^x, & \text{for } t^* > 0, \quad -\infty < x \le s(t^*), \\ p(x,0) = \max(1-e^x, 0), \qquad \text{for } -\infty < x < \infty, & \\ s(0) = \begin{cases} 0 & \text{if } r^* \ge q^* \\ \ln \dfrac{r^*}{q^*} & \text{if } r^* < q^* \end{cases}, & \end{cases} \tag{23}$$

where

$$r^* = \frac{2r}{\sigma^2}, \qquad q^* = \frac{2q}{\sigma^2}.$$

For simplicity of notation, from now on, we drop the star in variables t^*, r^* and q^* while keeping in mind that they denote the dimensionless time to maturity, interest rate and dividend yield. We denote by $\mathcal{L}$ the operator

$$\mathcal{L}[p] = p_{xx} + (r - q - 1)p_x - rp, \tag{24}$$

and by $\Gamma(x,t)$ the fundamental solution to the operator $\partial_t - \mathcal{L}$, more precisely

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{ -\frac{[x + (r-q-1)t]^2}{4t} - rt \right\}. \tag{25}$$

Applying Green's identity to equation (23) gives us a formula for the American put option price, $p(x,t)$, in terms of the free boundary $s(t)$,

$$p(x,t) = \int_{-\infty}^{0} (1-e^y)\Gamma(x-y,t)dy + \int_0^t \int_{-\infty}^{s(t-\tau)} (r - qe^y)\Gamma(x-y,\tau)dyd\tau, \quad x \in R, \ t > 0, \tag{26}$$

where the first term gives the Black-Scholes formula for the European put and the second term is the early-exercise premium. The double integration in the second term can not be carried out without knowledge of the free boundary. The formula is equivalent to equation (7) presented in the paper.

In order to solve the free boundary analytically, we need an equation for the boundary only. We now construct a few such equations. Since $\Gamma(\cdot, 0)$ is the Delta function centered at the origin, using

$$\begin{aligned}\Gamma_\tau(x-y,\tau) &= \Gamma_{xx}(x-y,\tau) + (r-q-1)\Gamma_x(x-y,\tau) - r\Gamma(x-y,\tau)\\ &= \Gamma_{yy}(x-y,\tau) - (r-q-1)\Gamma_y(x-y,\tau) - r\Gamma(x-y,\tau)\end{aligned}$$

and integrating by parts, we have the following equality:

$$\begin{aligned}\int_{-\infty}^{0}(1-e^y)\Gamma(x-y,t)dy &= \max(1-e^x,0) + \int_0^t\int_{-\infty}^{0}(1-e^y)\Gamma_\tau(x-y,\tau)dyd\tau\\ &= \max(1-e^x,0) + \int_0^t\left[\Gamma(x,\tau) - \int_{-\infty}^{0}\Gamma(x-y,\tau)(r-qe^y)dy\right]d\tau. \qquad (27)\end{aligned}$$

Substituting the identity into (26) gives us the following option pricing formula for $x \in R$ and $t > 0$:

$$p(x,t) = \max(1-e^x,0) + \int_0^t\left[\Gamma(x,\tau) - \int_{s(t-\tau)}^{0}(r-qe^y)\Gamma(x-y,\tau)dy\right]d\tau, \qquad (28)$$

where the first term is the intrinsic value of the American put and the second term is its time value. If the second term is zero, the American put should be exercised immediately. Differentiating (28) with respect to t yields

$$\begin{aligned}p_t(x,t) &= \Gamma(x,t) - \int_{s(0)}^{0}(r-qe^y)\Gamma(x-y,t)dy\\ &\quad + \int_0^t(r-qe^{s(t-\tau)})\Gamma(x-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \qquad (29)\end{aligned}$$

Differentiating (28) with respect to x and t yields

$$\begin{aligned}p_{xt}(x,t) &= \Gamma_x(x,t) - \int_{s(0)}^{0}(r-qe^y)\Gamma_x(x-y,t)dy\\ &\quad + \int_0^t(r-qe^{s(t-\tau)})\Gamma_x(x-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \qquad (30)\end{aligned}$$

At the free boundary, $p(x,t)$ and $p_x(x,t)$ are continuous. In fact,

$$p(x,t)|_{x=s^+(t)} = p(x,t)|_{x=s^-(t)} = (1-e^x)|_{x=s^-(t)} = 1-e^{s(t)}, \tag{31}$$

$$p_x(x,t)|_{x=s^+(t)} = p_x(x,t)|_{x=s^-(t)} = -e^x|_{x=s^-(t)} = -e^{s(t)}. \tag{32}$$

Taking total differentiation of (31) and (32) with respect to t at the boundary yields

$$\dot{s}(t)p_x(x,t)|_{x=s^+(t)} + p_t(x,t)|_{x=s^+(t)} = -\dot{s}(t)e^{s(t)}, \tag{33}$$

$$\dot{s}(t)p_{xx}(x,t)|_{x=s^+(t)} + p_{xt}(x,t)|_{x=s^+(t)} = -\dot{s}(t)e^{s(t)} \tag{34}$$

Substituting equation (32) into (33) gives

$$p_t(x,t)|_{x=s^+(t)} = 0. \tag{35}$$

Taking the limit $x \to s^+(t)$ of the first equation in (23) gives

$$\begin{aligned} p_{xx}(x,t)|_{x=s^+(t)} &= [p_t(x,t) - (r-q-1)p_x(x,t) + rp(x,t)]|_{x=s^+(t)} \\ &= 0 + (r-q-1)e^{s(t)} + r(1-e^{s(t)}) = r-(q+1)e^{s(t)}. \end{aligned} \tag{36}$$

Substituting equation (36) into (34) gives

$$p_{xt}(x,t)|_{x=s^+(t)} = -(r-qe^{s(t)})\dot{s}(t). \tag{37}$$

Applying the two equations (29, 30) at the free boundary, $x = s^+(t)$, and using the two conditions in equations (35) and (37), we have

$$\begin{aligned} \Gamma(s(t),t) &= \int_{s(0)}^0 (r-qe^y)\Gamma(s(t)-y,t)dy \\ &\quad - \int_0^t (r-qe^{s(t-\tau)})\Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau, \end{aligned} \tag{38}$$

$$\begin{aligned} (r-qe^{s(t)})\dot{s}(t) &= -2\Gamma_x(s(t),t) + 2\int_{s(0)}^0 (r-qe^y)\Gamma_x(s(t)-y,t)dy \\ &\quad -2\int_0^t (r-qe^{s(t-\tau)})\Gamma_x(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \end{aligned} \tag{39}$$

In taking the limit for p_{xt}, we need the following fact (see, e.g., Cannon 1984, Lemma 14.2.3.-14.2.5., pp. 218-223): for any continuous function f,

$$\lim_{x\to s^+(t)} \int_0^t f(t-\tau)\Gamma_x(x-s(t-\tau),\tau)d\tau = -\frac{f(t)}{2} + \int_0^t f(t-\tau)\Gamma_x(s(t)-s(t-\tau),\tau)d\tau.$$

Since

$$\Gamma_x(x,t) = -\frac{x+(r-q-1)t}{2t}\Gamma(x,t),$$

adding equations (39) and (38) multiplied by $[s(t)-s(0)+2(r-q-1)t]/(2t)$ gives

$$\begin{aligned}(r-qe^{s(t)})\dot{s}(t) &= \frac{s(t)+s(0)}{2t}\Gamma(s(t),t)\\ &\quad -\int_{s(0)}^{0}(r-qe^{y})\left[\frac{s(t)-y}{t}-\frac{s(t)-s(0)}{2t}\right]\Gamma(s(t)-y,t)dy\\ &\quad +\int_0^t (r-qe^{s(t-\tau)})\left[\frac{s(t)-s(t-\tau)}{\tau}-\frac{s(t)-s(0)}{2t}\right]\\ &\qquad \Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \end{aligned} \tag{40}$$

This is the integro-differential equation that we use to solve for the free boundary, $s(t)$. The equation for the special case without a dividend, i.e., $q=0$, was derived and used by Chen and Chadam (2007).

