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Author(s)	Ng, CO
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Mass transport in gravity waves revisited

Chiu-On Ng
Department of Mechanical Engineering, University of Hong Kong, Hong Kong

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[1] On re-examining the problem of mass transport due to partially standing waves in a domain that can be closed or open ended, it is pointed out that several aspects of the existing theories need to be clarified. Particular attention is paid to the free surface setup, whose expression is shown to be different when described by the Eulerian and Lagrangian approaches. In the Lagrangian system, it is the horizontal gradient of the setup, rather than the mean pressure gradient alone, that is implied by the condition of no net flux in a closed domain. In connection with this, one may determine for an unbounded domain the streamline value along the free surface, which cannot be fixed in the stream function formulation alone, by requiring the setup due to the unidirectional part of the mass transport to be zero. Three components of the free surface setup, which are of different orders of magnitude and arise owing to various mechanisms, are obtained in the process of deriving the solutions for the mass transport velocity in the boundary layers and the fluid core. *INDEX TERMS:* 4560 Oceanography: Physical: Surface waves and tides (1255); 4546 Oceanography: Physical: Nearshore processes; 4203 Oceanography: General: Analytical modeling; *KEYWORDS:* mass transport, gravity surface waves, free surface setup

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1. Introduction

[2] This paper is to revisit the classical problem of mass transport in gravity waves, with a view of re-examining some of the aspects that have not been fully made clear in the literature. Since the celebrated work of *Longuet-Higgins* [1953], who gave a detailed exposition of the basic theory, the problem has been extensively studied featuring all sorts of sophistication. Some notable works may include, just to mention a few, mass transport in a two-layer system [*Dore*, 1970], in water overlying a layer of non-Newtonian mud [*Sakakiyama and Bijker*, 1989] or above a bed with a hump [*Iskandarani and Liu*, 1991], in waves with viscous spatial or temporal attenuation [e.g., *Liu and Davis*, 1977; *Wen and Liu*, 1995; *Piedra-Cueva*, 1995], and so on. No matter what the specific problem is, the basic mechanisms that induce mass transport, also known as Lagrangian mean drift, under periodic waves are essentially the same.

[3] We shall focus here on the so-called "conduction solution" of *Longuet-Higgins* [1953], for which the problem is linear and closed-form analytical solutions are available. It corresponds to the case when the ratio a/s is very small, where a is the wave amplitude and s is the thickness of the wave boundary layer, thereby resulting in the viscous diffusion of $O(a^2)$ vorticity from the boundary layers into the entire fluid core. Such a creeping flow condition will limit the solution to rather small waves if the fluid, like water, has a small viscosity. Nevertheless, the conduction solution can still be of great pertinence in practical situations when, for example, one is concerned with pollutant transport in very viscous fluid mud under waves. Some hyperconcentrated muds can be several orders of magnitude more viscous than water, and the corresponding Stokes boundary layer thickness can be much larger than the wave amplitude. Moreover, some studies [e.g., *Liu and Davis*, 1977] have found that the conduction solution turns out to have a wider range of validity than is assumed, especially in the case of progressive waves.

[4] Our specific problem is to consider partially standing waves propagating in a domain that is either closed or open at the far field. This is a combination of cases investigated separately in the past. Our emphasis is, however, not on the solution itself, but on the following aspects that deserve clarification in the course of deriving the solution. First, we shall show that a free surface setup, which is the mean level of the free surface, will have different expressions when described by the Eulerian and Lagrangian approaches. Apparently not aware of this subtle difference, *Piedra-Cueva* [1995], in his equation (6.24), has expressed an Eulerian equation for a Lagrangian setup. This is in fact inconsistent. Second, we shall make clear that in the Lagrangian system it is the horizontal gradient of a free surface setup, rather than a mean pressure gradient alone [*Ünlüata and Mei*, 1970], that is responsible for a return current in the case of a closed system. Although the order of magnitude of this setup is very small, its existence is necessary in order to balance the flow in either direction. Third, we shall point out that the stream function along the free surface is not a constant whose value is entirely at one's disposal to choose, as stated by *Iskandarani and Liu* [1991]. Since *Longuet-Higgins* [1953], a Lagrangian stream function is commonly used as the variable in the formulation of

0148-0227/04/2003JC002121

a two-dimensional mass transport problem. The stream function cannot be uniquely determined unless its value along the free surface is specified. Conventionally, this value is for convenience taken to be zero (or the same value as on the bed), physically equivalent to a zero net flux that may occur in a closed system. For an open-ended system, the stream function formulation alone falls short in establishing the free surface streamline value. We argue that this value is not completely arbitrary, but can be fixed using the fact that in the absence of a return current, the free surface setup due to the unidirectional part of the mass transport in the core should be zero. To find the setup, however, requires one to solve the momentum equations in terms of the primitive variables. In addition, we also show that for a flat bottom and over a distance comparable to a wavelength so that wave refraction and decay are negligible, the free surface setup due to the inviscid motion induced by a standing wave dominates over the other setup components.

[5] Our analysis is based on the Lagrangian approach that was first used by *Ünlüata and Mei* [1970] for mass transport in water waves, and was later extended by *Piedra-Cueva* [1995] to a two-layer system. Both studies considered only mass transport in purely progressive waves. In order to be self-contained, we present in the following sections the entire process of our deduction starting from the basics. Those parts of the analysis and solutions that are already known in the literature will not be described in great detail. Emphasis will be placed on those parts that are related to the above-mentioned aspects.

[6] Before concluding, we shall also illustrate with some numerical results the effects of the reflection coefficient on the distribution of mass transport velocity and vorticity in the core field subject to either bounding condition of the domain. These kinds of comparison do not seem to have been reported in the past. *Iskandarani and Liu* [1991] have presented similar results, but only for a closed system.

2. Formulation of the Problem

[7] We consider a single layer of homogeneous viscous fluid of depth h lying on a horizontal rigid bottom. By Lagrangian description, the instantaneous horizontal/vertical positions of a fluid particle (x, z) are functions of the initial or undisturbed coordinates (α, δ) and time t. Fluid motion is essentially due to a prescribed displacement of the free surface $\delta = 0$ in the form of a superposition of forward- and backward-going waves,

$$\eta(\alpha, t) = \Re\left\{a\left[e^{i(k\alpha-\sigma t)} + Re^{i(k\alpha+\sigma t)}\right]\right\}, \tag{1}$$

where a is the amplitude of the forward-going wave, $0 \leq R \leq 1$ is the reflection coefficient or the ratio of the wave amplitudes, k is the wave number, and σ is the angular frequency. While a, R, and σ are supposed to be real known constants, k is a complex quantity to be determined by the dispersion relation. In general, the wave is a partial standing wave with $0 < R < 1$. In the two limits, it becomes purely progressive when $R = 0$, and purely standing when $R = 1$. In this work, the terms "closed system" and "open system" are used to distinguish between the cases where a return current is generated to counterbalance the flux due to the unidirectional component of the mass transport or not. Practically, a closed system may correspond to a finite tank or channel with a far-end boundary that may perfectly or partially reflect the incident wave. For a partial standing wave in an open system, we shall mean that our point of interest is in the middle of a very long but bounded domain so that the reflected wave has reached the point but the return current is not felt yet. This is possible because the propagation speed of the wave itself is 2 orders of magnitude faster than the induced return current. As shown by *Ünlüata and Mei* [1970], the second-order mean motions can be established well before the arrival of the return current in a natural setting.

