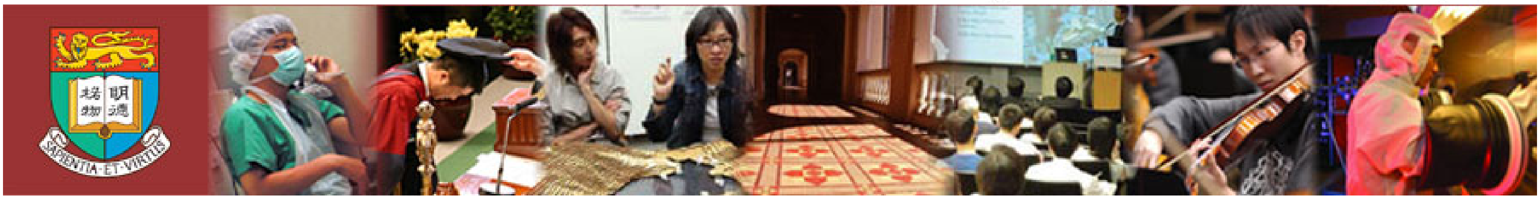

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Embeddings of hypersurfaces in affine spaces

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Abstract. In this paper, we address the following two general problems: given two algebraic varieties in $\mathbf{C}^n$, find out whether or not they are (1) isomorphic; (2) equivalent under an automorphism of $\mathbf{C}^n$. Although a complete solution of either of these problems is out of the question at this time, we give here some handy and useful invariants of isomorphic as well as of equivalent varieties. Furthermore, and more importantly, we give a universal procedure for obtaining all possible algebraic varieties isomorphic to a given one, and use it to construct numerous examples of isomorphic, but inequivalent algebraic varieties in $\mathbf{C}^n$. Among other things, we establish the following interesting fact: for isomorphic hypersurfaces $\{p(x_1, ..., x_n) = 0\}$ and $\{q(x_1, ..., x_n) = 0\}$, the number of zeros of $grad(p)$ might be different from that of $grad(q)$. This implies, in particular, that, although the fibers $\{p = 0\}$ and $\{q = 0\}$ are isomorphic, there are some other fibers $\{p = c\}$ and $\{q = c\}$ which are not. We construct examples like that for any $n \geq 2$.

1. Introduction

Let $\mathbf{C}[x_1, ..., x_n]$ be the polynomial algebra in n variables over the field $\mathbf{C}$. Any collection of polynomials $p_1, ..., p_m$ from $\mathbf{C}[x_1, ..., x_n]$ determines an algebraic variety $\{p_i = 0, i = 1, ..., m\}$ in the affine space $\mathbf{C}^n$. We shall denote this algebraic variety by $V(p_1, ..., p_m)$.

We say that two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ are *isomorphic* if the algebras of residue classes $\mathbf{C}[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$ and $\mathbf{C}[x_1, ..., x_n]/\langle q_1, ..., q_k\rangle$ are isomorphic. Here $\langle p_1, ..., p_m\rangle$ denotes the ideal of $\mathbf{C}[x_1, ..., x_n]$ generated by $p_1, ..., p_m$. Thus, isomorphism that we consider here is algebraic, not geometric, i.e., we actually consider isomorphism of what is called *affine schemes*. (For example, two polynomials x_1 and x_1^2 determine the same geometric set, but in our terminology, the corresponding algebraic varieties are not isomorphic.)

On the other hand, we say that two algebraic varieties (or, rather, embeddings of the same algebraic variety in $\mathbf{C}^n$) are *equivalent* if there is an automorphism of $\mathbf{C}^n$ that takes one of them onto the other. If any two embeddings of an algebraic variety in $\mathbf{C}^n$ are equivalent, we say that this algebraic variety has a *unique embedding* in $\mathbf{C}^n$.

In this paper, we address two principal problems:

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(I) How to find out whether or not two given algebraic varieties are isomorphic?

(II) How to find out whether or not two given algebraic varieties are equivalent?

Both problems have received a lot of attention over the years. For a survey on the problem (I), we refer to [19]. The most substantial contribution to the problem (II) are [2], [3], [18], [20] for equivalence in $\mathbf{C}^2$, and [9], [11], [17] for equivalence in higher dimensions. In the latter three papers, it is shown that if the codimension of an algebraic variety V in $\mathbf{C}^n$ is sufficiently large, then V has a unique embedding in $\mathbf{C}^n$. Examples of non-uniquely embedded varieties in higher dimensions are given in [8] (for $n = 5$) and [13] and [19] (for $n \geq 3$).