Now, we study the asymptotic solution of the free boundary near expiration for three cases: $r > q \geq 0$, $r = q$ and $r < q$.

A.1 Case 1: $r > q \geq 0$

For this case, $s(0)=0$, equation (40) becomes

$$\begin{aligned}(r-qe^{s(t)})\dot{s}(t) &= \frac{s(t)}{2t}\Gamma(s(t),t)+\int_0^t (r-qe^{s(t-\tau)})\left[\frac{s(t)-s(t-\tau)}{\tau}-\frac{s(t)}{2t}\right]\\ &\qquad \Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau, \end{aligned} \tag{41}$$

or

$$\begin{aligned} s(t)\left[\dot{s}(t)-\frac{s(t)}{2t(r-qe^{s(t)})}\Gamma(s(t),t)\right] = s(t)\int_0^t \frac{r-qe^{s(t-\tau)}}{r-qe^{s(t)}}\left[\frac{s(t)-s(t-\tau)}{\tau}-\frac{s(t)}{2t}\right]\\ \Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \end{aligned} \tag{42}$$

Setting

$$s(t) = -\sqrt{4u(\xi)t}, \qquad \xi = \ln\sqrt{4\pi(r-q)^2 t}, \tag{43}$$

by assuming $\tau = \frac{4zt}{(1+z)^2}$, we transform equation (42) into the new variables (u,ξ),

$$u' + 2u(1-e^{-u-\xi+a}) = -\int_0^1 \left(\frac{1+z^2}{z}\sqrt{u}-\frac{1-z^2}{z}\sqrt{\hat{u}}\right)\left(1+\frac{\hat{u}'}{2\hat{u}}\right)\frac{\sqrt{-\xi\hat{u}}}{\sqrt{\pi z}}e^{\xi z - b}dz \tag{44}$$

or

$$u=-\xi-\ln\left\{1+\frac{u'}{2u}+\frac{1}{2u}\int_0^1\left(\frac{1+z^2}{z}\sqrt{u}-\frac{1-z^2}{z}\sqrt{\hat{u}}\right)\left(1+\frac{\hat{u}'}{2\hat{u}}\right)\frac{\sqrt{-\xi\hat{u}}}{\sqrt{\pi z}}e^{\xi z-b}dz\right\}+a, \quad (45)$$

where $u=u(\xi)$, $u'=u'(\xi)$, $\hat{u}=u(\xi+\ln\frac{1-z}{1+z})$, $\hat{u}'=u'(\xi+\ln\frac{1-z}{1+z})$, and

$$\begin{aligned}
a &= \frac{(r-q-1)\sqrt{u}e^{\xi}}{2\sqrt{\pi}(r-q)}-\frac{(r-q-1)^2+4r}{16\pi(r-q)^2}e^{2\xi}-\ln(1+\frac{q}{r-q}(1-e^{-\frac{\sqrt{u}e^{\xi}}{\sqrt{\pi}(r-q)}})),\\
b &= \ln\sqrt{\frac{-\xi}{u}}+(u+\xi)z+\ln(1+z)+\frac{1-z}{2}(u-\hat{u})+\frac{1-z^2}{4z}(\sqrt{u}-\sqrt{\hat{u}})^2\\
&\quad -\frac{(r-q-1)e^{\xi}}{2\sqrt{\pi}(r-q)}(\sqrt{u}-\frac{1-z}{1+z}\sqrt{\hat{u}})+\frac{[(r-q-1)^2+4r]ze^{2\xi}}{4(r-q)^2\pi(1+z)^2}\\
&\quad -\ln(1+\frac{q}{r-q}(1-e^{-\frac{1-z}{1+z}\frac{\sqrt{\hat{u}}e^{\xi}}{\sqrt{\pi}(r-q)}}))+\ln(1+\frac{q}{r-q}(1-e^{-\frac{\sqrt{u}e^{\xi}}{\sqrt{\pi}(r-q)}})).
\end{aligned}$$

The problem becomes similar to the case without dividends that has been studied by Chen and Chadam (2007). By starting with $u=-\xi$ and successively replacing u on the right-hand side of (45) by its previous expansion, we obtain the asymptotic expansion for $u(\xi)$ near $t=0$:

$$u(\xi)=-\xi+\sum_{i=1}^{\infty}\frac{\alpha_i}{\xi^i}=-\xi-\frac{1}{2\xi}+\frac{1}{8\xi^2}+\frac{11}{24\xi^3}+O\left(\frac{1}{\xi^4}\right), \quad (46)$$

The key here is that the right-hand side of (45) produces a unique $n+1^{st}$ order expansion, if an n^{th} order expansion of u is given, because of the denominator $2u$.

A.2 Case 2: $r=q$

In this case, $r=q$, $s(0)=0$, equation (40) becomes

$$\begin{aligned}
r(1-e^{s(t)})\dot{s}(t) &= \frac{s(t)}{2t}\Gamma(s(t),t)+\int_0^t r(1-e^{s(t-\tau)})\left[\frac{s(t)-s(t-\tau)}{\tau}-\frac{s(t)}{2t}\right]\\
&\qquad \Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau, \quad (47)
\end{aligned}$$

or

$$\begin{aligned}
(1-e^{s(t)})\dot{s}(t)-\frac{s(t)}{2tr}\Gamma(s(t),t) &= \int_0^t (1-e^{s(t-\tau)})\left[\frac{s(t)-s(t-\tau)}{\tau}-\frac{s(t)}{2t}\right]\\
&\qquad \Gamma(s(t)-s(t-\tau),\tau)\dot{s}(t-\tau)d\tau. \quad (48)
\end{aligned}$$

Setting

$$s(t)=-\sqrt{4v(\eta)t}, \qquad \eta=\ln(4tr\sqrt{\pi}), \quad (49)$$

by assuming $\tau = \frac{2zt}{1+z}$, we transform equation (48) into the new variables (v, η),

$$H + 2\sqrt{v}\exp[-v-\eta+c] = -\int_0^1 (1+\frac{\hat{v}'}{\hat{v}})F dz \tag{50}$$

or

$$v = -\eta - \ln\left\{\frac{1}{2\sqrt{v}}\left[-H - \int_0^1 (1+\frac{\hat{v}'}{\hat{v}})F dz\right]\right\} + c, \tag{51}$$

where $v = v(\eta)$, $v' = v'(\eta)$, $\hat{v} = v(\eta + \ln\frac{1-z}{1+z})$, $\hat{v}' = v'(\eta + \ln\frac{1-z}{1+z})$, and

$$\begin{aligned}
c &= -\frac{\sqrt{v}e^{\frac{\eta}{2}}}{2\sqrt{r\sqrt{\pi}}} - \frac{1+4r}{16r\sqrt{\pi}}e^{\eta},\\
H &= 2\sqrt{r\sqrt{\pi}}e^{-\frac{\eta}{2}}(e^{-\frac{\sqrt{v}e^{\frac{\eta}{2}}}{\sqrt{r\sqrt{\pi}}}}-1)(\sqrt{v}+\frac{v'}{\sqrt{v}}),\\
F &= \sqrt{\frac{2r}{\sqrt{\pi}}}\frac{e^{-\frac{\eta}{2}}}{(1+z)\sqrt{z(1-z)}}\left(1-e^{-\frac{\sqrt{\hat{v}}}{\sqrt{r\sqrt{\pi}}}e^{\frac{\eta}{2}}\sqrt{\frac{1-z}{1+z}}}\right)\sqrt{\hat{v}}\left[\sqrt{v}+\frac{1+z}{z}\left(\sqrt{\frac{1-z}{1+z}}\sqrt{\hat{v}}-\sqrt{v}\right)\right]\\
&\quad \exp\left[-\frac{1+z}{2z}\left(\sqrt{\frac{1-z}{1+z}}\sqrt{\hat{v}}-\sqrt{v}\right)^2 + \frac{e^{\frac{\eta}{2}}}{2\sqrt{r\sqrt{\pi}}}\left(\sqrt{\frac{1-z}{1+z}}\sqrt{\hat{v}}-\sqrt{v}\right) - \frac{(1+4r)e^{\eta}}{8r\sqrt{\pi}}\frac{z}{1+z}\right].
\end{aligned}$$