[8] It is assumed that the wavelength is comparable in magnitude with the fluid depth, i.e., $kh = O(1)$, and is much larger than the wave amplitude. The small wave steepness

$$\epsilon \equiv ka \ll 1 \tag{2}$$

will be used as the ordering parameter in the following analysis. The equations of motion and boundary conditions in Lagrangian form are obtainable from *Pierson* [1962] and *Piedra-Cueva* [1995], and are recalled as follows. Using the standard notation for a Jacobian, the mass conservation equation is

$$\frac{\partial(x,z)}{\partial(\alpha,\delta)} = 1, \tag{3}$$

and the x- and z-momentum equations are

$$\ddot{x} = -\frac{1}{\rho}\frac{\partial(p,z)}{\partial(\alpha,\delta)} + \frac{1}{\rho}\left[\frac{\partial(\tau_{xx},z)}{\partial(\alpha,\delta)} + \frac{\partial(x,\tau_{xz})}{\partial(\alpha,\delta)}\right] \tag{4}$$

$$\ddot{z} = -\frac{1}{\rho}\frac{\partial(x,p)}{\partial(\alpha,\delta)} - g + \frac{1}{\rho}\left[\frac{\partial(\tau_{zx},z)}{\partial(\alpha,\delta)} + \frac{\partial(x,\tau_{zz})}{\partial(\alpha,\delta)}\right], \tag{5}$$

and the stress components are

$$\tau_{xx} = 2\mu\frac{\partial(\dot{x},z)}{\partial(\alpha,\delta)}, \tag{6}$$

$$\tau_{zz} = 2\mu\frac{\partial(x,\dot{z})}{\partial(\alpha,\delta)}, \tag{7}$$

$$\tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial(x,\dot{x})}{\partial(\alpha,\delta)} + \frac{\partial(\dot{z},z)}{\partial(\alpha,\delta)}\right]. \tag{8}$$

In the equations above, an overdot is used to denote time derivative, g is the acceleration due to gravity, p is the pressure, ρ is the fluid density, and μ is the dynamic viscosity. As we do not consider effects of surface tension, both shear and normal stress components are zero along the free surface,

$$(\tau_{zz} - \tau_{xx})\frac{\partial x}{\partial\alpha}\frac{\partial z}{\partial\alpha} + \tau_{xz}\left[\left(\frac{\partial x}{\partial\alpha}\right)^2 - \left(\frac{\partial z}{\partial\alpha}\right)^2\right] = 0 \qquad \delta = 0 \tag{9}$$

$$-p\left[\left(\frac{\partial x}{\partial \alpha}\right)^2+\left(\frac{\partial z}{\partial \alpha}\right)^2\right]+\tau_{xx}\left(\frac{\partial z}{\partial \alpha}\right)^2+\tau_{zz}\left(\frac{\partial x}{\partial \alpha}\right)^2 - 2\tau_{xz}\frac{\partial x}{\partial \alpha}\frac{\partial z}{\partial \alpha}=0 \qquad \delta=0. \tag{10}$$

On the bottom, the particles are not displaced and the velocity components vanish,

$$\dot{x}=\dot{z}=0 \qquad \delta=-h. \tag{11}$$

On the basis of small-amplitude displacements, the variables may be expanded as follows [*Pierson*, 1962]:

$$(x,z,p)=(\alpha,\delta,-\rho g\delta)\ +\ \epsilon\left(x^{(1)},z^{(1)},p^{(1)}\right) + \epsilon^2\left(x^{(2)},z^{(2)},p^{(2)}\right)+\cdots. \tag{12}$$

Perturbation equations are obtainable for the $O(\epsilon)$ and $O(\epsilon^2)$ variables on substituting equation (12) into the Lagrangian equations above and collecting terms of equal powers of ϵ. From here on, we shall use the subscripts α, δ to denote the derivatives with respect to the corresponding spatial variables.

3. First-Order Problem

[9] At $O(\epsilon)$, the equations of motion (3)–(5) yield

$$x^{(1)}_\alpha+z^{(1)}_\delta=0, \tag{13}$$

$$\ddot{x}^{(1)}+gz^{(1)}_\alpha+\frac{1}{\rho}p^{(1)}_\alpha-\nu\left(\dot{x}^{(1)}_{\alpha\alpha}+\dot{x}^{(1)}_{\delta\delta}\right)=0, \tag{14}$$

$$\ddot{z}^{(1)}+gz^{(1)}_\delta+\frac{1}{\rho}p^{(1)}_\delta-\nu\left(\dot{z}^{(1)}_{\alpha\alpha}+\dot{z}^{(1)}_{\delta\delta}\right)=0, \tag{15}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. On the free surface, the kinematic boundary condition requires $z^{(1)}(\alpha, \delta = 0, t)$ to be equal to $\eta(\alpha, t)$ given by (1). Also, the dynamic free surface boundary conditions (9) and (10) give

$$\dot{x}^{(1)}_\delta+\dot{z}^{(1)}_\alpha=0 \qquad \delta=0 \tag{16}$$

$$-p^{(1)}+2\mu\dot{z}^{(1)}_\delta=0 \qquad \delta=0. \tag{17}$$

At the bottom the boundary conditions are simply $\dot{x}^{(1)} = \dot{z}^{(1)} = 0$ on $\delta = -h$.

[10] Solutions to this leading-order problem can readily be found on separating the variables into forward and backward propagating modes,

$$\left(x^{(1)},z^{(1)},p^{(1)}\right)=\Re\left[(\tilde{x},\tilde{z},\tilde{p})e^{i(k\alpha-\sigma t)}+R(-\tilde{x}^*,\tilde{z}^*,\tilde{p}^*)e^{i(k\alpha+\sigma t)}\right], \tag{18}$$

where an asterisk is used to denote complex conjugate, and $\tilde{x}$, $\tilde{z}$, and $\tilde{p}$ are complex functions of δ. *Dalrymple and Liu* [1978] have solved a similar first-order problem for a two-layer viscous system, and their steps can be followed here to arrive at an analytical solution,

$$\tilde{x}(\delta)=iA\left[\cosh k(\delta+h)-k\lambda^{-1}\sinh k(\delta+h).-e^{-\lambda(\delta+h)}\right] + i\lambda k^{-1}Ce^{\lambda\delta} \tag{19}$$

$$\tilde{z}(\delta)=A\left[\sinh k(\delta+h)-k\lambda^{-1}\cosh k(\delta+h).+k\lambda^{-1}e^{-\lambda(\delta+h)}\right] +Ce^{\lambda\delta}, \tag{20}$$

where A, C, and λ are complex constants given by

$$A=a\left(\frac{\lambda^2+k^2}{\lambda^2-k^2}\right)\left(\sinh kh-k\lambda^{-1}\cosh kh\right)^{-1}, \tag{21}$$

$$C=-\frac{2k^2a}{\lambda^2-k^2}, \tag{22}$$

$$\lambda=\frac{1-i}{s}, \tag{23}$$

in which s is the Stokes boundary layer thickness given by

$$s=\sqrt{2\nu/\sigma}. \tag{24}$$

The Stokes boundary layer thickness is typically much smaller than the fluid depth and the wavelength. Hence

$$|k/\lambda|=O(ks)\ll 1, \quad \textit{and} \quad |\lambda|h=O(h/s)\gg 1. \tag{25}$$

Therefore in equations (19) and (20), the terms multiplied by $e^{\lambda\delta}$ are appreciable only in the free surface boundary layer ($\delta \sim 0$), while the terms with $e^{-\lambda(\delta+h)}$ are significant only in the bottom boundary layer ($\delta \sim -h$). The other terms that contain hyperbolic functions of $k(\delta + h)$ correspond to the inviscid flow solutions.

[11] In addition, we may determine the wave number k as an eigen-value from the normal stress free surface boundary condition (17),

$$-i\sigma\left[\nu k^{-1}\lambda^2A\left(\cosh kh-k\lambda^{-1}\sinh kh\right)+2\nu\lambda C\right]+ga=0. \tag{26}$$

Expanding $k = k_1 + k_2 + \cdots$, where $k_2/k_1 = O(ks)$, we get at the leading order a real wave number satisfying the familiar dispersion relation

$$\sigma^2=gk_1\tanh k_1h, \tag{27}$$

and at the next order a complex wave number

$$k_2=(1+i)\frac{k_1^2s}{\sinh 2k_1h+2k_1h}, \tag{28}$$

the imaginary part of which corresponds to the wave attenuation rate. Since the present focus is on a horizontal length-scale comparable to the wavelength, we may ignore any effects due to k_2 that are significant only over a much longer distance. From here on, we shall not distinguish k_1 from k, which is simply taken as a real number.