In the present paper, we give a simple generic procedure for constructing examples of isomorphic, but inequivalent varieties in any dimension. Furthermore, we give a very simple but efficient criterion for distinguishing isomorphic, but inequivalent *hypersurfaces.* This criterion allows us to show that isomorphic hypersurfaces that we construct, are actually inequivalent even under any *holomorphic* automorphism of the ambient space $\mathbf{C}^n$.

However, we start by addressing the problem (I). Our contribution to (I) is the following Theorem 1.1. Before we give the statement, we introduce three types of isomorphism-preserving "elementary" transformations that can be applied to an arbitrary algebra of residue classes $K[x_1, ..., x_n]/R$, where R is an ideal of $K[x_1, ..., x_n]$, and K an arbitrary ground field.

(P1) *Introducing a new variable*: replace $K[x_1, ..., x_n]/R$ by $K[x_1, ..., x_n, y]/R + \langle y - p(x_1, ..., x_n)\rangle$, where $p(x_1, ..., x_n)$ is an arbitrary polynomial.

(P2) *Canceling a variable* (this is the converse of (P1)): if we have an algebra of residue classes of the form $K[x_1, ..., x_n, y]/\langle p_1, ..., p_m, q\rangle$, where q is of the form $y - p(x_1, ..., x_n)$, and $p_1, ..., p_m \in K[x_1, ..., x_n]$, replace this by $K[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$.

(P3) *Renaming the variables*: replace variables $(x_1, ..., x_n)$ by $(x_{i_1}, ..., x_{i_n})$, where $i_1, ..., i_n$ are arbitrary distinct indices, not necessarily the integers in $\{1, ..., n\}$.

Then we have:

Theorem 1.1. Two algebras $K[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$ and $K[x_1, ..., x_n]/\langle q_1, ..., q_k\rangle$ are isomorphic if and only if one can get from one of them to the other by a sequence of transformations (P1)–(P3).

In particular, two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ in $\mathbf{C}^n$ are isomorphic if and only if one can get from the algebra $\mathbf{C}[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$ to the algebra $\mathbf{C}[x_1, ..., x_n]/\langle q_1, ..., q_k\rangle$ by a sequence of transformations (P1)–(P3).

This result alone does not solve the problem (I) since it does not give a hint on how to choose the polynomials $p(x_1, ..., x_n)$ in (P1). Still, the result is quite useful because, on one hand, it yields invariants (=necessary conditions of isomorphism) of algebraic varieties, and, on the other hand, gives an easy practical way of constructing "exotic" examples of isomorphic algebraic varieties thus providing, in particular, potential examples of isomorphic, but inequivalent varieties.

Theorem 1.1 also has the following interesting corollary which says that, although isomorphic algebraic varieties may not be equivalent, they are always *stably equivalent* (the precise meaning of that is clear from the statement below). Moreover, isomorphic varieties turn out to be stably equivalent under a *tame* automorphism. We call an automorphism tame if it is a product of elementary and linear automorphisms, where elementary automorphisms are those that change only one variable.

Corollary 1.2. (cf. [11], [17]) If two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ in $\mathbf{C}^n$ are isomorphic, then the varieties $V(p_1, ..., p_m, x_{n+1}, ..., x_{2n})$ and $V(q_1, ..., q_k, x_{n+1}, ..., x_{2n})$ in $\mathbf{C}^{2n}$ are equivalent under a tame automorphism of $\mathbf{C}^{2n}$.

We now describe some invariants of isomorphic varieties that can be obtained based on Theorem 1.1. Given a variety $V = V(q_1, ..., q_m)$ (in $\mathbf{C}^n$) that contains another variety $V' = V(r_1, ..., r_s)$, consider the Jacobian matrix $J(q_1, ..., q_m) = \left(\frac{\partial q_i}{\partial x_j}\right), 1 \le i \le m, 1 \le j \le n$. Then, consider the image of this matrix under the natural homomorphism from $\mathbf{C}[x_1, ..., x_n]$ onto $\mathbf{C}[x_1, ..., x_n]/\langle r_1, ..., r_s\rangle$. We denote this image by $A_R(V)$, and call it the *Alexander matrix* of the variety $V = V(q_1, ..., q_m)$ with respect to the ideal $R = \langle r_1, ..., r_s\rangle$. This resembles the Alexander matrix of a finitely presented group which plays an important role in algebraic topology – see e.g. [5] or [6]. Continuing this parallel with the classical Alexander matrix, we consider *elementary ideals* E_k, $k \ge 0$, of the Jacobian matrix $J(q_1, ..., q_m)$ defined as follows:

– if $0 < n - k \le m$, then E_k is the ideal of $\mathbf{C}[x_1, ..., x_n]$ generated by all $(n-k) \times (n-k)$ minors of the matrix $J(q_1, ..., q_m)$;
– if $n - k > m$, then $E_k = 0$;
– if $n - k \le 0$, then $E_k = \mathbf{C}[x_1, ..., x_n]$.

Denote by $\bar{E}_k$ the image of E_k in $\mathbf{C}[x_1, ..., x_n]/R$. Then Theorem 1.1 allows us to show that $\bar{E}_k$ are invariants of isomorphic algebraic varieties. More precisely:

Theorem 1.3. Let V_1 and V_2 be two algebraic varieties in $\mathbf{C}^n$, corresponding to the ideals R_1 and R_2, respectively, of the algebra $\mathbf{C}[x_1, ..., x_n]$. Let R be another ideal that contains $R_1 + R_2$. Consider Jacobian matrices J_1 and J_2 of the varieties V_1 and V_2. Let $E_k^{(1)}$ and $E_k^{(2)}$ be elementary ideals of J_1 and J_2, respectively, defined as above. Then, if the varieties V_1 and V_2 are isomorphic, one has $E_k^{(1)} + R = E_k^{(2)} + R$, after possibly renaming the variables in $E_k^{(1)} + R$.

In Section 2, we give examples of applying Theorem 1.3 to distinguishing non-isomorphic algebraic varieties. Note that the condition $E_k^{(1)} + R = E_k^{(2)} + R$ is always actually verifiable since checking if two ideals of a polynomial algebra are equal can be done by using Gröbner basis algorithm (see [4]). As far as renaming the variables is concerned, there are only finitely many ways of doing it without changing the whole set of variables.

Then, we consider another application of Theorem 1.1, to constructing isomorphic, but inequivalent algebraic varieties. The most practical way of doing it is illustrated by the following

Example 1. Let $f(x, y, z)$ be any polynomial, and let $p(x, y, z) = x - f(x^k, y, z)$; $q(x, y, z) = x - f^k(x, y, z)$; $k \geq 1$. Then the varieties $V(p)$ and $V(q)$ are isomorphic.

Indeed, the algebra $K[x, y, z]/\langle p\rangle$ can be generated by $u = x^k$, y, and z since in this algebra, we have $x = f(x^k, y, z)$. These new generators u, y, z are subject to the relation $u = f^k(u, y, z)$ (i.e., the minimal polynomial of u, y, and z in the algebra $K[x, y, z]/\langle p\rangle$ is $u - f^k(u, y, z)$). Therefore, the varieties $V(p)$ and $V(q)$ are isomorphic. (In Section 3, we give a more formal and rigorous proof of this fact based on our Theorem 1.1; what we have just given here, is a somewhat informal, "fast" version of the same method).

Now that we have a procedure for constructing non-trivial examples of isomorphic algebraic varieties, we need invariants of *equivalent* varieties. It appears that in many situations, one can get away with very simple invariants. From now on, we are going to concentrate on *hypersurfaces*, i.e., on varieties of the form $V(p)$.

It turns out that *the number of zeros of the gradient* $grad(p)$ is a rather sharp invariant of equivalent hypersurfaces. (The fact that it *is* an invariant, follows immediately from the "chain rule" for partial derivatives). Note that this number is, in general, different from the number of singularities of a hypersurface since a point where the gradient vanishes, may not belong to the hypersurface. For the lack of an established name for those points, we introduce one here:

Definition. Let $V(p)$ be a hypersurface in $\mathbf{C}^n$. A point in $\mathbf{C}^n$ where $grad(p)$ vanishes, is called a *quasi-singular* point of the hypersurface $V(p)$.