In the same way as in Case 1, we solve (51) by starting with $v = -\eta$ and successively replacing v on the right-hand side by its previous expansion. We then obtain the asymptotic expansion for $v(\eta)$ near $t = 0$:

$$v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1-\frac{5}{4\sqrt{2\pi}}}{\eta} + o\left(\frac{1}{\eta}\right). \tag{52}$$

A.3 Case 3: $r < q$

In this case, $s(0) = \ln r - \ln q < 0$, equation (40) becomes

$$\begin{aligned}
(1-e^{\tilde{s}(t)})\dot{\tilde{s}}(t) &= \frac{\tilde{s}(t)+2s(0)}{2rt}\Gamma(\tilde{s}(t)+s(0),t)\\
&\quad + \int_0^t (1-e^{\tilde{s}(t-\tau)})\left[\frac{\tilde{s}(t)-\tilde{s}(t-\tau)}{\tau} - \frac{\tilde{s}(t)}{2t}\right]\Gamma(\tilde{s}(t)-\tilde{s}(t-\tau),\tau)\dot{\tilde{s}}(t-\tau)d\tau\\
&\quad - \int_0^{-s(0)}(1-e^y)\left[\frac{\tilde{s}(t)}{2t} - \frac{y}{t}\right]\Gamma(\tilde{s}(t)-y,t)dy,
\end{aligned} \tag{53}$$

where $s(t) = s(0) + \tilde{s}(t)$. We adopt the following singular perturbation scheme for small t:

$$\tilde{s}(t) = -2\sqrt{t}\, w(\sqrt{t}). \tag{54}$$

Since, for any fixed $\epsilon \in (0,1)$

$$\frac{\tilde{s}(t)+2s(0)}{2rt}\Gamma(\tilde{s}(t)+s(0),t) = O(t^{-\frac{3}{2}})\ e^{-\frac{\epsilon s^2(0)}{4t}}, \tag{55}$$

we obtain from (53) the asymptotic expansion for $w(\sqrt{t})$ near $t=0$

$$w(\sqrt{t}) = \beta_0 + \beta_1\sqrt{t} + \beta_2 t + \beta_3 t^{3/2} + O(t^2), \tag{56}$$

where β_i, $i=0,1,2,3$ satisfy the equations in Appendix C. Solving these equations gives

$$\beta_0 = 0.451723, \qquad \beta_1 = 0.144914\ (r-q),$$
$$\beta_2 = -0.009801 - 0.041764\ (r+q) + 0.014829\ (r-q)^2,$$
$$\beta_3 = -0.000618 - 0.002087\ (r-q) - 0.015670\ (r^2-q^2) - 0.001052\ (r-q)^3.$$

B Proof of Theorem 2

The price of an American call option, $C(S,t)$, satisfies the PDE, boundary conditions and final conditions in equations (12, 13, 14, 15, 16). By applying the transformation

$$S = \frac{K^2}{\tilde{S}}, \qquad C(S,t) = \frac{K}{\tilde{S}}P(\tilde{S},t), \qquad B^c(t) = \frac{K^2}{B^p(t)},$$

we convert the problem of pricing an American call to a problem of pricing an American put with the same strike price but with a new interest rate, $\tilde{r} = q$, and a new dividend yield, $\tilde{q} = r$, i.e.,

$$\begin{aligned}
&P_t + \frac{1}{2}\sigma^2\tilde{S}^2 P_{\tilde{S}\tilde{S}} + (\tilde{r}-\tilde{q})\tilde{S}P_{\tilde{S}} - \tilde{r}P = 0, \qquad \text{for}\ \ t<T,\ \ \tilde{S} > B^p_t,\\
&P(\tilde{S},t) = K - \tilde{S}, \qquad P_{\tilde{S}}(\tilde{S},t) = -1, \qquad \text{for}\ \ t<T,\ \ 0<\tilde{S}\le B^p_t,\\
&P(\tilde{S},T) = \max(K-\tilde{S},0),\\
&B^p_T = \begin{cases} K & \text{if}\ \ \tilde{r}\ge\tilde{q}\\ \frac{\tilde{r}}{\tilde{q}}K & \text{if}\ \ \tilde{r}<\tilde{q}\end{cases}.
\end{aligned}$$

With the solution of the critical stock price, $B^p_t(q,r,\sigma,K,T)$, of an American put given by Theorem 1, we can obtain the critical stock price for an American call by

$$B^c_t(r,q,\sigma,K,T) = \frac{K^2}{B^p_t(q,r,\sigma,K,T)}.$$

Note. The duality relation

$$C(S,t;r,q,\sigma,K,T) = \frac{S}{K}P\left(\frac{K^2}{S},t;q,r,\sigma,K,T\right) = SKP\left(\frac{1}{S},t;q,r,\sigma,\frac{1}{K},T\right)$$

is also called *put-call symmetry*. It was first discovered by Grabbe (1983) in the case of foreign-exchange options, where it has a natural interpretation. Building on the earlier work of Grabbe (1983), McDonald and Schroder (1990, 1998) recognized the relationship for American options in the binomial model. A review is offered by Carr and Chesney (1996). To the best of our knowledge, a clear proof in the PDE framework has not been offered before in the literature.

C Equations for β_i, $i = 0, 1, 2, 3$

The equation to determine β_0 is

$$-2\beta_0^2 = \frac{e^{-\beta_0^2}}{\sqrt{\pi}}\beta_0 - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{-\beta_0} e^{-z^2}dz\ (1+\beta_0^2) + \int_0^1 \frac{1}{\sqrt{\pi z}}e^{-\frac{z\beta_0^2}{(1+\sqrt{1-z})^2}}\frac{\beta_0^3 z}{(1+\sqrt{1-z})^2}dz.$$

The equation to determine β_1 is

$$\begin{aligned}-6\beta_0\beta_1 = &-\frac{e^{-\beta_0^2}}{\sqrt{\pi}}(2r-2q-3\beta_1) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\beta_0} e^{-z^2}dz\ (4\beta_0\beta_1 - 3\beta_0 r + 3\beta_0 q)\\ &+\int_0^1 \frac{1}{\sqrt{\pi z}}e^{-\frac{z\beta_0^2}{(1+\sqrt{1-z})^2}}\left[\beta_0^4(r-q-2\beta_1)\frac{(1-\sqrt{1-z})^2}{1+\sqrt{1-z}}\right.\\ &\left.+3\beta_0^2\beta_1\frac{\sqrt{1-z}-1+z}{1+\sqrt{1-z}} + \beta_0^2\beta_1\right]dz.\end{aligned}$$