4. Second-Order Problem

[12] The $O(\epsilon^2)$ governing equations are as follows:

$$\dot{x}_\alpha^{(2)} + \dot{z}_\delta^{(2)} = -\left(x_\alpha^{(1)} z_\delta^{(1)}\right)_t + \left(x_\delta^{(1)} z_\alpha^{(1)}\right)_t, \tag{29}$$

$$\ddot{x}^{(2)} + g z_\alpha^{(2)} + \frac{1}{\rho} p_\alpha^{(2)} - \nu\left(\dot{x}_{\alpha\alpha}^{(2)} + \dot{x}_{\delta\delta}^{(2)}\right) = X^{(2)}, \tag{30}$$

$$\ddot{z}^{(2)} + g z_\delta^{(2)} + \frac{1}{\rho} p_\delta^{(2)} - \nu\left(\dot{z}_{\alpha\alpha}^{(2)} + \dot{z}_{\delta\delta}^{(2)}\right) = Z^{(2)}, \tag{31}$$

where the forcing terms

$$\begin{aligned} X^{(2)} = & -\ddot{x}^{(1)} x_\alpha^{(1)} - \ddot{z}^{(1)} z_\alpha^{(1)} + \nu\Big[x_\alpha^{(1)}\left(-\dot{x}_{\alpha\alpha}^{(1)} + 3\dot{x}_{\delta\delta}^{(1)}\right) \\ & + z_\alpha^{(1)}\left(\dot{z}_{\alpha\alpha}^{(1)} + \dot{z}_{\delta\delta}^{(1)}\right) - 2\dot{x}_{\alpha\delta}^{(1)}\left(z_\alpha^{(1)} + x_\delta^{(1)}\right) - \dot{x}_\alpha^{(1)}\left(x_{\alpha\alpha}^{(1)} + x_{\delta\delta}^{(1)}\right) \\ & - \dot{x}_\delta^{(1)}\left(z_{\alpha\alpha}^{(1)} + z_{\delta\delta}^{(1)}\right)\Big] \end{aligned} \tag{32}$$

$$\begin{aligned} Z^{(2)} = & -\ddot{x}^{(1)} x_\delta^{(1)} - \ddot{z}^{(1)} z_\delta^{(1)} + \nu\Big[x_\delta^{(1)}\left(\dot{x}_{\alpha\alpha}^{(1)} + \dot{x}_{\delta\delta}^{(1)}\right) \\ & + z_\delta^{(1)}\left(3\dot{z}_{\alpha\alpha}^{(1)} - \dot{z}_{\delta\delta}^{(1)}\right) - 2\dot{z}_{\alpha\delta}^{(1)}\left(z_\alpha^{(1)} + x_\delta^{(1)}\right) - \dot{z}_\alpha^{(1)}\left(x_{\alpha\alpha}^{(1)} + x_{\delta\delta}^{(1)}\right) \\ & - \dot{z}_\delta^{(1)}\left(z_{\alpha\alpha}^{(1)} + z_{\delta\delta}^{(1)}\right)\Big] \end{aligned} \tag{33}$$

consist of products of the first-order variables.

[13] The steady component of the $O(\epsilon^2)$ Lagrangian drift is called the mass transport velocity, and we may define

$$(u_L, w_L) \equiv \left(\bar{\dot{x}}^{(2)}, \bar{\dot{z}}^{(2)}\right), \tag{34}$$

where the overbar denotes time average over a period. Now, suppose a steady second-order field of Lagrangian streaming has been established in the entire depth. Then, the mass transport velocity components are governed by the time-averaged equations (29)–(31), which read as follows:

$$(u_L)_\alpha + (w_L)_\delta = 0, \tag{35}$$

$$\frac{1}{\rho}\bar{p}_\alpha^{(2)} + g\bar{z}_\alpha^{(2)} - \nu\left[(u_L)_{\alpha\alpha} + (u_L)_{\delta\delta}\right] = \bar{X}^{(2)}, \tag{36}$$

$$\frac{1}{\rho}\bar{p}_\delta^{(2)} + g\bar{z}_\delta^{(2)} - \nu\left[(w_L)_{\alpha\alpha} + (w_L)_{\delta\delta}\right] = \bar{Z}^{(2)}. \tag{37}$$

Note that the time average of the right side of equation (29) is zero because the steady terms in the brackets have been differentiated to zero. The time-averaged forcing terms, after some manipulation, can be written as

$$\begin{aligned} \bar{X}^{(2)} = & R\sigma^2 k\left(|\tilde{x}|^2 - |\tilde{z}|^2\right)\sin 2k\alpha + \nu\sigma k\Big\{\left(1 - R^2\right)\Big[\frac{1}{2}k^2|\tilde{z}|^2 \\ & - 2\Re(\tilde{x}^*\tilde{x}'') - k\Im(\tilde{z}^*\tilde{x}') - \frac{3}{2}|\tilde{x}'|^2\Big] + 4R\sin 2k\alpha[k\Re(\tilde{z}^*\tilde{x}') \\ & + \Im(\tilde{x}^*\tilde{x}'')]\Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} \bar{Z}^{(2)} = & \frac{1}{2}\left(1 + R^2\right)\sigma^2[\Re(\tilde{x}^*\tilde{x}') - k\Im(\tilde{x}^*\tilde{z})] \\ & - R\sigma^2\cos 2k\alpha[\Re(\tilde{x}^*\tilde{x}') + k\Im(\tilde{x}^*\tilde{z})] \\ & + \nu\sigma\Big\{\frac{1}{2}\left(1 + R^2\right)[\Im(\tilde{x}^{*\prime}\tilde{x}'') - k\Re(\tilde{z}\tilde{x}^{*\prime\prime}) + 3k^2\Im(\tilde{x}^*\tilde{x}') \\ & - k^3\Re(\tilde{x}^*\tilde{z})] - R\cos 2k\alpha[\Im(\tilde{x}^{*\prime}\tilde{x}'') - k\Re(\tilde{z}\tilde{x}^{*\prime\prime}) \\ & + 3k^2\Im(\tilde{x}^*\tilde{x}') + 7k^3\Re\left(\tilde{x}^{*\tilde{z}}\right)]\Big\}. \end{aligned} \tag{39}$$

Taking the time average of the $O(\epsilon^2)$ boundary conditions gives

$$(u_L)_\delta + (w_L)_\alpha = 3\overline{\dot{x}_\alpha^{(1)} z_\alpha^{(1)}} - 2\overline{\dot{x}_\delta^{(1)} x_\alpha^{(1)}} + \overline{\dot{x}_\alpha^{(1)} x_\delta^{(1)}} \qquad \delta = 0 \tag{40}$$

for zero shear stress on the free surface, and

$$-\bar{p}^{(2)} + 2\mu(w_L)_\delta = 0 \qquad \delta = 0 \tag{41}$$

for zero normal stress on the free surface. On the bottom, the particles are not in motion, and hence $u_L = w_L = 0$ at $\delta = -h$.

[14] To solve the two-dimensional flow, as in most previous studies, a stream function could be introduced at this point, thereby immediately eliminating $\bar{p}^{(2)}$ and $\bar{z}^{(2)}$ from equations (36) and (37). We, however, choose not to do so at this stage, and show how these terms will lead to results that have been ignored in the past.

4.1. Inviscid Solution

[15] Let us first obtain the inviscid part of the solution to the problem above. On setting $\nu = 0$ (which implies $s = 0$, and so on), equations (36) and (37) reduce to

$$\frac{1}{\rho}\bar{p}_\alpha^{(2)} + g\bar{z}_\alpha^{(2)} = \bar{X}_I^{(2)} \tag{42}$$

$$\frac{1}{\rho}\bar{p}_\delta^{(2)} + g\bar{z}_\delta^{(2)} = \bar{Z}_I^{(2)}, \tag{43}$$

where

$$\bar{X}_I^{(2)} = R\frac{\sigma^2 k a^2}{\sinh^2 kh}\sin 2k\alpha \tag{44}$$

$$\bar{Z}_I^{(2)} = \left(1 + R^2\right)\frac{\sigma^2 k a^2}{\sinh^2 kh} \times \sinh k(\delta + h)\cosh k(\delta + h). \tag{45}$$

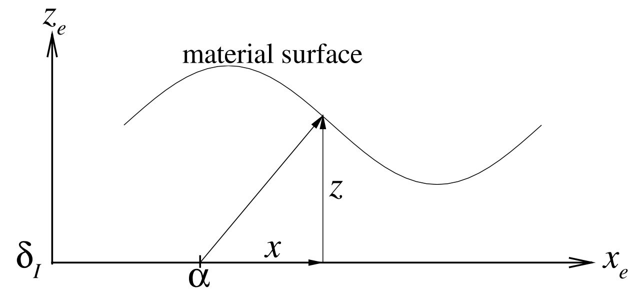

Figure 1. Lagrangian and Eulerian descriptions of a material surface, where $\delta = \delta_I$ is the undisturbed position of the material surface, α is the original horizontal position of a surface particle, (x, z) are the displacements of the surface particle, and (x_e, z_e) are the Eulerian Cartesian coordinates.