Whereas the number of singular points is an intrinsic characteristic of a hypersurface (i.e., is invariant under an isomorphism), the number of quasi-singular points is not, and therefore it can be used to distinguish isomorphic, but inequivalent hypersurfaces. This can be illustrated by the following simple

Example 2. Let $p(x, y, z) = x + y + z - xyz$, and $q(x, y, z) = x + y^2 + yz - xyz$. Then $grad(p)$ has 2 zeros, whereas $grad(q)$ vanishes nowhere. Therefore, the hypersurfaces $V(p)$ and $V(q)$ are inequivalent. The fact that they are isomorphic, can be established by the same method that we used in Example 1. The algebra $K[x, y, z]/\langle p\rangle$ can be generated by $u = xy$, y and z since in this algebra, $x = xyz - y - z$. These new generators are subject to the relation $u = (uz - y - z)y = uyz - y^2 - yz$, whence the isomorphism.

This example can be generalized to the following

Proposition 1.4. For any $n \geq 3$, the hypersurface $x_1 + x_2 + ... + x_n - x_1 x_2 ... x_n = 0$ has at least two embeddings in $\mathbf{C}^n$ inequivalent under any algebraic or even just holomorphic automorphism of $\mathbf{C}^n$.

We emphasize this particular result here because it contrasts sharply with a result of Jelonek [10] saying that the "n-cross" $x_1x_2...x_n = 0$ has a unique embedding in $\mathbf{C}^n$ for any $n \geq 3$. Our Section 3 contains many other examples of isomorphic, but inequivalent hypersurfaces.

One of our motivations for addressing these issues was the following well-known conjecture of Abhyankar [1] and Sathaye [16]:

The Embedding conjecture. If a hypersurface $V(p)$ in $\mathbf{C}^n$ is isomorphic to the coordinate hyperplane $V(x_1)$, then it is equivalent to it.

This conjecture was proved by Abhyankar and Moh for $n = 2$ [2], and it remains open for $n \geq 3$, where the prevalent opinion is that it is false, although for $n = 3$, there are substantial partial results in the positive direction [15]. In order to construct a counterexample, one needs two things: a procedure for constructing non-trivial examples of hypersurfaces isomorphic to a hyperplane, and also invariants of equivalent hypersurfaces, sharp enough to detect hypersurfaces inequivalent to a hyperplane.

The procedure described above (based on Theorem 1.1) is rather flexible and simple. Moreover, the "only if" part of Theorem 1.1 ensures that if a counterexample to the Embedding conjecture exists, then it can be found by our method. Hence, the main problem is to find sufficiently sharp invariants of equivalent hypersurfaces. The question whether or not the number of quasi-singular points can possibly be useful in that situation, seems interesting in its own right:

Problem. If a hypersurface $V(p)$ in $\mathbf{C}^n$ is isomorphic to a coordinate hyperplane, is it true that $grad(p)$ vanishes nowhere?

If the answer to this problem is negative, that would also give an example of a polynomial some of whose fibers are isomorphic to $V(x_1)$, and some are not. Indeed, suppose $X_0 \in \mathbf{C}^n$ is a quasi-singular point of $V(p)$, and let $p(X_0) = c$. Then the hypersurface $V(p-c)$ has a singular point, and therefore cannot be isomorphic to $V(x_1)$.

There is no polynomial like that in $\mathbf{C}[x_1, x_2]$ – see [2], but for polynomials in more than two variables the situation is unknown.

2. Invariants of isomorphic varieties

We start with

Proof of Theorem 1.1. Since the "if" part of the theorem is obvious, we proceed with the "only if" part.

We are going to show that if the algebras $K[X] = K[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$ and $K[x_1, ..., x_n]/\langle q_1, ..., q_k\rangle$ are isomorphic, then we can take them both to the same algebra by a sequence of transformations (P1)–(P3).

We denote the former algebra by $K[X]/R$, and the latter by $K[X]/S$, where R and S are ideals of $K[x_1, ..., x_n]$ generated by $p_1, ..., p_m$ and $q_1, ..., q_k$, respectively. Upon applying the transformation (P3), we may assume that the latter algebra is of the form $K[Y]/S$, where the set $Y = \{y_1, ..., y_n\}$ does not overlap with X.

Then, by repeatedly applying the transformation (P1), we can get

$$K[X]/R \;\cong\; K[X \cup Y]/R + \sum_{i=1}^{n} \langle y_i - u_i(x_1, ..., x_n)\rangle,$$

where u_i are polynomials constructed as follows. Since $K[X]/R$ is isomorphic to $K[Y]/S$, there is a mapping $\varphi : K[X]/R \to K[Y]/S$, such that $Ker(\varphi) = 0 + R$, and

$$y_i + S = u_i(\varphi(x_1 + R), ..., \varphi(x_n + R)) + S = \varphi(u_i(x_1, ..., x_n) + R).$$

Denote the ideal $\sum_{i=1}^{n} \langle y_i - u_i(x_1, ..., x_n)\rangle$ by U. Now comes the crucial point: we are going to show that $S \subseteq R + U$.