The equation to determine β_2 is

$$\begin{aligned}&-\frac{4}{3}(\beta_0^4 - 6\beta_0^2\beta_1 + 3\beta_1^2 + 6\beta_0\beta_2)\\ &= \frac{e^{-\beta_0^2}}{12\sqrt{\pi}}\left[8\beta_0^3 + (5+18r-30q-3r^2+6rq-3q^2)\beta_0\right.\\ &\qquad \left.-12(4-r+q)\beta_0\beta_1 - 12\beta_0\beta_1^2 + 36\beta_2\right]\\ &-\frac{1}{3\sqrt{\pi}}\int_{-\infty}^{-\beta_0} e^{-z^2}dz\ \left[4\beta_0^4 - 6\beta_0^2(-1-r+2q+3\beta_1) + 3(-r-q+r^2-2rq+q^2)\right.\\ &\qquad \left.+3(-2-3r+3q)\beta_1 + 6\beta_1^2 + 12\beta_0\beta_2\right]\\ &+\int_0^1 \frac{1}{\sqrt{\pi z}}e^{-\frac{z\beta_0^2}{(1+\sqrt{1-z})^2}}\ \left[\frac{1}{3}\beta_0(3\sqrt{1-z}(-\beta_0^2+\beta_1)\right.\end{aligned}$$

$$
\begin{aligned}
&(\beta_1 - \frac{1}{z}(-2 + 2\sqrt{1-z} + z)(2\sqrt{1-z}\beta_1 + (-1 + \sqrt{1-z})\beta_0^2(1 - r + q + 2\beta_1))) \\
&+ \frac{1}{z}(-1+z)(-2 + 2\sqrt{1-z} + z)\beta_0(2\beta_0^3 - 6\beta_0\beta_1 + 3\beta_2) \\
&+ 3(-\beta_1(-2\sqrt{1-z}\beta_1 - (-1 + \sqrt{1-z})\beta_0^2(1 - r + q + 2\beta_1)) \\
&- \frac{1}{z}(-2 + 2(1-z)^{3/2} + z)\beta_0\beta_2 + \frac{1}{z}((-2 + 2\sqrt{1-z} + z)\beta_0(\\
&- 2(-1 + \sqrt{1-z})\sqrt{1-z}\beta_0\beta_1(1 - r + q + 2\beta_1) + 3(-1+z)\beta_2 \\
&- \frac{1}{2}\beta_0((-1 + \sqrt{1-z})^2\beta_0^2(1 - r + q + 2\beta_1)^2 + 2(-rz - \frac{z}{4}(1 - r + q + 2\beta_1)^2 \\
&- \frac{2}{z}(2 - 2\sqrt{1-z} + (-2 + \sqrt{1-z})z + z^2)\beta_0\beta_2))))))\Big]\, dz.
\end{aligned}
$$

The equation to determine β_3 is

$$
\begin{aligned}
&\frac{2}{3}(\beta_0^5 - 10\beta_0^3\beta_1 + 15\beta_0^2\beta_2 - 15\beta_1\beta_2 + 15\beta_0(\beta_1^2 - \beta_3)) \\
&\quad = \frac{e^{-\beta_0^2}}{12\sqrt{\pi}}\Big[-4\beta_0^4 + (5 + 9q^2 + 6r + 9r^2 - 6q(7+3r))\beta_1 + 12(-3 + q - r)\beta_1^2 \\
&\quad + 4\beta_1^3 + \beta_0^2(-5 + q^3 - 3q^2(-1+r) - 11r + 3r^2 - r^3 + q(23 - 6r + 3r^2) \\
&\quad + 2(19 + 3q^2 - 6q(-1+r) - 6r + 3r^2)\beta_1 + 12(1 + q - r)\beta_1^2 + 8\beta_1^3) \\
&\quad -12\beta_0(4 + q - r + 2\beta_1)\beta_2 + 2(q + q^3 + 3qr^2 - 3q^2(2+r) - r(1 - 6r + r^2)18\beta_3))\Big] \\
&\quad + \frac{1}{6\sqrt{\pi}}\int_{-\infty}^{-\beta_0} e^{-z^2} dz\ \Big[4\beta_0^5 - 4\beta_0^3(-2 + 5q - 2r + 8\beta_1) + 36\beta_0^2\beta_2 \\
&\quad - 6(-2 + 3q - 3r + 4\beta_1)\beta_2 \\
&\quad +3\beta_0(-4q + 5q^2 - 4qr - r^2 + 8(-1 + 2q - r)\beta_1 + 12\beta_1^2 - 8\beta_3)\Big] \\
&\quad + \int_0^1 \frac{1}{\sqrt{\pi z}} e^{-\frac{z\beta_0^2}{(1+\sqrt{1-z})^2}}\ \Big[\frac{1}{3}(-1+z)\beta_0 \\
&\quad (\beta_1 - \frac{1}{z}(-2 + 2\sqrt{1-z} + z)(2\sqrt{1-z}\beta_1 + (-1 + \sqrt{1-z})\beta_0^2(1 + q - r + 2\beta_1))) \\
&\quad (-2\beta_0^3 + 6\beta_0\beta_1 - 3\beta_2) \\
&\quad + \sqrt{1-z}(-\beta_0^2 + \beta_1)(-\beta_1(-2\sqrt{1-z}\beta_1 - (-1 + \sqrt{1-z})\beta_0^2(1 + q - r + 2\beta_1)) \\
&\quad - \frac{1}{z}(-2 + 2(1-z)^{3/2} + z)\beta_0\beta_2 \\
&\quad + \frac{1}{z}((-2 + 2\sqrt{1-z} + z)\beta_0(-2(-1 + \sqrt{1-z})\sqrt{1-z}\beta_0\beta_1(1 + q - r + 2\beta_1) \\
&\quad + 3(-1+z)\beta_2 - \frac{1}{2}\beta_0((-1 + \sqrt{1-z})^2\beta_0^2(1 + q - r + 2\beta_1)^2 \\
&\quad + 2(-rz - \frac{z}{4}(1 + q - r + 2\beta_1)^2 - \frac{2}{z}(2 - 2\sqrt{1-z} + (-2 + \sqrt{1-z})z + z^2)\beta_0\beta_2)))))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3z}(1-z)^{3/2}(-2+2\sqrt{1-z}+z)\beta_0^2(\beta_0^4-6\beta_0^2\beta_1+3\beta_1^2+6\beta_0\beta_2-3\beta_3)\\
&+\beta_0(\frac{1}{z}(-2+2(1-z)^{3/2}+z)(-2\sqrt{1-z}\beta_1-(-1+\sqrt{1-z})\beta_0^2(1+q-r+2\beta_1))\beta_2\\
&-\beta_1(-2(-1+\sqrt{1-z})\sqrt{1-z}\beta_0\beta_1(1+q-r+2\beta_1)\\
&+3(-1+z)\beta_2-\frac{1}{2}\beta_0((-1+\sqrt{1-z})^2\beta_0^2(1+q-r+2\beta_1)^2\\
&+2(-rz-\frac{z}{4}(1+q-r+2\beta_1)^2-\frac{2}{z}(2-2\sqrt{1-z}+(-2+\sqrt{1-z})z+z^2)\beta_0\beta_2)))\\
&-(-3+2z)\beta_0\beta_3+\frac{1}{z}((-2+2\sqrt{1-z}+z)\beta_0(3(-1+\sqrt{1-z})(-1+z)\beta_0\\
&(1+q-r+2\beta_1)\beta_2-\sqrt{1-z}\beta_1((-1+\sqrt{1-z})^2\beta_0^2(1+q-r+2\beta_1)^2\\
&+2(-rz-\frac{z}{4}(1+q-r+2\beta_1)^2-\frac{2}{z}(2-2\sqrt{1-z}+(-2+\sqrt{1-z})z+z^2)\beta_0\beta_2))\\
&-4(1-z)^{3/2}\beta_3-\frac{1}{6}\beta_0(4(-1+\sqrt{1-z})\beta_0(1+q-r+2\beta_1)\\
&(-rz-\frac{z}{4}(1+q-r+2\beta_1)^2-\frac{2}{z}(2-2\sqrt{1-z}+(-2+\sqrt{1-z})z+z^2)\beta_0\beta_2)\\
&+(-1+\sqrt{1-z})\beta_0(1+q-r+2\beta_1)((-1+\sqrt{1-z})^2\beta_0^2(1+q-r+2\beta_1)^2\\
&+2(-rz-\frac{z}{4}(1+q-r+2\beta_1)^2-\frac{2}{z}(2-2\sqrt{1-z}+(-2+\sqrt{1-z})z+z^2)\beta_0\beta_2))\\
&+6((-1+(1-z)^{3/2})(1+q-r+2\beta_1)\beta_2-2(-1+\sqrt{1-z})(-2+z)\beta_0\beta_3)))))\Big]\,dz.
\end{aligned}
$$

References

[1] Barone-Adesi, G., and R. E. Whaley (1987): Efficient Analytic Approximation of American Option Values, *J. Finance* 42, 301-320.