Now, integrating equation (43) with respect to δ gives

$$\frac{1}{\rho}\bar{p}^{(2)} + g\bar{z}^{(2)} = g\bar{\eta}_I^{(2)} + \int_0^\delta \bar{Z}_I^{(2)} d\delta, \tag{46}$$

where the free surface boundary condition $\bar{p}^{(2)}(\delta = 0) = 0$ has been used, and

$$\bar{\eta}_I^{(2)} \equiv \bar{z}^{(2)}(\delta = 0) \tag{47}$$

is the free surface setup due to the inviscid wave motion. Substituting equation (46) into equation (42), noting that $\bar{Z}_I^{(2)}$ is independent of α, we get an equation for the wave setup $g(\bar{\eta}_I^{(2)})_\alpha = \bar{X}_I^{(2)}$, which can readily be integrated to give

$$\bar{\eta}_I^{(2)} = -R\frac{ka^2}{\sinh 2kh}\cos 2k\alpha + \text{constant}, \tag{48}$$

which accords with the well-known fact that the surface setup exists only for a standing wave if the bottom is horizontal. One may however question if equation (48) is correct since it looks different from the classical expression derived by *Longuet-Higgins and Stewart* [1964],

$$\bar{\zeta} = ka^2 \coth 2kh \cos 2kx, \tag{49}$$

for a pure standing wave. We remark that this apparent difference can be reconciled on noting that the Lagrangian description of the displacement of a material surface essentially differs from the Eulerian description. A general relation can be deduced as follows between these two quantities when the displacement of the material surface is small in amplitude, as in the present case.

[16] Referring to Figure 1, we suppose there exists a material surface which lies on $\delta = \delta_I$ when undisturbed. We let (x, z) be the horizontal/vertical displacements of a surface particle that is originally at a horizontal position α along the undisturbed surface. Also, we let $\zeta(x_e, t) = z_e$ be the Eulerian description of the surface displacement, where (x_e, z_e) are the Eulerian Cartesian coordinates. Then, obviously,

$$z = \zeta(\alpha + x, t) = \zeta(\alpha) + x\frac{\partial\zeta}{\partial\alpha} + \cdots, \tag{50}$$

where in the second step, small-amplitude displacements are assumed. The smallness being $O(\epsilon)$, we may expand

$$(x, z, \zeta) = \epsilon\left(x^{(1)}, z^{(1)}, \zeta^{(1)}\right) + \epsilon^2\left(x^{(2)}, z^{(2)}, \zeta^{(2)}\right) + \cdots, \tag{51}$$

which are then substituted into equation (50) to give

$$z^{(1)} = \zeta^{(1)} \tag{52}$$

$$z^{(2)} = \zeta^{(2)} + x^{(1)}\frac{\partial z^{(1)}}{\partial\alpha}. \tag{53}$$

By now, it is clear that while to the first order either description will give the same displacement, there is a difference in the second-order displacement as described by the two approaches. Taking the time average of equation (53) gives us a desired general relation between the Lagrangian and the Eulerian descriptions of the setup of a material surface,

$$\bar{\eta}^{(2)} = \bar{\zeta}^{(2)} + \overline{\left(x^{(1)}\frac{\partial z^{(1)}}{\partial\alpha}\right)_{\delta_I}}, \tag{54}$$

where we have denoted $z(\delta = \delta_I)$ by $\eta(\alpha, t)$.

[17] Let us now go back and check if equation (54) holds for the present problem. After some algebra, one can evaluate on the free surface the second term on the right-hand side of equation (54) to be, for $R = 1$,

$$\overline{\left(x^{(1)}\frac{\partial z^{(1)}}{\partial\alpha}\right)_{\delta=0}} = ka^2 \coth kh(1 - \cos 2k\alpha). \tag{55}$$

Adding this to equation (49) gives

$$\bar{\zeta} + \overline{\left(x^{(1)}\frac{\partial z^{(1)}}{\partial\alpha}\right)_{\delta=0}} = -\frac{ka^2}{\sinh 2kh}\cos 2k\alpha + ka^2 \coth kh, \tag{56}$$

which obviously matches $\bar{\eta}_I^{(2)}$ given in equation (48) provided that the wave is purely standing ($R = 1$) and the constant is equal to $ka^2\coth kh$. Therefore, by this exercise the integration constant is also fixed, and $\bar{\eta}_I^{(2)}$ may now be written as

$$\bar{\eta}_I^{(2)} = Rka^2\left[-\frac{\cos 2k\alpha}{\sinh 2kh} + \coth kh\right]. \tag{57}$$

It is interesting to further note that the horizontal spatial mean of the Lagrangian setup $\bar{\eta}_I^{(2)}$ is not equal to zero, even though its Eulerian counterpart $\bar{\zeta}$ has a zero spatial mean. This subtle difference seems to have been ignored in previous studies [e.g., *Piedra-Cueva*, 1995].

4.2. Boundary Layers

[18] Having found the inviscid part of the solution, we may now subtract the non-viscous forcings $\bar{X}_I^{(2)}$ and $\bar{Z}_I^{(2)}$ from $\bar{X}^{(2)}$ and $\bar{Z}^{(2)}$, respectively. Let us proceed to consider flow in the surface and bottom Stokes boundary layers, in which $ks \ll 1$ is the small parameter. Invoking the standard

boundary layer approximation, the momentum equations can be simplified to

$$\frac{1}{\rho}\bar{p}_{\alpha}^{(2)} + g\bar{z}_{\alpha}^{(2)} - \nu(u_L)_{\delta\delta} = \bar{X}_{bl}^{(2)} + O(ks)^2 \tag{58}$$

$$\frac{1}{\rho}\bar{p}_{\delta}^{(2)} + g\bar{z}_{\delta}^{(2)} = \bar{Z}_{bl}^{(2)} + O(ks), \tag{59}$$

where $\bar{X}_{bl}^{(2)}$ and $\bar{Z}_{bl}^{(2)}$ contain only the dominant terms in the boundary layers, and will be given below specifically for each boundary layer. To eliminate $\bar{p}^{(2)}$ and $\bar{z}^{(2)}$ from the two equations above, we may integrate equation (59) from $\delta = 0$ to a general δ, which is then differentiated with respect to α before substitution into equation (58). We remark that it is not inconsistent to have ignored the $O(ks)$ terms in equation (59) since by the processes of integration with respect to $\delta = O(s)$ and differentiation with respect to $\alpha = O(k^{-1})$, the order of such omitted terms will be raised to $O(ks)^2$, the same as in equation (58). Without writing the error term $O(ks)^2$, we may express the equation for the mass transport velocity in a boundary layer as follows:

$$\nu\frac{\partial^2 u_L}{\partial\delta^2} = g\frac{\partial\bar{\eta}_{sbl}^{(2)}}{\partial\alpha} + \frac{\partial}{\partial\alpha}\int_0^{\delta}\bar{Z}_{bl}^{(2)}d\delta - \bar{X}_{bl}^{(2)}, \tag{60}$$

where $\bar{p}^{(2)}(\delta = 0) = O(ks)^2$ has been dropped, and

$$\bar{\eta}_{sbl}^{(2)} = \bar{z}^{(2)}(\delta = 0) \tag{61}$$

is a higher order free surface setup that is of dominance only in the surface boundary layer; see section 4.2.2 below. After some lengthy algebra involving identification of orders of magnitude of individual terms, we keep only those leading terms, which are $O(1)$ in the bottom boundary layer and $O(ks)$ in the surface boundary layer, for the right-hand side of equation (60), for near the bottom ($\delta \sim -h$) and near the free surface ($\delta \sim 0$).

4.2.1. Bottom Boundary Layer

[19] Near the bottom, equation (60) reads

$$\nu\frac{\partial^2 u_L}{\partial\delta^2} = R\frac{\sigma^2 ka^2}{\sinh^2 kh}\sin 2k\alpha\left[3e^{-2(\delta+h)/s} - 4e^{-(\delta+h)/s}\cos(\delta+h)/s\right] + \left(1-R^2\right)\frac{\sigma^2 ka^2}{\sinh^2 kh}\left[\frac{3}{2}e^{-2(\delta+h)/s} - 2e^{(\delta+h)/s}\sin(\delta+h)/s\right]. \tag{62}$$

Integrating this equation with respect to δ twice, and making use of the boundary conditions $\partial u_L/\partial\delta \to 0$ as $(\delta + h)/s \gg 1$ and $u_L = 0$ at $\delta = -h$, we get the mass transport velocity profile in the bottom boundary layer,

$$u_L(\alpha,\delta) = R\frac{\sigma ka^2}{\sinh^2 kh}\sin 2k\alpha\left[\frac{3}{2}e^{-2(\delta+h)/s} - \frac{3}{2} + 4e^{-(\delta+h)/s}\sin(\delta+h)/s\right] + \left(1-R^2\right)\frac{\sigma ka^2}{\sinh^2 kh}\left[\frac{3}{4}e^{-2(\delta+h)/s} + \frac{5}{4} - 2e^{-(\delta+h)/s}\cos(\delta+h)/s\right]. \tag{63}$$

At the outer edge of the bottom boundary layer,

$$u_L|_{\delta+h\gg s} = -\frac{3}{2}R\frac{\sigma ka^2}{\sinh^2 kh}\sin 2k\alpha + \frac{5}{4}\left(1-R^2\right)\frac{\sigma ka^2}{\sinh^2 kh}. \tag{64}$$

These results are well known in the literature [e.g., *Mei*, 1989].