Suppose we have an arbitrary polynomial $w(y_1, ..., y_n) \in S$. Then $w(y_1, ..., y_n) = w((y_1 - u_1) + u_1, ..., (y_n - u_n) + u_n) \equiv w(u_1, ..., u_n)$ $(mod\ U)$. On the other hand, $\varphi(w(u_1, ..., u_n)+R) = w(y_1, ..., y_n)+S$ since $y_i+S = \varphi(u_i+R)$. Thus, $w(u_1, ..., u_n)+R \in Ker(\varphi) = 0 + R$, and therefore $w(y_1, ..., y_n) \in R + U$.

Again, since $K[X]/R$ is isomorphic to $K[Y]/S$, there is another mapping (the inverse of φ) $\psi : K[Y]/S \to K[X]/R$, such that $Ker(\psi) = 0 + S$;

$$x_i + R = v_i(\psi(y_1 + S), ..., \psi(y_n + S)) + R = \psi(v_i(y_1, ..., y_n) + S),$$

and, furthermore, $\psi(\varphi(x_i + R)) = x_i + R$, which implies

$$v_i(u_1, ..., u_n) \equiv x_i\ (mod\ R).$$

Then we have: $x_i - v_i(y_1, ..., y_n) = x_i - v_i((y_1 - u_1) + u_1, ..., (y_n - u_n) + u_n) \equiv x_i - v_i(u_1, ..., u_n)\ (mod\ U) \equiv x_i - x_i\ (mod\ R) \equiv 0\ (mod\ R)$. Therefore, $x_i - v_i(y_1, ..., y_n) \in R + U$.

Thus, we have shown

$$R + U = R + S + U + \sum_{i=1}^{n} \langle x_i - v_i\rangle,$$

and therefore that the algebra $K[X]/R$ can be taken to $K[X \cup Y]/R+S+U+\sum_{i=1}^{n}\langle x_i - v_i\rangle$ by a sequence of transformations (P1), (P3). By the symmetry, the algebra $K[Y]/S$ can be taken to the same algebra by a sequence of (P1), (P3), and this completes the proof. $\square$

Corollary 1.2 follows immediately from the proof of Theorem 1.1 since transformations of the type (P1) that we have used, are actually tame automorphisms of $K[X \cup Y]$. $\square$

Proof of Theorem 1.3 basically goes along the same lines as the standard proof of invariance of Alexander ideals – see e.g. [6]. Namely, our transformations (P1) and

(P2) are analogous to Tietze transformations applied to group presentations, so that the invariance of $E_k + R$ can be established similarly.

However, there is one subtlety here. We have to show that $E_k + R$ do not depend on a particular choice of generators of the ideal $\langle q_1, ..., q_m\rangle$, i.e., that the ideals $E_k + R$ are well-defined. Suppose $\{u_1, ..., u_b\}$ is another generating set of $\langle q_1, ..., q_m\rangle$, and let $u_j = \sum_{i=1}^{m} q_i \cdot w_{ij},\ 1 \le j \le b$, where w_{ij} are some polynomials.

Add b zero rows to the Jacobian matrix $J(q_1, ..., q_m)$. This does not change any of the E_k, which can be easily seen from the definition. Enumerate those zero rows somehow, and add to the row number j the following combination of other rows: $\sum_{i=1}^{m} Q_i \cdot w_{ij}$, where by Q_i we denote the row of partial derivatives of the polynomial q_i. Again, this operation does not affect any of the E_k. Now, modulo the ideal $\langle q_1, ..., q_m\rangle$, the above combination of rows equals $\sum_{i=1}^{m} \partial(q_i \cdot w_{ij}) = \partial(u_j)$, where by $\partial(z)$ we denote the row of partial derivatives of z. By the symmetry, we can obtain exactly the same $(m+b)\times(m+b)$ matrix over $\mathbf{C}[x_1, ..., x_n]/\langle q_1, ..., q_m\rangle$ from the Jacobian matrix $J(u_1, ..., u_b)$, without changing any of the $E_k + R$. This completes the proof. $\Box$

We now illustrate Theorem 1.3 by a couple of examples.