[2] Black, F., and M. Scholes (1973): The Pricing of Options and Corporate Liabilities, *J. Polit. Econ.* 81, 637-654.

[3] Brennan, M. J., and E. S. Schwartz (1977): The Valuation of American Options, *J. Finance* 32, 449-462.

[4] Broadie, M., and J. Detemple (1996): American Option Valuation: New bounds, Approximations, and a Comparison of Existing Methods, *Rev. Financ. Stud.* 9, 1211-1250.

[5] Bunch, D. S., and H. Johnson (1992): A Simple and Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach, *J. Finance* 47, 809-816.

[6] Bunch, D. S., and H. Johnson (2000): The American Put Option and Its Critical Stock Price, *J. Finance* 55, 2333-2356.

[7] Cannon, J. R. (1984): *The one-dimensional heat equation*, Encyclopedia of mathematics and its application, Vol. 23, Addison-Wesley Publishing Company.

[8] Carr, P. (1998): Randomization and the American Put, *Rev. Financ. Stud.* 11, 597-626.

[9] Carr, P., M. Chesney (1996): *American Put Call Symmetry*, Working paper, Morgan Stanley New York and Groupe H.E.C. France.

[10] Carr, P., R. Jarrow, and R. Myneni (1992): Alternative Characterizations of American Put Otions, *Math. Finance* 2, 87-106.

[11] Chen, X.-F., J. Chadam (2007): A Mathematical Analysis for the Optimal Exercise Boundary of American Put Options, *SIAM J. Math. Analysis* 38, 1613-1641.

[12] Cox, J., S. Ross, and M. Rubinstein (1979): Option Pricing: A Simplified Approach, *J. Financ. Economics* 7, 229-263.

[13] Evans, J. D., R. A. Kuske, and J. B. Keller (2002): American Options on Assets with Divdidends Near Expiry, *Math. Finance* 12, 219-237.

[14] Geske, R., and H. Johnson (1984): The American Put Option Valued Analytically, *J. Finance* 39, 1511-1524.

[15] Grabbe, J. (1983): The Pricing of Call and Put Options on Foreign Exchange, *J. International Money and Finance* 2, 239-253.

[16] Huang, J.-Z., M. G. Subrahmanyam, and G. G. Yu (1996): Pricing and Hedging American Options: A Recursive Integration Method, *Rev. Financ. Stud.* 9, 277-300.

[17] Jacka, S. D. (1991): Optimal Stopping and the American Put, *Math. Finance* 1, 1-14.

[18] Johnson, H. E. (1983): An Analytic Approximation for the American Put Price, *J. Financ. Quant. Analysis* 18, 141-148.

[19] Ju, N.-J. (1998): Pricing an American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function, *Rev. Financ. Stud.* 11, 627-646.

[20] Kim, I. J. (1990): The Analytic Valuation of American Options, *Rev. Financ. Stud.* 3, 547-572.

[21] Kolodner, I. I. (1956): Free Boundary Problem for the Heat Equation with Applications to Problems of Change of Phase, *Communications in Pure and Applied Mathematics* 9, 1-31.

[22] Kuske, R. A., and J. B. Keller (1998): Optimal Exercise Boundary for an American Put Option, *Appl. Math. Finance* 5, 107-116.

[23] Longstaff, F. A., and E. S. Schwartz (2001): Valuing American Options by Simulation: A Simple Least-squares Approach, *Rev. Financ. Stud.* 14, 113-147.

[24] MacMillan, L. W. (1986): Analytic Approximation for the American Put Option, *Advances in Futures and Options Research* 1, 119-139.

[25] McDonald, R. L., and M. D. Schroder (1990): *A Parity Result for American Options*, Working paper, Northwestern University.

[26] McDonald, R. L., and Mark D. Schroder (1998): A Parity Result for American Options, *J. Computational Finance* 1, 5-13.

[27] McKean, H. P., Jr. (1965): Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics, *Industrial Management Review* 6, 32-39.

[28] Merton, R. (1973): The Theory of Rational Option Pricing, *Bell Journal of Economics* 4, 141-183.

[29] Parkinson, M. (1977): Option Pricing: the American Put, *J. Business* 50, 21-36.

[30] Stamicar, R., D. Ševčovič, and J. Chadam (1999): The Early Exercise Boundary for the American Put Near Expiry: Numerical Approximation, *Canadian Applied Mathematics Quarterly* 7, 427-444.

[31] Yu, G. G. (1993): *Essays on the Valuation of American Options*, Ph.D. dissertation, New York University.

[32] Zhang, J. E. (2003): Pricing Continuously Sampled Asian Options with Perturbation Method, *J. Futures Markets* 23, 535-560.

[33] Zhu, S.-P. (2006): An Exact and Explicit Solution for the Valuation of American Put Options, *Quant. Finance* 6, 229-242.

Table 1: The critical stock price, B_t^p, of an American put option with a strike price $K = 100$ — Convergence of the numerical values computed by solving the integral equation.

$$
\begin{aligned}
K - B_t^p &= Ke^{-r(T-t)}N(-d_2(B_t^p, K, T-t)) - B_t^p e^{-q(T-t)}N(-d_1(B_t^p, K, T-t)) \\
&\quad + \int_t^T [rKe^{-rs}N(-d_2(B_t^p, B_s^p, s)) - qB_t^p e^{-qs}N(-d_1(B_t^p, B_s^p, s))]ds.
\end{aligned}
$$

Parameters: $r = 0.05$, $q = 0$ and $\sigma = 0.3$

$T-t$ \ Δt	1/52	1/104	1/208	1/416	1/832	1/1664	1/3328
0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
1/52	91.4974	91.3575	91.3407	91.3356	91.3342	91.3339	91.3338
2/52	88.7423	88.7417	88.7417	88.7423	88.7427	88.7430	88.7431
3/52	86.9467	86.9524	86.9556	86.9572	86.9579	86.9582	86.9583
4/52	85.5495	85.5587	85.5627	85.5645	85.5652	85.5655	85.5657
5/52	84.3953	84.4053	84.4095	84.4112	84.4119	84.4122	84.4124
6/52	83.4060	83.4160	83.4201	83.4218	83.4225	83.4228	83.4229
7/52	82.5372	82.5470	82.5510	82.5526	82.5533	82.5535	82.5536
8/52	81.7612	81.7706	81.7744	81.7760	81.7766	81.7768	81.7769
9/52	81.0589	81.0680	81.0716	81.0731	81.0736	81.0739	81.0740
10/52	80.4169	80.4256	80.4291	80.4305	80.4310	80.4312	80.4313
11/52	79.8253	79.8336	79.8369	79.8382	79.8387	79.8390	79.8390
12/52	79.2764	79.2844	79.2875	79.2888	79.2893	79.2894	79.2895
13/52	78.7643	78.7719	78.7749	78.7761	78.7766	78.7768	78.7768
14/52	78.2842	78.2915	78.2944	78.2955	78.2960	78.2962	78.2962
15/52	77.8323	77.8393	77.8421	77.8432	77.8436	77.8438	77.8438
16/52	77.4054	77.4121	77.4148	77.4158	77.4162	77.4164	77.4165
17/52	77.0009	77.0073	77.0099	77.0109	77.0113	77.0114	77.0115

Table 2: The price, $P(S,t)$, of an American put option with a strike price $K = 100$ at time t — Convergence of the numerical values computed by solving the integral equation where B_t^p is given by Table 1.

