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AN INTEGRAL INEQUALITY OF
AN INTRINSIC MEASURE ON
BOUNDED DOMAINS IN \mathbf{C}^n

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1. Introduction. Let D be a complete hyperbolic bounded domain in \mathbf{C}^n in the sense of [7]. We denote by M_D^E the differential Eisenman-Kobayashi n -measure (defined with respect to the unit ball) on D . Since we may endow D with a global coordinate system, M_D^E can therefore be viewed as a function. The main goal of this paper is to prove the following theorem.

Theorem. *If we assume there exists a neighborhood N of ∂D in \mathbf{C}^n where M_D^E satisfies the growth condition*

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{m+s}},$$

where $k =$ positive constant, $r(z) =$ the euclidean distance from z to ∂D , $z \in N \cap D$, m and s positive numbers, then we can find a neighborhood U of ∂D in \mathbf{C}^n such that for all $z_0 \in U \cup D$, whenever the closed disk $\{z_0 + \rho z_1 : \rho \in \mathbf{C}, |\rho| \leq 1\}$, $z_1 \in \mathbf{C}^n$, lies in $U \cap D$, the inequality

$$\ln |M_D^E(z_0)| \leq \frac{n}{m\pi} \int_0^{2\pi} \ln |M_D^E(z_0 + e^{i\theta} z_1)| d\theta$$

for $|M_D^E|$ holds.

Typical examples satisfying conditions of our theorem include analytic polyhedra, strongly pseudoconvex domains and certain domains of holomorphy with smooth real analytic boundary [1]. For the case of strongly pseudoconvex domains we have the following corollary.

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Corollary. *Let D be a strongly pseudoconvex domain in \mathbf{C}^n with smooth boundary. Then there exists a neighborhood U of ∂D in \mathbf{C}^n such that for all $z_0 \in U \cap D$, whenever the closed disk*

$$\{z_0 + \rho z_1; \rho \in \mathbf{C}, |\rho| \leq 1\}, \quad z_1 \in \mathbf{C}^n,$$

lies in $U \cap D$, the inequality

$$\ln |M_D^E(z_0)| \leq \frac{1}{\pi} \int_0^{2\pi} \ln |M_D^E(z_0 + e^{i\theta} z_1)| d\theta$$

holds.

Our proof rests on the boundary assumption of the intrinsic measure and the classical Hartogs' construction of analytic family of disks [11]. It is quite clear from a minor modification of the proof that our integral inequality also holds on an analytically embedded disk (i.e., the image of a holomorphic embedding $f : B \rightarrow D$, here $B = \{z \in \mathbf{C} : |z| < 1\}$, which is homeomorphic up to the boundary ∂B with $f(\partial B) \subset D$ and $f(0) = z_0$). One can see this inequality imposes a restriction on M_D^E when the analytic disk is large and sufficiently close to the boundary. This type of inequality can be generalized to other low dimensional intrinsic measures. A sharpened result can also probably be derived along our line. The arrangement of our paper can be summarized as below.

Section 2. Definition of Eisenman-Kobayashi measure. Section 3. A boundary estimate of Eisenman-Kobayashi measure on strongly pseudoconvex domain and proof of the Corollary. Section 4. Proof of our main statement.

2. Definition of Eisenman-Kobayashi measure. For the basic definitions and a survey of this subject, one should consult [7, 8]. Since our *Eisenman-Kobayashi n -measures* are defined with respect to the ball in \mathbf{C}^n , which is somewhat different from what had been done in [7, 8], we shall include our definition here.

Let N be a complex manifold of dimension n . The Eisenman Kobayashi n -measure M_N^E is an (n, n) -form $|M_N^E| \cdot (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge$

$dz_n \wedge d\bar{z}_n$) on N , such that $|M_N^E|$ is defined for all $x \in N$ as

$$|M_N^E(x)| = \inf \left\{ \frac{1}{R^{2n}} : \exists f \in \text{Hol}(B_n^R, N) \text{ such that } f(0) = x, \det(Jf(0)) = 1 \right\},$$

where B_n^R is the euclidean ball with center 0 and radius R in \mathbf{C}^n , $\text{Hol}(B_n^R, N)$ the set of all holomorphic maps from B_n^R to N , and $Jf(0)$ the Jacobian matrix of f at 0.

It is easy to check that for a bounded domain D in \mathbf{C}^n , $|M_D^E(x)| \neq 0$ for all $x \in D$. $|M_D^E|$ is in general a semicontinuous function [12]. When D is a complete hyperbolic bounded domain, $|M_D^E|$ can be proved to be continuous. All complete hyperbolic domains in \mathbf{C}^n are pseudoconvex [7].

3. A boundary estimate of M_D^E on S.P.C. domains and proof of the corollary. Let D be a strongly pseudoconvex domain in \mathbf{C}^n with smooth boundary. The following will be proved in this section.

Theorem. *There exist a neighborhood U of ∂D in \mathbf{C}^n and a positive constant c such that for all $z \in U \cap D$,*

$$|M_D^E(z)| \geq \frac{c}{(r(z))^{n+1}},$$

where $r(z)$ is the euclidean distance function from z to the boundary of D .