Example 2.1. Let $p = p(x, y) = x^2 + xy + y^3$; $V_1 = V(p)$, and $q = q(x, y) = x^2 + y^3$; $V_2 = V(q)$. Then $J(p) = grad(p) = (2x + y, x + 3y^2)$; $J(q) = grad(q) = (2x, 3y^2)$. Therefore, for the matrix $J(p)$ we have $E_1^{(1)} + R = \langle 2x + y, x + 3y^2, p\rangle = \langle x, y\rangle$ (here we choose $R = \langle p, q\rangle$). If we rename the variables in $E_1^{(1)}$ (by switching x and y), we shall get the same ideal $\langle x, y\rangle$.

On the other hand, for the matrix $J(q)$ we have $E_1^{(2)} + R = \langle 2x, 3y^2, p, q\rangle = \langle x, y^2\rangle$. Thus, V_1 and V_2 are not isomorphic.

The point of the next example is to present a situation where our invariants can distinguish algebraic varieties with the same topology. (We are appealing to an intuitively clear fact that minor perturbations of coefficients of a polynomial do not change the topology of a general fiber). We also note that Makar-Limanov [14] has come up with another way to distinguish varieties with the same topology by using purely algebraic invariants.

Example 2.2. Let $p = p(x, y, z) = x + yz + z^2$; $q_k = kx^2 + y^3,\ k \in \mathbf{C},\ k \neq 0$. By using our invariants, we can show that $V(p, q_k)$ and $V(p, q_m)$ are non-isomorphic algebraic curves in $\mathbf{C}^3$ if $k \neq m$.

Let $R = \langle p, q_k, q_m\rangle$. Then the ideal $E_1^{(1)} + R$ for the Jacobian matrix $J(p, q_k)$ has the following (reduced) Gröbner basis with respect to *deglex* term ordering (see [4]): $\{x^2, xz^2, x + yz + z^2, 3y^2 - 2kxz, xy + 2xz, z^3 + 3xz\}$. The ideal $E_1^{(2)} + R$ for the Jacobian matrix $J(p, q_m)$ has the reduced Gröbner basis (with respect to the same term ordering) which is obtained from the one above upon replacing k by m.

Since the reduced Gröbner basis with respect to a particular term ordering is unique, it follows that the ideal $E_1^{(2)} + R$ is different from $E_1^{(1)} + R$ if $k \neq m$ (obviously, renaming the variables cannot change this fact), whence non-isomorphism.

3. Non-equivalent isomorphic varieties

In this section, we give various examples of inequivalent embeddings of hypersurfaces in $\mathbf{C}^n$. In all of these examples, we use the number of zeros of the gradient to establish inequivalence, as described in the Introduction. Therefore, in each case, we actually establish a stronger type of inequivalence than just being inequivalent under any algebraic automorphism of $\mathbf{C}^n$. All examples here are those of isomorphic hypersurfaces in $\mathbf{C}^n$ inequivalent under any *holomorphic* (i.e., complex analytic) automorphism of $\mathbf{C}^n$. Also, in all of these examples, whenever we show that a given hypersurface $V(p)$ has inequivalent embeddings in $\mathbf{C}^n$, it will follow that $V(p)$ has inequivalent embeddings in $\mathbf{C}^m$ for any $m > n$ as well (in which case $V(p)$ is a cylindrical hypersurface).

Finally, we note that by the examples below, we try to illustrate all possible combinations of numbers of zeros of $grad(p)$ and $grad(q)$ for isomorphic hypersurfaces $V(p)$ and $V(q)$. That is, we have examples where $grad(p)$ vanishes nowhere, but $grad(q)$ has infinitely many zeros; examples where both $grad(p)$ and $grad(q)$ have finitely many, but different number of zeros, and so on.

We start with the proof of Proposition 1.4, but before we proceed, we give a rigorous proof of isomorphism claimed in Examples 1 and 2 in the Introduction.

In **Example 1**, we start with the algebra $K[x,y,z]/\langle p\rangle$, where $p = p(x,y,z) = x - f(x^k,y,z)$. We are now going to give a sequence of elementary transformations (P1)–(P3) that will bring us to the algebra $K[x,y,z]/\langle q\rangle$. It will be technically more convenient to write those algebras of residue classes as "algebras with relations", i.e., for example, instead of $K[x,y,z]/\langle p\rangle$ we shall write $\langle x,y,z \mid p = 0\rangle$. The symbol $\cong$ below means "is isomorphic to".