4.2.2. Surface Boundary Layer

[20] Near the free surface, equation (60) reads

$$\nu\frac{\partial^2 u_L}{\partial\delta^2} = g\frac{\partial\bar{\eta}_{sbl}^{(2)}}{\partial\alpha} - 2R\frac{\sigma^2 k^2 a^2 s}{\tanh kh}\sin 2k\alpha \times\left[2e^{\delta/s}(\cos\delta/s + \sin\delta/s) - 1\right] - 2\left(1-R^2\right)\frac{\sigma^2 k^2 a^2 s}{\tanh kh}e^{\delta/s} \times(\cos\delta/s - \sin\delta/s). \tag{65}$$

The setup term and the constant term inside the square brackets will become secular on approaching the outer edge of the boundary layer ($\delta/s \to -\infty$), and therefore they must balance each other. Hence we get

$$g\frac{\partial\bar{\eta}_{sbl}^{(2)}}{\partial\alpha} = -2R\frac{\sigma^2 k^2 a^2 s}{\tanh kh}\sin 2k\alpha. \tag{66}$$

Clearly this setup term is sub-dominant in the bottom boundary layer. Also, $\bar{\eta}_{sbl}^{(2)}$ is $O(k^2a^2s)$, i.e., an order of magnitude smaller than $\bar{\eta}_l^{(2)}$, which is of the order ka^2. Nevertheless, this setup term did not show up in the work by *Ünlüata and Mei* [1970], who considered only a pure progressive wave.

[21] With the velocity gradient at the free surface $\partial u_L/\partial\delta(\delta = 0) = O(ks)^2$, thereby negligible, we now integrate equation (65) to get

$$\frac{\partial u_L}{\partial\delta} = -8R\frac{\sigma k^2 a^2}{\tanh kh}\sin 2k\alpha\left(e^{\delta/s}\sin\delta/s\right) + 4\left(1-R^2\right)\frac{\sigma k^2 a^2}{\tanh kh}\left(1 - e^{\delta/s}\cos\delta/s\right), \tag{67}$$

which tends to the following limit at a distance far below the boundary layer:

$$\frac{\partial u_L}{\partial\delta} = 4\left(1-R^2\right)\frac{\sigma k^2 a^2}{\tanh kh} \qquad -\delta \gg s. \tag{68}$$

4.3. Core Region

[22] We have come to a point where a Lagrangian stream function $\psi(\alpha,\delta)$ should be introduced such that the continuity equation (35) is satisfied identically by

$$u_L = -\frac{\partial\psi}{\partial\delta} \qquad w_L = \frac{\partial\psi}{\partial\alpha}. \tag{69}$$

Then equations (36) and (37) can be combined to give

$$\nu(\psi_{\alpha\alpha\alpha\alpha} + 2\psi_{\alpha\alpha\delta\delta} + \psi_{\delta\delta\delta\delta}) = \bar{X}_{\delta}^{(2)} - \bar{Z}_{\alpha}^{(2)}. \tag{70}$$

The full expressions for $\bar{X}^{(2)}$ and $\bar{Z}^{(2)}$ in equations (38) and (39) must be used; fortunately, only the inviscid parts of $\tilde{x}$

and $\tilde{z}$ contribute to the leading-order terms in these expressions for outside the boundary layers. One can easily show that to the leading order, $\bar{Z}_{\alpha}^{(2)} = 0$, and

$$\bar{X}_{\delta}^{(2)} = -\left(1 - R^2\right)\frac{4\nu\sigma k^4 a^2}{\sinh^2 kh}\sinh 2k(\delta + h). \tag{71}$$

The stream function ψ is fixed to be zero along the bottom $\delta = -h$, and is also subject to, following from equations (64) and (68),

$$-\psi_{\delta} = -\frac{3}{2}R\frac{\sigma k a^2}{\sinh^2 kh}\sin 2k\alpha, + \frac{5}{4}\left(1 - R^2\right)\frac{\sigma k a^2}{\sinh^2 kh} \qquad \delta = -h, \tag{72}$$

$$-\psi_{\delta\delta} = 4\left(1 - R^2\right)\frac{\sigma k^2 a^2}{\tanh kh} \qquad \delta = 0. \tag{73}$$

[23] A far field bounding condition needs to be specified at this point. If the domain is closed, a return current will be established so that the net discharge due to the mass transport across the entire depth is zero. This is equivalent to further specifying $\psi = 0$ on $\delta = 0$. Solution to this zero-flux problem, which was first obtained by *Longuet-Higgins* [1953], can be written as

$$\psi = \frac{\sigma a^2}{4\sinh^2 kh}\Big\{\left(1 - R^2\right)\left[Z^{(p)}(\delta) - \sinh 2k(\delta + h)\right] + 2RZ^{(s)}(\delta)\sin 2k\alpha\Big\}, \tag{74}$$

where

$$\begin{aligned} Z^{(p)} = {} & \sinh 2kh - 3k\delta - k^2h^2\left(\delta^3/h^3 + 2\delta^2/h^2 + \delta/h\right)\sinh 2kh \\ & - \frac{1}{2}\left(\delta^3/h^3 - 3\delta/h\right)\left(\sinh 2kh + 3kh\right) \end{aligned} \tag{75}$$

$$\begin{aligned} Z^{(s)} = {} & -3(2kh\cosh 2kh\sinh 2k\delta - 2k\delta\cosh 2k\delta\sinh 2kh) \\ & \div (\sinh 4kh - 4kh). \end{aligned} \tag{76}$$

[24] One may note that the mass transport due to a standing wave forms a recirculation cell over a quarter of the wavelength [*Longuet-Higgins*, 1953], and the associated net flux is zero irrespective of the domain being bounded or not. The bounding condition, however, becomes a matter of importance when the wave is partially standing, in which the mass transport is composed of unidirectional and recirculating parts. It is only the unidirectional part of the mass transport that is to be counterbalanced by a return current if the domain is completely bounded. Therefore, if the domain is open ended (in the sense discussed earlier), a return current is either not established or not felt yet at the point of interest. It is, however, not possible to judge from equation (74) alone which terms are contributed by the return current and should be removed to recover that for an opened-ended domain. In this regard, we may resort to the work of *Ünlüata and Mei* [1970], who have solved the one-dimensional mass transport due to a pure progressive wave, and found that in the absence of the return current the unidirectional mass transport velocity in the core is

$$u_L = \frac{\sigma k a^2}{4\sinh^2 kh}[3 + 2\cosh 2k(\delta + h) + 4k(\delta + h)\sinh 2kh]. \tag{77}$$

[25] Therefore, taking into account all the results given above, we may obtain a general expression, encompassing both bounding conditions, for the horizontal mass transport velocity in the core,

$$\begin{aligned} u_L(\alpha, \delta) = {} & \frac{\sigma k a^2}{4\sinh^2 kh}\Bigg\{\left(1 - R^2\right)\Bigg[3 + 2\cosh 2k(\delta + h) \\ & + 4k(\delta + h)\sinh 2kh + 3Y\left(\frac{\delta^2}{h^2} - 1\right)\left(\frac{3}{2} + kh\sinh 2kh \right. \\ & \left. + \frac{\sinh 2kh}{2kh}\right)\Bigg] + \frac{12R\sin 2k\alpha}{\sinh 4kh - 4kh}\times\Bigg[\cosh 2k\delta \\ & (2kh\cosh 2kh - \sinh 2kh) - 2k\delta\sinh 2k\delta\sinh 2kh\Bigg]\Bigg\}, \end{aligned} \tag{78}$$

where Y is an integer parameter switching on or off the closed-system condition,

$$Y = \begin{cases} 0 & \text{if domain is open ended} \\ 1 & \text{if domain is bounded.} \end{cases} \tag{79}$$

The vertical mass transport velocity is simply given by

$$w_L(\alpha, \delta) = R\frac{\sigma k a^2}{\sinh^2 kh}\cos 2k\alpha Z^{(s)}(\delta), \tag{80}$$

where $Z^{(s)}$ is given in equation (76). While *Longuet-Higgins* [1953] provided the parts of the solution corresponding to $Y = 1$, *Ünlüata and Mei* [1970] gave those for $R = 0$. We present here a combined solution, for which the stream function is thus written as

$$\begin{aligned} \psi(\alpha, \delta) = {} & \frac{\sigma a^2}{4\sinh^2 kh}\Big\{\left(1 - R^2\right)[-3k(\delta + h) - \sinh 2k(\delta + h) \\ & - 2k^2h^2(\delta/h + 1)^2\sinh 2kh \\ & - Ykh\left(\delta^3/h^3 - 3\delta/h - 2\right)(3/2 + kh\sinh 2kh \\ & + \sinh 2kh/2kh)] + 2RZ^{(s)}(\delta)\sin 2k\alpha\Big\}. \end{aligned} \tag{81}$$