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)}N(-d_2(S,K,T-t)) - Se^{-q(T-t)}N(-d_1(S,K,T-t)) \\ &\quad + \int_t^T [rKe^{-rs}N(-d_2(S,B_s^p,s)) - qSe^{-qs}N(-d_1(S,B_s^p,s))]ds, \end{aligned}$$

Parameters: $r = 0.05$, $q = 0$ and $\tau \doteq T - t = 1/12$

(S,σ) \ $\Delta t/\tau$	1/20	1/40	1/80	1/160	1/320
(90, 0.2)	9.9905	9.9950	9.9974	9.9987	10.0000
(95, 0.2)	5.3572	5.3567	5.3565	5.3564	5.3564
(100, 0.2)	2.1280	2.1273	2.1271	2.1270	2.1269
(105, 0.2)	0.5808	0.5804	0.5802	0.5801	0.5801
(110, 0.2)	0.1054	0.1053	0.1052	0.1052	0.1052
(90, 0.3)	10.2316	10.2312	10.2310	10.2309	10.2309
(95, 0.3)	6.2098	6.2091	6.2088	6.2087	6.2087
(100, 0.3)	3.2726	3.2718	3.2715	3.2714	3.2713
(105, 0.3)	1.4763	1.4757	1.4755	1.4754	1.4754
(110, 0.3)	0.5678	0.5674	0.5673	0.5672	0.5672
(90, 0.4)	10.8325	10.8318	10.8316	10.8315	10.8314
(95, 0.4)	7.1948	7.1940	7.1937	7.1936	7.1935
(100, 0.4)	4.4183	4.4174	4.4171	4.4170	4.4169
(105, 0.4)	2.5018	2.5011	2.5008	2.5007	2.5007
(110, 0.4)	1.3070	1.3065	1.3063	1.3063	1.3062
(90, 0.5)	11.6092	11.6085	11.6081	11.6080	11.6080
(95, 0.5)	8.2325	8.2316	8.2313	8.2312	8.2311
(100, 0.5)	5.5638	5.5630	5.5626	5.5625	5.5625
(105, 0.5)	3.5842	3.5834	3.5831	3.5830	3.5830
(110, 0.5)	2.2039	2.2033	2.2031	2.2030	2.2030

Table 3: The critical stock price, B_t^p, of an American put option with a strike price $K = 100$ — Accuracy of the present truncated analytical formulas for the case $r > q$.

$$B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)u(\xi)}}, \qquad \xi = \ln\sqrt{8\pi(r-q)^2(T-t)/\sigma^2},$$

$$\begin{aligned}
\text{TA1}: &\quad u(\xi) = -\xi, \\
\text{TA2}: &\quad u(\xi) = -\xi - \frac{1}{2\xi}, \\
\text{TA3}: &\quad u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2}, \\
\text{TA4}: &\quad u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3}.
\end{aligned}$$

Parameters: $r = 0.05$, $q = 0$ and $\sigma = 0.3$

$T-t$	Highly Accurate	TA1	TA2	TA3	TA4
0	100.0000	100.0000	100.0000	100.0000	100.0000
1/52	91.3338	91.7250	91.3105	91.2638	91.3433
2/52	88.7431	89.4132	88.6789	88.5817	88.7797
3/52	86.9583	87.8849	86.8398	86.6860	87.0404
4/52	85.5657	86.7374	85.3807	85.1640	85.7151
1/12	85.1584	86.4111	84.9496	84.7105	85.3369
5/52	84.4124	85.8224	84.1493	83.8635	84.6555
6/52	83.4229	85.0670	83.0706	82.7091	83.7910
7/52	82.5536	84.4289	82.1009	81.6573	83.0839
8/52	81.7769	83.8816	81.2128	80.6802	82.5135
1/6	81.3009	83.5580	80.6564	80.0608	82.2039
9/52	81.0740	83.4070	80.3873	79.7590	82.0696
10/52	80.4313	82.9922	79.6109	78.8798	81.7490
11/52	79.8390	82.6276	78.8736	78.0325	81.5547
12/52	79.2895	82.3060	78.1677	77.2091	81.4951
1/4	78.7768	82.0219	77.4870	76.4032	81.5853
14/52	78.2962	81.7706	76.8266	75.6098	81.8486
15/52	77.8438	81.5485	76.1825	74.8246	82.3196
16/52	77.4165	81.3528	75.5512	74.0438	83.0513
17/52	77.0115	81.1807	74.9298	73.2643	84.1288
1/3	76.8810	81.1283	74.7245	73.0043	84.5875

The "Highly Accurate" value is the converged numerical result obtained by numerically solving the following integral equation:

$$\begin{aligned}
K - B_t^p &= Ke^{-r(T-t)}N(-d_2(B_t^p, K, T-t)) - B_t^p e^{-q(T-t)}N(-d_1(B_t^p, K, T-t)) \\
&\quad + \int_t^T [rKe^{-rs}N(-d_2(B_t^p, B_s^p, s)) - qB_t^p e^{-qs}N(-d_1(B_t^p, B_s^p, s))]ds.
\end{aligned}$$

Table 4: The price, $P(S,t)$, of an American put option with a strike price $K = 100$ at time t — Accuracy of the present truncated analytical formulas for the case $r > q$.

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)}N(-d_2(S,K,T-t)) - Se^{-q(T-t)}N(-d_1(S,K,T-t)) \\ &\quad + \int_t^T [rKe^{-rs}N(-d_2(S,B_s^p,s)) - qSe^{-qs}N(-d_1(S,B_s^p,s))]ds, \end{aligned}$$

$$\text{where} \quad B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)u(\xi)}}, \qquad \xi = \ln\sqrt{8\pi(r-q)^2(T-t)/\sigma^2},$$

$$\begin{aligned} \text{TA1}: &\quad u(\xi) = -\xi, \\ \text{TA2}: &\quad u(\xi) = -\xi - \frac{1}{2\xi}, \\ \text{TA3}: &\quad u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2}, \\ \text{TA4}: &\quad u(\xi) = -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3}. \end{aligned}$$

Parameters: $r = 0.05$, $q = 0$ and $\tau \doteq T - t = 1/12$

(S, σ)	Highly Accurate	TA1	TA2	TA3	TA4
(90, 0.2)	10.0000	10.0000	10.0000	10.0000	10.0000
(95, 0.2)	5.3564	5.3770	5.3547	5.3516	5.3615
(100, 0.2)	2.1269	2.1310	2.1267	2.1262	2.1276
(105, 0.2)	0.5801	0.5807	0.5801	0.5800	0.5802
(110, 0.2)	0.1052	0.1053	0.1052	0.1052	0.1052
(90, 0.3)	10.2309	10.2484	10.2291	10.2268	10.2321
(95, 0.3)	6.2087	6.2154	6.2081	6.2072	6.2090
(100, 0.3)	3.2713	3.2737	3.2712	3.2709	3.2714
(105, 0.3)	1.4754	1.4761	1.4753	1.4752	1.4754
(110, 0.3)	0.5672	0.5674	0.5672	0.5672	0.5672
(90, 0.4)	10.8314	10.8386	10.8301	10.8292	10.8308
(95, 0.4)	7.1935	7.1969	7.1929	7.1925	7.1932
(100, 0.4)	4.4169	4.4185	4.4167	4.4165	4.4168
(105, 0.4)	2.5007	2.5013	2.5006	2.5005	2.5006
(110, 0.4)	1.3062	1.3065	1.3062	1.3061	1.3062
(90, 0.5)	11.6080	11.6117	11.6067	11.6062	11.6070
(95, 0.5)	8.2311	8.2332	8.2304	8.2302	8.2305
(100, 0.5)	5.5625	5.5636	5.5621	5.5620	5.5622
(105, 0.5)	3.5830	3.5835	3.5828	3.5827	3.5828
(110, 0.5)	2.2030	2.2032	2.2029	2.2028	2.2029
RMSE		0.0067	0.0007	0.0016	0.0012
MAE		0.0206	0.0018	0.0048	0.0051

The "Highly Accurate" value is obtained by numerically computing the integration with the highly accurate critical stock prices presented in Table 3. RMSE is the root of the mean squared errors. MAE is the maximum absolute error.