$\langle x,y,z \mid x = f(x^k,y,z)\rangle \cong \langle x,y,z,u \mid x = f(x^k,y,z),\ u = x^k\rangle = \langle x,y,z,u \mid x = f(u,y,z),\ u = x^k\rangle = \langle x,y,z,u \mid x = f(u,y,z),\ u = f^k(u,y,z)\rangle \cong \langle y,z,u \mid u = f^k(u,y,z)\rangle \cong \langle x,y,z \mid x = f^k(u,y,z)\rangle$. $\Box$

For **Example 2**, the sequence of elementary transformations looks as follows:

$\langle x,y,z \mid x = xyz - y - z\rangle \cong \langle x,y,z,u \mid x = xyz - y - z,\ u = xy\rangle = \langle x,y,z,u \mid x = uz - y - z,\ u = xy\rangle \cong \langle y,z,u \mid u = (uz - y - z)y\rangle \cong \langle x,y,z \mid x = (xz - y - z)y\rangle = \langle x,y,z \mid x = xyz - y^2 - yz\rangle$. $\Box$

From now on, we are going to use a faster, slicker way of establishing isomorphism, as it is done in the Introduction. We are now ready for

Proof of Proposition 1.4. Let $p = p(x_1,...,x_n) = x_1 + x_2 + ... + x_n - x_1x_2...x_n$. Then, since the algebra $\mathbf{C}[x_1,...,x_n]/\langle p\rangle$ can be generated by $x_1x_2...x_{n-1}, x_2,...,x_n$, we have, as in Example 2, the hypersurface $V(p)$ isomorphic to $V(q)$, where $q = x_1 - (x_1x_n - x_2 - ... - x_n)x_2x_3...x_{n-1} = x_1 + x_2^2x_3...x_n + x_2x_3^2...x_{n-1} + ... + x_2...x_n - x_1x_2...x_n$.

Now compute the gradients: $grad(p) = (1 - x_2...x_n, 1 - x_1x_3...x_n, ..., 1 - x_1x_2...x_{n-1})$. This gradient has $(n-1)$ zeros since, for $grad(p) = 0$, one must have $x_1 = x_2 = ... = x_n$, and hence $x_1^{n-1} = 1$.

On the other hand, $grad(q) = (1 - x_2...x_n, 2x_2x_3...x_n + ... + x_3...x_{n-1}x_n - x_1x_3...x_n, ..., x_2...x_{n-1} - x_1x_2...x_{n-1})$. We are going to show that this gradient never vanishes.

Indeed, from the first component we see that none of the $x_2, ..., x_n$ can be equal to 0. Then, from the last component, we get $x_1 = 1$. Now, if we divide each component except the first and the last ones by $x_3...x_{n-1}$, we shall get a homogeneous system of linear equations in $x_2, x_3, ..., x_{n-1}$ (note that x_n will cancel out since $x_1 = 1$). The first equation is $2x_2 + x_3 + ... + x_{n-1} = 0$, and other equations are obtained from it by repeatedly applying the cyclic permutation on variables. A system like that is easily seen to have only the trivial solution $x_2 = x_3 = ... = x_{n-1} = 0$, and then the first component of $grad(q)$ is not zero. $\Box$

One more example that seems to be worth emphasizing, is that of the tom Dieck – Petrie hypersurface in $\mathbf{C}^n$, $n \geq 3$, determined by the polynomial $p(x, y, z) = x^2z - y^3z^2 - 3y^2z + 2x - 3y - 1$ (see [7]). This hypersurface has several remarkable properties – see [12]. Here we prove:

Proposition 3.1. Let $V(p)$ be the hypersurface in $\mathbf{C}^n$, $n \geq 3$, determined by $p(x, y, z) = x^2z - y^3z^2 - 3y^2z + 2x - 3y - 1$. Then there is a hypersurface $V(q)$ isomorphic to $V(p)$, but such that there exists neither an algebraic nor a holomorphic automorphism of $\mathbf{C}^n$ that takes $V(p)$ onto $V(q)$.

We note that the part about an algebraic automorphism was established in [13] by using different, somewhat more complicated, techniques.