The vorticity may also be readily found to be

$$\begin{aligned} \omega(\alpha, \delta) & = \frac{\partial w_L}{\partial \alpha} - \frac{\partial u_L}{\partial \delta} \\ & = -\frac{\sigma k^2 a^2}{\sinh^2 kh}\Bigg\{\left(1 - R^2\right)\Bigg[\sinh 2k(\delta + h). + \sinh 2kh \\ & \quad + Y\left(\frac{3\delta}{2kh^2}\right)\left(\frac{3}{2} + kh\sinh 2kh + \frac{\sinh 2kh}{2kh}\right)\Bigg] \\ & \quad + \frac{6R\sin 2k\alpha}{\sinh 4kh - 4kh}\times[\sinh 2k\delta(2kh\cosh 2kh - \sinh 2kh) \\ & \quad - \sinh 2k\delta\sinh 2kh - 2k\delta\cosh 2k\delta\sinh 2kh] \\ & \quad + 2RZ^{(s)}(\delta)\sin 2k\alpha\Bigg\}. \end{aligned} \tag{82}$$

[26] A remark on the production of the return flow is now in order. When solving for one-dimensional mass transport in progressive waves, *Ünlüata and Mei* [1970] made several assumptions including (1) $\bar{z}_\alpha^{(2)} = 0$, and (2) $\bar{p}_\alpha^{(2)}$ is a constant, which turns out to be responsible for setting up a return current in the case of a closed system. We comment that these two conditions are not exactly true when a return current is generated. Indeed, it is contradictory to have a finite horizontal pressure gradient on the one hand, and to have a zero horizontal gradient of the mean vertical displacement on the other hand. As rightly pointed out by *Piedra-Cueva* [1995], a non-zero free surface setup must be in existence when a horizontal pressure gradient is established to balance the radiation stress of the progressive wave and thus to produce the return flow. To understand this better, one may readily obtain the following relationship from the time-averaged $O(\epsilon^2)$ vertical momentum equation, if the wave is purely progressive and wave attenuation is neglected,

$$\frac{1}{\rho}\bar{p}_\alpha^{(2)} + g\bar{z}_\alpha^{(2)} = g\bar{\eta}_\alpha^{(2)}, \tag{83}$$

where $\bar{\eta}^{(2)} = \bar{z}^{(2)}(\delta = 0)$ is the free surface setup. Therefore, it is the combined action of the horizontal gradients of the mean pressure and the mean vertical displacement of particles, resulting in a free surface setup, that is responsible for the return current. Since $\bar{z}_\alpha^{(2)}$ varies with depth, $\bar{p}_\alpha^{(2)}$ cannot be a constant, as supposed by *Ünlüata and Mei* [1970]. Nevertheless, the analysis and results presented by *Ünlüata and Mei* [1970] are perfectly all right as long as the pressure gradient term in their deduction is replaced by the gradient of the free surface setup.

[27] Returning our focus back to the present study, we may as well determine the free surface setup $\bar{\eta}_{rc}^{(2)}$ that is associated with the return current under a progressive wave in a closed system, and with the recirculating cells under a standing wave. The method involves substituting the mass transport velocity components (78) and (80) back into the governing equations (36) and (37), followed by steps similar to those shown earlier in getting equation (60). Without providing the tedious details, we present here the final expression for this component of free surface setup,

$$g\frac{\partial\bar{\eta}_{rc}^{(2)}}{\partial\alpha} = \frac{\sigma^2k^3a^2s^2}{2\sinh^2 kh}\left\{Y\left(1-R^2\right)\frac{3}{2k^2h^2}\times\left(\frac{3}{2}+kh\sinh 2kh + \frac{\sinh 2kh}{2kh}\right) - R\sin 2k\alpha\frac{48kh\cosh 2kh}{\sinh 4kh - 4kh}\right\}, \tag{84}$$

which confirms that if the system is unclosed ($Y = 0$), the free surface setup due to the unidirectional part of the mass transport will vanish. Conversely, one may use this condition to determine the constant value of the stream function along the free surface when the domain is unbounded.

[28] Thus far in the course of our deduction, three components of the free surface setup have been obtained arising from the inviscid standing wave motion, the surface boundary layer, and the return current. These components are of different orders of magnitude,

$$\bar{\eta}^{(2)} = \bar{\eta}_I^{(2)} + \bar{\eta}_{sbl}^{(2)} + \bar{\eta}_{rc}^{(2)}, \tag{85}$$

where $\bar{\eta}_I^{(2)} = O(ka^2)$, $\bar{\eta}_{sbl}^{(2)}/\bar{\eta}_I^{(2)} = O(ks)$, and $\bar{\eta}_{rc}^{(2)}/\bar{\eta}_I^{(2)} = O(ks)^2$. That is, the setup due to the inviscid wave motion is the dominant one among the three. Since wave decay is ignored and the bottom is horizontal, a progressive wave will not contribute to the free surface setup unless a return current is generated.

5. Numerical Results

[29] Let us briefly examine the effects of the bounding condition on the mass transport under various values of R. Numerical results shown below are in terms of the following normalized variables (distinguished by a hat):

$$\begin{aligned}\hat{\alpha} &= k\alpha,\\ \hat{\delta} &= \delta/h,\\ (\hat{u}_L, \hat{w}_L) &= (u_L, w_L)/\sigma ka^2,\\ \hat{\psi} &= \psi/\sigma a^2,\\ \hat{\omega} &= \omega/\sigma k^2a^2.\end{aligned} \tag{86}$$

We show in Figures 2 and 3 the Lagrangian streamlines together with velocity vectors for a range of the reflection coefficient when the system is, respectively, closed and unclosed. When the wave is purely standing (R = 1), the classical cellular structure shown by *Longuet-Higgins* [1953] is reproduced, which is independent of the far end bounding condition. The left and right cells rotate in the clockwise and counterclockwise sense, respectively, resulting in vertical jets shooting, respectively, upward and downward under the antinodes ($\hat{\alpha} = n\pi$, $n = 0, 1, \cdots$) and the nodes ($\hat{\alpha} = (2n + 1)\pi/2$) of the free surface. The recirculation structure persists as R decreases when the system is closed (Figures 2b–2d). The return current tends to lift up the clockwise cell, and to press down the counterclockwise cell. Meanwhile, the flow in the clockwise cell is much more weakened than the counterclockwise cell by the return current. Eventually, as the progressive wave limit is approached, a reversed current forms in the middle of the fluid layer, which balances the forward streams at the top and the bottom of the layer. Flow stops to reverse on the bottom when the reflection coefficient drops to 0.5. Recirculation, however, continues to exist even when the reflection coefficient is as low as 0.1. It is remarkable that for all values of R the flow patterns exhibit symmetry about the vertical lines midway between the nodes and antinodes ($\hat{\alpha} = n\pi/4$, $n = 1, 3, 5, \cdots$); the centers of the cells remain to lie on these lines. Such symmetry will disappear when the ratio a/s becomes very large (i.e., when the Stuart boundary layer forms outside the Stokes boundary layer), as presented by *Iskandarani and Liu* [1991]. Their results show that the centers of the recirculating cells will shift horizontally as the reflection coefficient varies.