Table 5: The critical stock price, B_t^p, of an American put option with a strike price $K = 100$ — Accuracy of the present truncated analytical formulas for the case $r = q$.

$$B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)v(\eta)}}, \qquad \eta = \ln[4\sqrt{\pi}r(T-t)],$$

$$\begin{aligned}
\text{TA1}: &\quad v(\eta) = -\eta, \\
\text{TA2}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta), \\
\text{TA3}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta), \\
\text{TA4}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1-\frac{5}{4\sqrt{2\pi}}}{\eta}.
\end{aligned}$$

Parameters: $r = 0.05$, $q = 0.05$ and $\sigma = 0.3$

$T-t$	Highly Accurate	TA1	TA2	TA3	TA4
0	100.0000	100.0000	100.0000	100.0000	100.0000
1/52	88.2254	87.6863	88.6604	88.5583	88.4323
2/52	84.7617	84.1608	85.4591	85.3005	85.0857
3/52	82.3911	81.7929	83.3107	83.1055	82.8084
4/52	80.5499	79.9855	81.6695	81.4233	81.0467
1/12	80.0129	79.4655	81.1967	80.9378	80.5350
5/52	79.0311	78.5192	80.3353	80.0520	79.5972
6/52	77.7323	77.2856	79.2097	78.8923	78.3598
7/52	76.5946	76.2225	78.2362	77.8871	77.2769
8/52	75.5807	75.2903	77.3790	77.0002	76.3121
1/6	74.9605	74.7280	76.8599	76.4623	75.7220
9/52	74.6651	74.4622	76.6139	76.2072	75.4407
10/52	73.8299	73.7193	75.9236	75.4906	74.6450
11/52	73.0617	73.0473	75.2955	74.8377	73.9122
12/52	72.3503	72.4357	74.7198	74.2387	73.2322
1/4	71.6877	71.8759	74.1889	73.6858	72.5975
14/52	71.0676	71.3613	73.6969	73.1731	72.0017
15/52	70.4848	70.8865	73.2387	72.6955	71.4400
16/52	69.9351	70.4470	72.8105	72.2491	70.9081
17/52	69.4149	70.0390	72.4088	71.8305	70.4027
1/3	69.2474	69.9095	72.2803	71.6966	70.2396

The "Highly Accurate" value is the converged numerical result obtained by numerically solving the following integral equation:

$$\begin{aligned}
K - B_t^p &= Ke^{-r(T-t)}N(-d_2(B_t^p, K, T-t)) - B_t^p e^{-q(T-t)}N(-d_1(B_t^p, K, T-t)) \\
&\quad + \int_t^T [rKe^{-rs}N(-d_2(B_t^p, B_s^p, s)) - qB_t^p e^{-qs}N(-d_1(B_t^p, B_s^p, s))]ds.
\end{aligned}$$

Table 6: The price, $P(S,t)$, of an American put option with a strike price $K = 100$ at time t — Accuracy of the present truncated analytical formulas for the case $r = q$.

$$
\begin{aligned}
P(S,t) &= Ke^{-r(T-t)}N(-d_2(S,K,T-t)) - Se^{-q(T-t)}N(-d_1(S,K,T-t)) \\
&\quad + \int_t^T [rKe^{-rs}N(-d_2(S,B_s^p,s)) - qSe^{-qs}N(-d_1(S,B_s^p,s))]ds, \\
\text{where} \quad & B_t^p = Ke^{-\sqrt{2\sigma^2(T-t)v(\eta)}}, \qquad \eta = \ln[4\sqrt{\pi}r(T-t)],
\end{aligned}
$$

$$
\begin{aligned}
\text{TA1}: &\quad v(\eta) = -\eta, \\
\text{TA2}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta), \\
\text{TA3}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta), \\
\text{TA4}: &\quad v(\eta) = -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1 - \frac{5}{4\sqrt{2\pi}}}{\eta}.
\end{aligned}
$$

Parameters: $r = 0.05$, $q = 0.05$ and $\tau \doteq T - t = 1/12$

$(S,\ \sigma)$	Highly Accurate	TA1	TA2	TA3	TA4
(90, 0.2)	10.0535	10.0522	10.0554	10.0549	10.0543
(95, 0.2)	5.5603	5.5598	5.5607	5.5606	5.5604
(100, 0.2)	2.2947	2.2947	2.2949	2.2949	2.2948
(105, 0.2)	0.6537	0.6537	0.6537	0.6537	0.6537
(110, 0.2)	0.1242	0.1242	0.1242	0.1242	0.1242
(90, 0.3)	10.4167	10.4157	10.4178	10.4176	10.4172
(95, 0.3)	6.4202	6.4197	6.4205	6.4204	6.4203
(100, 0.3)	3.4415	3.4414	3.4417	3.4417	3.4416
(105, 0.3)	1.5812	1.5812	1.5812	1.5812	1.5812
(110, 0.3)	0.6197	0.6198	0.6198	0.6198	0.6198
(90, 0.4)	11.0410	11.0400	11.0418	11.0416	11.0413
(95, 0.4)	7.3994	7.3989	7.3997	7.3996	7.3995
(100, 0.4)	4.5875	4.5875	4.5878	4.5878	4.5877
(105, 0.4)	2.6239	2.6238	2.6240	2.6240	2.6239
(110, 0.4)	1.3851	1.3850	1.3851	1.3851	1.3850
(90, 0.5)	11.8191	11.8182	11.8199	11.8197	11.8194
(95, 0.5)	8.4307	8.4301	8.4310	8.4309	8.4308
(100, 0.5)	5.7326	5.7325	5.7330	5.7330	5.7329
(105, 0.5)	3.7164	3.7163	3.7165	3.7165	3.7165
(110, 0.5)	2.2998	2.2999	2.3000	2.3000	2.3000
RMSE		0.0005	0.0006	0.0005	0.0003
MAE		0.0013	0.0019	0.0014	0.0008

The "Highly Accurate" value is obtained by numerically computing the integration with the highly accurate critical stock prices presented in Table 5. RMSE is the root of the mean squared errors. MAE is the maximum absolute error.

Table 7: The critical stock price, B_t^p, of an American put option with a strike price $K = 100$ — Accuracy of the present truncated analytical formulas for the case $r < q$.

$$B_t^p = \frac{r}{q} K e^{-2\sqrt{\tau^*} w(\sqrt{\tau^*})}, \qquad \tau^* = \frac{1}{2}\sigma^2(T-t), \qquad r^* = \frac{r}{\frac{1}{2}\sigma^2}, \qquad q^* = \frac{q}{\frac{1}{2}\sigma^2},$$

$$\text{TA1}: \quad w(\sqrt{\tau^*}) = \beta_0, \qquad \text{TA2}: \quad w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*},$$

$$\text{TA3}: \quad w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^*, \qquad \text{TA4}: \quad w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\tau^{*3/2},$$

$$\beta_0 = 0.451723, \qquad \beta_1 = 0.144914\ (r^* - q^*),$$
$$\beta_2 = -0.009801 - 0.041764\ (r^* + q^*) + 0.014829\ (r^* - q^*)^2,$$
$$\beta_3 = -0.000618 - 0.002087\ (r^* - q^*) - 0.015670\ (r^{*2} - q^{*2}) - 0.001052\ (r^* - q^*)^3.$$