Note that since ∂D is compact and everywhere strongly pseudoconvex, the above statement can be reduced to the following local problem:

For all $p \in \partial D$, there exists a neighborhood V of p in \mathbf{C}^n and a positive constant k such that for all $z \in D \cap V$,

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{n+1}}.$$

3.1 Localization lemma. Let D be a bounded domain in \mathbf{C}^n , D_1 another domain such that $D \cap D_1$ is nonempty.

Definitions. (1) Let z, w belong to D . Then $d(z, w) = \inf\{P(a, b) : \exists f \in \text{Hol}(B_n, D) \text{ s.t. } f(a) = z, f(b) = w, \text{ where } P \text{ is the Kobayashi metric on } B_n\}$; here B_n is the unit ball in \mathbf{C}^n .

(2) Let z belong to $D \cap D_1$, then $d(z) = \inf\{d(z, w) : w \in D - D_1\}$.

Lemma A. *Let us denote $\hat{D} = D \cap D_1$; then for all $z \in \hat{D}$, we have*

$$|M_{\hat{D}}^E(z)| \leq (\coth d(z))^{2n} \cdot |M_D^E(z)|,$$

where $M_{\hat{D}}^E$ and M_D^E denote the Eisenman-Kobayashi measures on \hat{D} and D , respectively.

Proof. First of all, let us fix $z \in \hat{D}$ and let

$$r = \sup\{r' : \exists f \in \text{Hol}(B_n^{r'}, \hat{D}) \text{ s.t. } f(0) = z, \det(Jf(0)) = 1\}.$$

Then we choose a number R which is slightly larger than r . From our choice of r it is obvious that there is an $f \in \text{Hol}(B_n^R, D)$ such that $f(0) = z$, $\det(Jf(0)) = 1$ and it maps a boundary point of B_n^R to a point belonging to $D - \hat{D}$. One can see that if w is such a point belonging to $D - \hat{D}$, then $d(z) \leq d(z, w)$. From our definition of d , we observe that

$$d(z) \leq d(z, w) \leq (1/2) \ln[(1 + r/R)/(1 - r/R)]$$

(distance-decreasing property under holomorphic mappings; consider f to be a holomorphic map from B_n^R to D [7]). Hence,

$$\begin{aligned} 1/r &\leq \coth d(z) \cdot (1/R), \\ (1/r)^{2n} &\leq (\coth d(z))^{2n} \cdot (1/R)^{2n}. \end{aligned}$$

This inequality is true for all the R 's satisfying the properties mentioned above. Considering the definition of $M_D^E(z)$, one can now conclude our desired inequality

$$|M_{\hat{D}}^E(z)| \leq (\coth d(z))^{2n} \cdot |M_D^E(z)|. \quad \square$$

Lemma B. *Suppose D is a strongly pseudoconvex bounded domain in \mathbf{C}^n and D_1 is a neighborhood of a boundary point of D . Let this*

boundary point be p , $\hat{D} = D \cap D_1$, and $M_{\hat{D}}^E, M_D^E$ the Eisenman-Kobayashi measures of \hat{D} and D , respectively. Then we have

$$\lim_{z \rightarrow p} \frac{|M_{\hat{D}}^E(z)|}{|M_D^E(z)|} = 1 \quad \text{for all } z \in \hat{D}.$$

Proof. We divide our proof into two steps.

(1) Since the inclusion map $\hat{D} \rightarrow D$ is holomorphic, by the volume-decreasing property [7] we have

$$|M_{\hat{D}}^E(z)| \geq |M_D^E(z)| \quad \text{i.e., } \frac{|M_{\hat{D}}^E(z)|}{|M_D^E(z)|} \geq 1.$$

(2) From Lemma (A) we have

$$|M_{\hat{D}}^E(z)| \leq (\coth d(z))^{2n} \cdot |M_D^E(z)|.$$

Now it is clear from our definitions that

$$d(z, w) \geq d_k(z, w) \quad \forall z, w \in D,$$

where d_k is the Kobayashi metric on D [7]. However, if we set

$$d_k(z, D - \hat{D}) = \inf \{d_k(z, w) : w \in D - \hat{D}\},$$

it is known that

$$\lim_{z \rightarrow p} d_k(z, D - \hat{D}) = \infty$$

if D is strongly pseudoconvex (this statement can easily be derived from the result of I. Graham [5]). Thus, $d(z)$ will go to infinity as $z \in \hat{D}$ approaches p ; consequently, $\coth d(z)$ will tend to 1 as $z \in \hat{D}$ tends to p . At this point we obtain another inequality:

$$\overline{\lim}_{z \rightarrow p} \frac{|M_{\hat{D}}^E(z)|}{|M_D^E(z)|} \leq 1.$$

Combining these two inequalities, we thereby complete the proof of our localization lemma. \square

3.2 Proof of our estimate. Before embarking on our proof, we first make two remarks here.

Remark 1. An analytic ellipsoid is a strongly pseudoconvex domain A in \mathbf{C}^n which can locally be described as: If $p \in \partial A$ and W is a sufficiently small open neighborhood of p in \mathbf{C}^n , then

$$A \cap W = \left\{ z \in \mathbf{C}^n : g(z) = -z_1 - \bar{z}_1 + \sum_{i,j=1}^n b_{ij} z_i \bar{z}_j < 0 \right\},$$

where $[b_{ij}]_{i,j=1}^n$ is a hermitian positive definite matrix.