[30] In sharp contrast, when the system is unclosed, the recirculation regions will be extensively wiped out as soon as the reflection coefficient drops below unity (Figures 3b–3d). Although not shown in this figure, it is found that the counterclockwise recirculation will

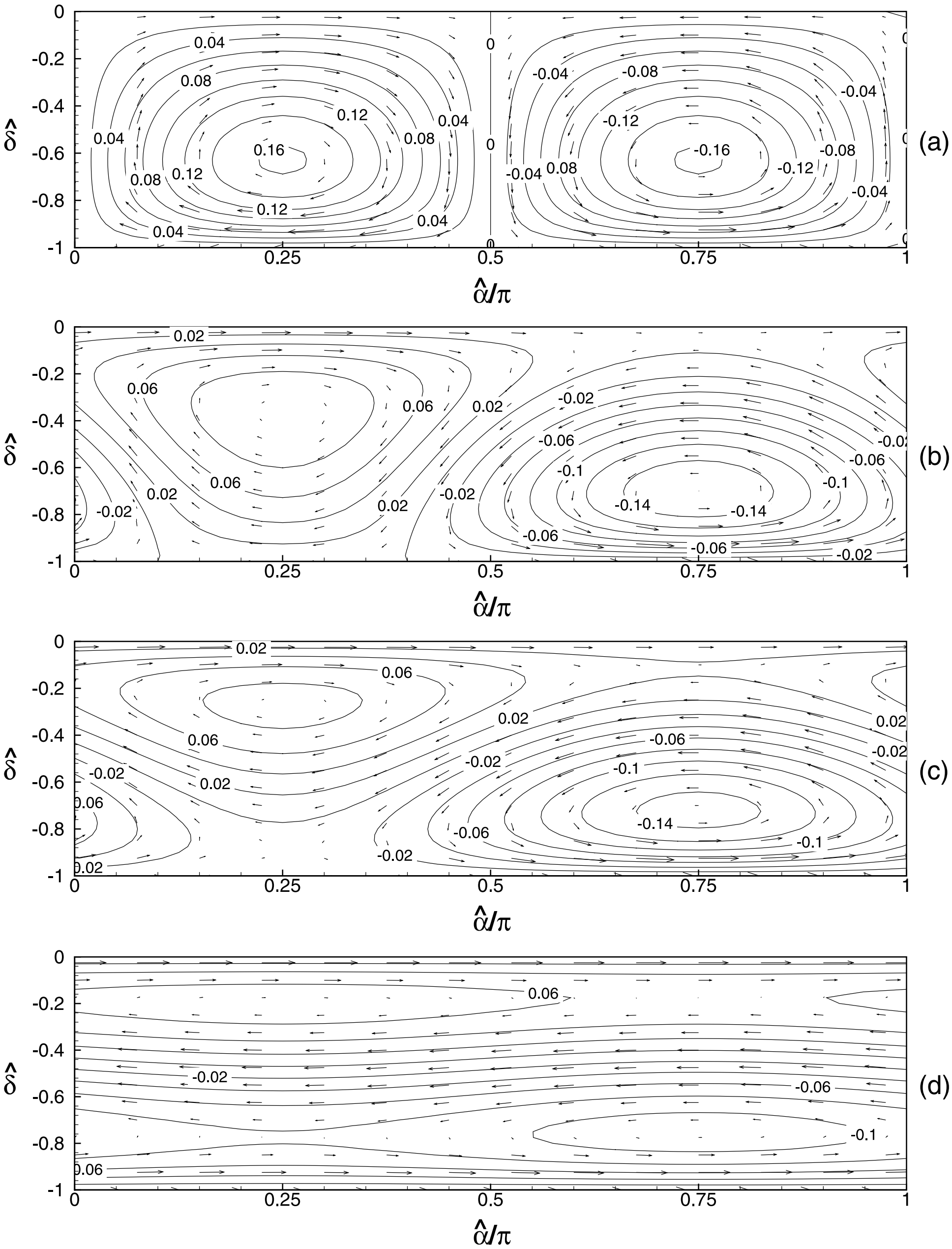

Figure 2. Mass transport streamlines and velocity vectors in partially standing waves in a closed system ($Y = 1$) for $kh = 1.0$, and (a) $R = 1.0$, (b) $R = 0.7$, (c) $R = 0.5$, and (d) $R = 0.1$.

disappear completely when R drops just by 5% from unity. The clockwise cell, which is then much diminished and pressed down to be confined near the bottom, may persist only until $R \approx 0.7$. The flow appears to be almost unidirectional when R gets down to 0.5.

[31] The corresponding vorticity contours (Figures 4 and 5) provide a further comparison between the two cases. In the case of a closed system, lowering the reflection coefficient has an effect of enhancing the vorticity of the flow in the counterclockwise cell that eventually intrudes itself into the region originally occupied by the clockwise cell. At the pure progressive wave limit, the vorticity is zero at some mid-depth, and is the strongest (with opposite sense) at the free surface and the bottom.

[32] For an unclosed system, lowering the reflection coefficient will increase the vorticity in the region above the counterclockwise cell, making it locally into a nearly uniform distribution across the depth. Meanwhile, the vorticity structure associated with the counterclockwise cell, though diminished, remains to survive and retains its identity even when the recirculation itself has already disappeared. This suggests that the vorticity is not as sensitive as the mass transport velocity to a small departure from a perfect standing wave. Finally, as the wave becomes purely progressive, the vorticity increases

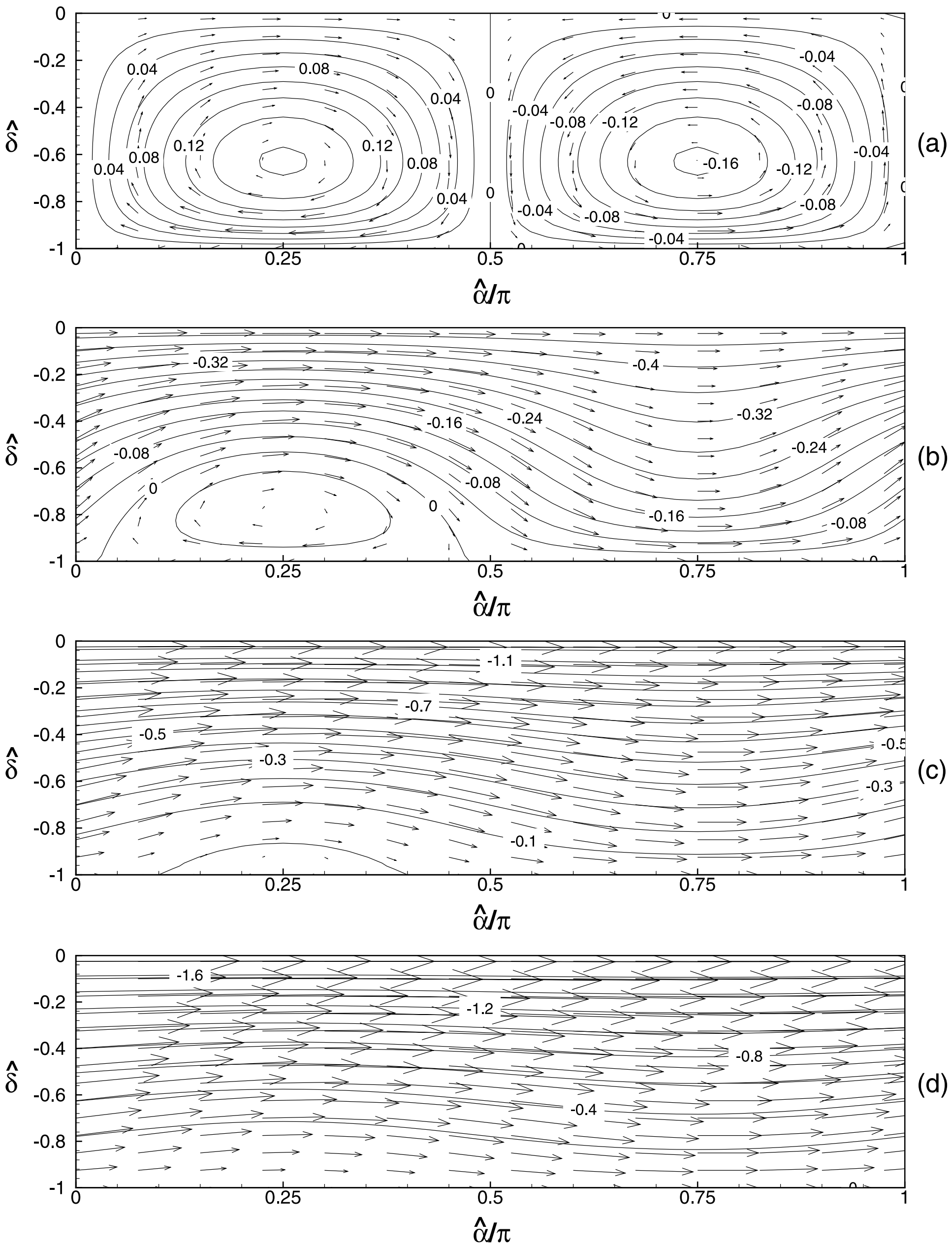

Figure 3. Mass transport streamlines and velocity vectors in partially standing waves in an unclosed system ($Y = 0$) for $kh = 1.0$, and (a) $R = 1.0$, (b) $R = 0.9$, (c) $R = 0.7$, and (d) $R = 0.5$.