Parameters: $r = 0.05$, $q = 0.07$ and $\sigma = 0.3$

$T-t$	Highly Accurate	TA1	TA2	TA3	TA4
0	71.4286	71.4286	71.4286	71.4286	71.4286
1/52	69.5641	69.5552	69.5630	69.5634	69.5634
2/52	68.8106	68.7937	68.8090	68.8102	68.8102
3/52	68.2401	68.2150	68.2379	68.2400	68.2400
4/52	67.7622	67.7310	67.7612	67.7645	67.7644
1/12	67.6155	67.5841	67.6168	67.6205	67.6204
5/52	67.3340	67.3074	67.3449	67.3494	67.3494
6/52	66.9237	66.9267	66.9715	66.9774	66.9773
7/52	66.5095	66.5785	66.6305	66.6379	66.6378
8/52	66.0811	66.2560	66.3152	66.3242	66.3241
1/6	65.7871	66.0530	66.1168	66.1270	66.1268
9/52	65.6382	65.9546	66.0208	66.0315	66.0314
10/52	65.1857	65.6708	65.7440	65.7565	65.7563
11/52	64.7297	65.4019	65.4822	65.4966	65.4963
12/52	64.2757	65.1461	65.2333	65.2496	65.2494
1/4	63.8278	64.9016	64.9958	65.0141	65.0138
14/52	63.3889	64.6673	64.7683	64.7887	64.7884
15/52	62.9608	64.4420	64.5498	64.5724	64.5720
16/52	62.5446	64.2248	64.3395	64.3643	64.3638
17/52	62.1408	64.0150	64.1365	64.1635	64.1630
1/3	62.0090	63.9466	64.0703	64.0981	64.0976

The "Highly Accurate" value is the converged numerical result obtained by numerically solving the following integral equation:

$$\begin{aligned} K - B_t^p &= K e^{-r(T-t)} N(-d_2(B_t^p, K, T-t)) - B_t^p e^{-q(T-t)} N(-d_1(B_t^p, K, T-t)) \\ &\quad + \int_t^T [rKe^{-rs} N(-d_2(B_t^p, B_s^p, s)) - qB_t^p e^{-qs} N(-d_1(B_t^p, B_s^p, s))] ds. \end{aligned}$$

Table 8: The price, $P(S,t)$, of an American put option with a strike price $K = 100$ at time t — Accuracy of the present truncated analytical formulas for the case $r < q$.

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)}N(-d_2(S,K,T-t)) - Se^{-q(T-t)}N(-d_1(S,K,T-t)) \\ &\quad + \int_t^T [rKe^{-rs}N(-d_2(S,B_s^p,s)) - qSe^{-qs}N(-d_1(S,B_s^p,s))]ds, \end{aligned}$$

where $B_t^p = \frac{r}{q}Ke^{-2\sqrt{\tau^*}w(\sqrt{\tau^*})}$, $\quad \tau^* = \frac{1}{2}\sigma^2(T-t)$, $\quad r^* = \frac{r}{\frac{1}{2}\sigma^2}$, $\quad q^* = \frac{q}{\frac{1}{2}\sigma^2}$,

TA1 : $w(\sqrt{\tau^*}) = \beta_0$, TA2 : $w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*}$,
TA3 : $w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^*$, TA4 : $w(\sqrt{\tau^*}) = \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\tau^{*3/2}$,

$\beta_0 = 0.451723$, $\quad \beta_1 = 0.144914\ (r^* - q^*)$,
$\beta_2 = -0.009801 - 0.041764\ (r^* + q^*) + 0.014829\ (r^* - q^*)^2$,
$\beta_3 = -0.000618 - 0.002087\ (r^* - q^*) - 0.015670\ (r^{*2} - q^{*2}) - 0.001052\ (r^* - q^*)^3$.

Parameters: $r = 0.05$, $q = 0.07$ and $\tau \doteq T - t = 1/12$

$(S,\ \sigma)$	Highly Accurate	TA1	TA2	TA3	TA4
(90, 0.2)	10.1755	10.1755	10.1755	10.1755	10.1755
(95, 0.2)	5.6814	5.6814	5.6814	5.6814	5.6814
(100, 0.2)	2.3753	2.3754	2.3754	2.3754	2.3754
(105, 0.2)	0.6875	0.6875	0.6875	0.6875	0.6875
(110, 0.2)	0.1329	0.1329	0.1329	0.1329	0.1329
(90, 0.3)	10.5333	10.5333	10.5333	10.5333	10.5333
(95, 0.3)	6.5264	6.5263	6.5263	6.5263	6.5263
(100, 0.3)	3.5201	3.5202	3.5202	3.5202	3.5202
(105, 0.3)	1.6285	1.6285	1.6285	1.6285	1.6285
(110, 0.3)	0.6430	0.6430	0.6430	0.6430	0.6430
(90, 0.4)	11.1485	11.1485	11.1485	11.1485	11.1485
(95, 0.4)	7.4961	7.4961	7.4961	7.4961	7.4961
(100, 0.4)	4.6644	4.6645	4.6645	4.6645	4.6645
(105, 0.4)	2.6781	2.6781	2.6781	2.6781	2.6781
(110, 0.4)	1.4193	1.4193	1.4193	1.4193	1.4193
(90, 0.5)	11.9189	11.9188	11.9188	11.9188	11.9188
(95, 0.5)	8.5209	8.5209	8.5209	8.5209	8.5209
(100, 0.5)	5.8077	5.8079	5.8079	5.8079	5.8079
(105, 0.5)	3.7743	3.7744	3.7744	3.7744	3.7744
(110, 0.5)	2.3416	2.3417	2.3417	2.3417	2.3417
RMSE		0.0001	0.0001	0.0001	0.0001
MAE		0.0002	0.0002	0.0002	0.0002

The "Highly Accurate" value is obtained by numerically computing the integration with the highly accurate critical stock prices presented in Table 7. RMSE is the root of the mean squared errors. MAE is the maximum absolute error.

Table 9: The price, $P(S,t)$, of an American put option with a strike price $K = 100$ at time t — A comparison between three formulas (19), (20) and (21) for $|r-q| \ll 1$.

Parameters: $r = 0.05$ and $\tau \doteq T - t = 1/12$

$(S,\, \sigma)$	Highly Accurate	Formula1	Formula2	Formula3
(90, 0.2)	10.0535	10.0440	10.0543	10.0731
(95, 0.2)	5.5603	5.5573	5.5604	5.5746
(100, 0.2)	2.2947	2.2939	2.2948	2.2986
(105, 0.2)	0.6537	0.6535	0.6537	0.6545
(110, 0.2)	0.1242	0.1241	0.1242	0.1243
(90, 0.3)	10.4167	10.4099	10.4172	10.4432
(95, 0.3)	6.4202	6.4173	6.4203	6.4349
(100, 0.3)	3.4415	3.4404	3.4416	3.4473
(105, 0.3)	1.5812	1.5807	1.5812	1.5832
(110, 0.3)	0.6197	0.6196	0.6198	0.6204
(90, 0.4)	11.0410	11.0350	11.0413	11.0696
(95, 0.4)	7.3994	7.3964	7.3995	7.4150
(100, 0.4)	4.5875	4.5862	4.5877	4.5952
(105, 0.4)	2.6239	2.6232	2.6239	2.6274
(110, 0.4)	1.3851	1.3847	1.3850	1.3865
(90, 0.5)	11.8191	11.8134	11.8194	11.8484
(95, 0.5)	8.4307	8.4274	8.4308	8.4475
(100, 0.5)	5.7326	5.7310	5.7329	5.7422
(105, 0.5)	3.7164	3.7154	3.7165	3.7215
(110, 0.5)	2.2998	2.2994	2.3000	2.3026

The "Highly Accurate" value is obtained by numerically computing the integration with the highly accurate critical stock prices presented in Table 5 for $q = 0.05$. The "Formula1" value is obtained by using formula (19) for the region $r > q \geq 0$ with $q = 0.0499$. The "Formula2" value is obtained by using formula (20) for the region $r = q$ with $q = 0.05$. The "Formula3" value is obtained by using formula (21) for the region $r < q$ with $q = 0.0501$.