Proof of Proposition 3.1. Let $p = p(x, y, z) = x^2z - y^3z^2 - 3y^2z + 2x - 3y - 1$. Since in the algebra $\mathbf{C}[x, y, z]/\langle p\rangle$, one has $x = \frac{1}{2}y^3z^2 + \frac{3}{2}y^2z - \frac{1}{2}x^2z + \frac{3}{2}y + \frac{1}{2}$, this algebra can be generated by x^2, y, z. Therefore, as in Example 1, the hypersurface $V(p)$ is isomorphic to $V(q)$, where $q = x - (\frac{1}{2}y^3z^2 + \frac{3}{2}y^2z - \frac{1}{2}xz + \frac{3}{2}y + \frac{1}{2})^2$.

Now compute the gradients: $grad(p) = (2 + 2xz, -3 - 3y^2z^2 - 6yz, x^2 - 2y^3z - 3y^2)$. This gradient has infinitely many zeros since the common zero locus of the three components is the same as that of $(x - y, xz + 1)$.

On the other hand, $grad(q) = (1 + z \cdot Q, -(\frac{3}{2}y^2z^2 + 3yz + \frac{3}{2})Q, -(y^3z + \frac{3}{2}y^2 - \frac{1}{2}x)Q)$, where $Q = Q(x, y, z) = (\frac{1}{2}y^3z^2 + \frac{3}{2}y^2z - \frac{1}{2}xz + \frac{3}{2}y + \frac{1}{2})$. This gradient never vanishes.

Indeed, the second component is $-\frac{3}{2}(yz + 1)^2Q$. Since $Q \neq 0$ (otherwise, the first component would be 1), we get $yz = -1$. Now from the last component, we get $x = y^2$. Plug this into the first component and get $z = 0$, which contradicts $yz = -1$. $\Box$

To construct similar examples in $\mathbf{C}^2$ is somewhat more difficult since there is "less room" there. However, we managed to do that as well.

Example 3.2. Let $p = p(x, y) = x^2 + y^2 - 1$. We claim that the curve $V(p)$ has at least two inequivalent embeddings in $\mathbf{C}^2$. (A probably difficult question is whether or not it has a unique embedding in $\mathbf{R}^2$).

Since $x^2+y^2=(x+iy)(x-iy)$, where $i^2=-1$, the algebra $\mathbf{C}[x,y]/\langle p\rangle$ is isomorphic to $\mathbf{C}[x,y]/\langle xy-1\rangle$. In this latter algebra, one has $x=x^2y$, so that $\mathbf{C}[x,y]/\langle xy-1\rangle$ can be generated by x^2, y. Therefore, the curve $V(p)$ is isomorphic to $V(q)$, where $q=x^2y-1$.

Obviously, $grad(p)$ vanishes only at the origin, whereas $grad(q)$ has infinitely many zeros. □

Example 3.3. Let $p=p(x,y)=x-x^2-x^2y-1$. Since in the algebra $\mathbf{C}[x,y]/\langle p\rangle$, one has $x=x^2+x^2y+1$, this algebra can be generated by x^2, y. Therefore, the curve $V(p)$ is isomorphic to $V(q)$, where $q=x-(1+x+xy)^2$.

Clearly, $grad(p)=(1-2x-2xy,-x^2)$ vanishes nowhere, whereas $grad(q)=(1-2(1+y)(1+x+xy),-2x(1+x+xy)$ vanishes at the point $(0,-\frac{1}{2})$. □

Example 3.4. Let $p=p(x,y)=x-y-x^2y-x^2y^2$. In the algebra $\mathbf{C}[x,y]/\langle p\rangle$, one has $x=y+x^2y+x^2y^2$, hence this algebra can be generated by x^2y, y. Therefore, the curve $V(p)$ is isomorphic to $V(q)$, where $q=x-(x+y+xy)^2y$.

To find zeros of $grad(p)$ and $grad(q)$ is not so easy in this example, but constructing Gröbner bases for both ideals $\langle p_x, p_y\rangle$ and $\langle q_x, q_y\rangle$ facilitates the process. The Gröbner basis (with respect to the lexicographic term ordering with $y>x$) for $\langle p_x, p_y\rangle$ turns out to be $\{4y^3+6^2+2y+x+2,\ 4y^4+8y^3+4y^2+2y+1\}$, whereas for $\langle q_x, q_y\rangle$ it is $\{36y^2+24y+8x+27,\ 4y^3+4y^2+3y+1\}$.

Now we see that $grad(p)$ has 4 zeros, whereas $grad(q)$ has 3 zeros. □

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