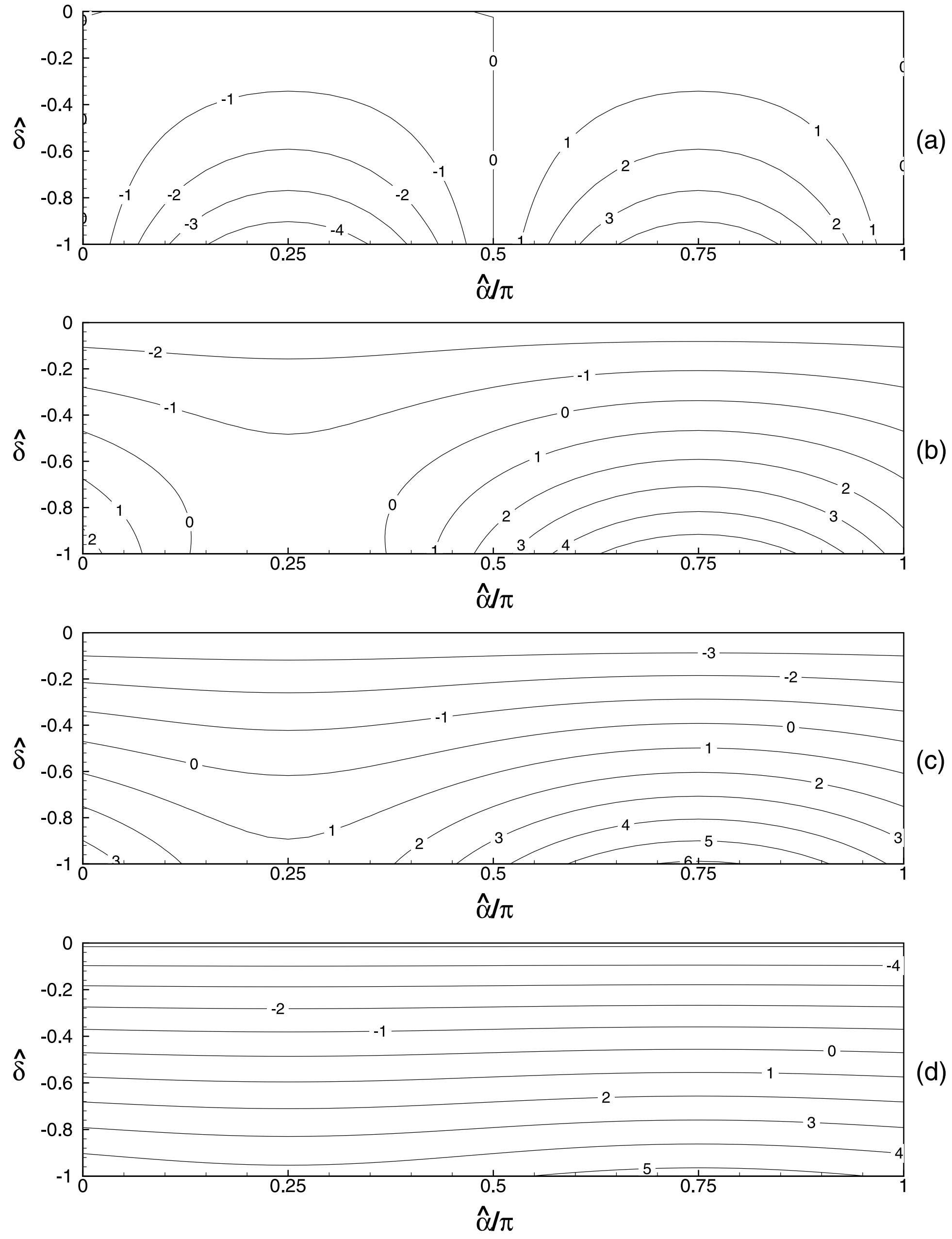

Figure 4. Vorticity contours in partially standing waves in a closed system ($Y = 1$) for $kh = 1.0$, and (a) $R = 1.0$, (b) $R = 0.7$, (c) $R = 0.5$, and (d) $R = 0.1$.

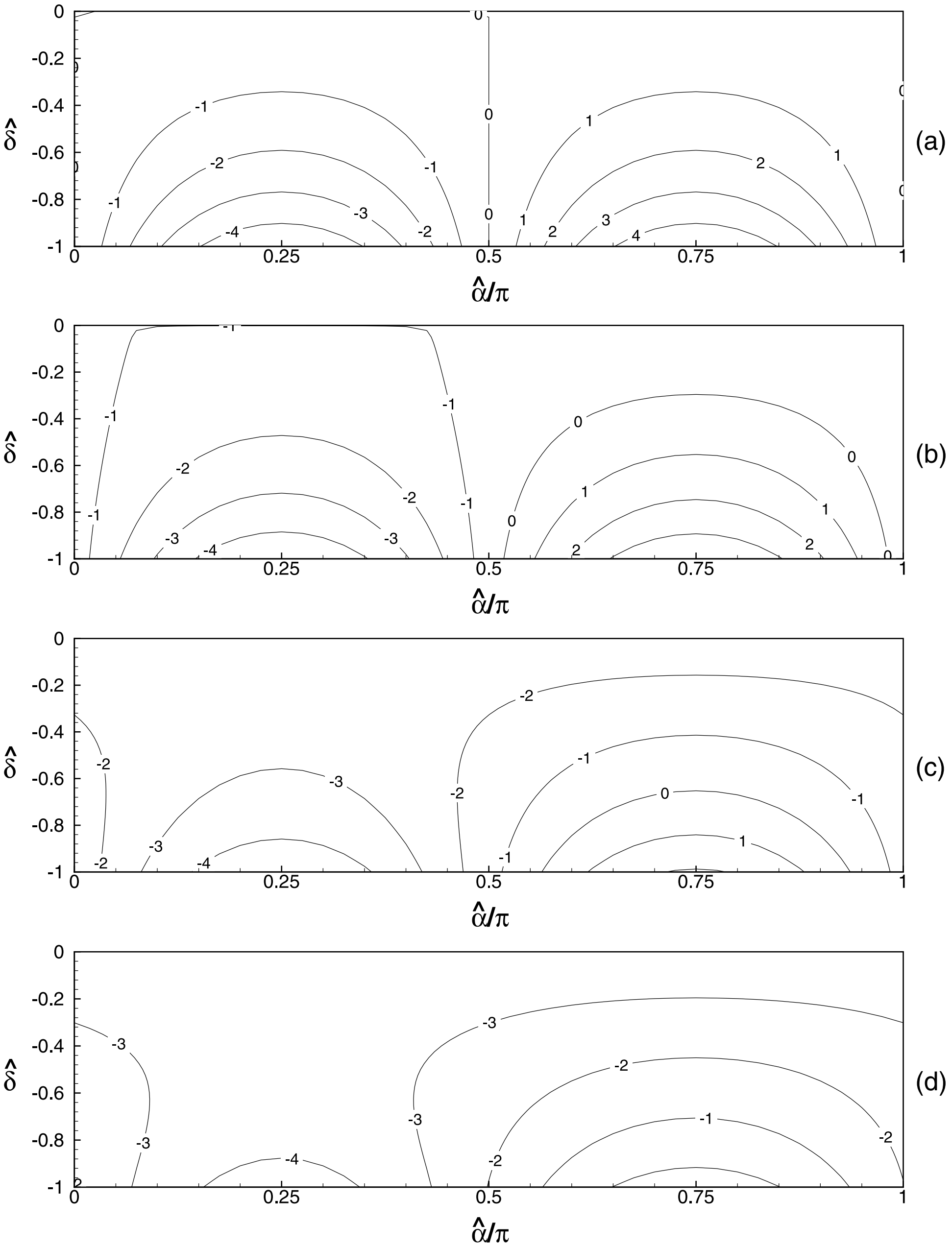

Figure 5. Vorticity contours in partially standing waves in an unclosed system ($Y = 0$) for $kh = 1.0$, and (a) $R = 1.0$, (b) $R = 0.9$, (c) $R = 0.7$, and (d) $R = 0.5$.

mildly from the bottom to a maximum at the free surface without change in sign. This implies a much more uniform vertical distribution of vorticity than when the system is closed.

6. Concluding Remarks

[33] An effort has been made in this work to patch up several unnoticed holes in the existing theories for mass transport in gravity waves. To demonstrate our points without excessive complication, only the conduction solution under a simple configuration has been considered. Nevertheless, we have shown a way by which the mass transport due to partially standing waves in an unbounded domain can be derived in general. It is our intention to extend the present work to more complex situations, say, in a two-layer system, and/or when the fluid is non-Newtonian. The results will be presented shortly.

[34] **Acknowledgments.** The work was supported by the Research Grants Council of the Hong Kong Special Administrative Region, China, through projects HKU 7081/02E and HKU 7199/03E.

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C.-O. Ng, Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong. (cong@hku.hk)