In our expression p is the origin of the coordinates $\{z_1, z_2, \dots, z_n\}$, z_1 is the complex normal of ∂A at p , and $\{z_2, \dots, z_n\}$ is the basis of the maximal complex tangent space $T_p(\partial A)$. It is known that any analytic ellipsoid is biholomorphically equivalent to the unit ball in \mathbf{C}^n , and the Eisenman-Kobayashi measure on the unit ball is equal to the volume form of the Bergman metric (with the reservation of the multiple of a constant). The following estimate can thus be obtained from the explicit formula of the Bergman metric on B_n (see [13], for example).

There exists a sufficiently small open neighborhood W_1 of ∂A in \mathbf{C}^n and a positive constant c_1 such that

$$|M_A^E(z)| \approx \frac{c_1}{(r(z))^{n+1}} \quad \forall z \in W_1 \cap A.$$

Furthermore, by the volume-decreasing property again we have the following estimate.

There exists an open neighborhood W_2 of p in \mathbf{C}^n and a positive constant c_1 such that

$$|M_{W_2}^E(z)| \geq \frac{c_2}{(r(z))^{n+1}} \quad \forall z \in \hat{W}_2 = W_2 \cap A.$$

Remark 2. Let D be a strongly pseudoconvex boundary domain in \mathbf{C}^n with smooth boundary and p be a given boundary point of D as before. With a similar coordinate system $\{z_1, z_2, \dots, z_n\}$ as in *Remark 1*, in which we can locally characterize D around p as

$$D \cap U_1 = \{z \in D; G(z) < 0\},$$

where U_1 is a neighborhood of p in \mathbf{C}^n , and

$$G(z) = -z_1 - \bar{z}_1 + \sum a_{ij} z_i \bar{z}_j + 2\operatorname{Re} \left\{ \sum_{i \geq 2} \frac{\partial^2 G}{\partial z_i \partial \bar{z}_j}(p) z_i \bar{z}_j \right\} + O(|z|^3)$$

with respect to the above coordinate system. Since D is strongly pseudoconvex, (a_{ij}) is a hermitian positive definite matrix. We can apply our localization in Lemma B to make the assertion: For all $\varepsilon > 0$, there exists a neighborhood U_2 of p in \mathbf{C}^n such that

$$\left| \frac{|M_{\hat{U}_2}^E(z)|}{|M_D^E(z)|} - 1 \right| < \varepsilon$$

for all $z \in \hat{U}_2 = U_2 \cap D$.

Proof of our main estimate. To start, we fix a boundary point $p \in \partial D$ and choose the coordinate system $\{z_1, \dots, z_n\}$ as in *Remark 2*. Now we construct an analytic ellipsoid A_s whose defining equation around the point p is given by

$$g_s = -z_1 - \bar{z}_1 + \sum (a_{ij} - s \cdot \delta_{ij}) z_i \bar{z}_j,$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

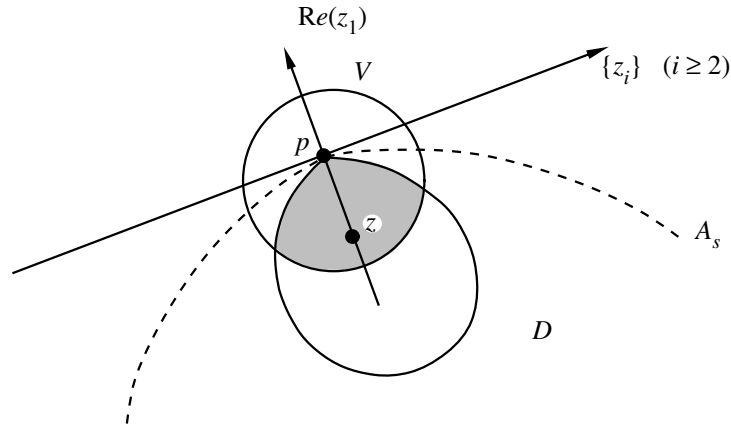
and s is a well-chosen positive constant such that A_s satisfies the following properties:

- (i) there exists a neighborhood V of p in \mathbf{C}^n such that $V \cap A_s = \{z : g_s(z) < 0\}$;
- (ii) with the same V in (i), we have $V \cap A_s \supset V \cap D$.

Graphically, our situation can be illustrated by the picture on the top of the next page.

If we denote by $V_1 = A_s \cap V$ (white area) and $V_2 = D \cap V$ (shaded area), we have by volume decreasing property

$$|M_{V_2}^E(z)| \geq |M_{V_1}^E(z)| \quad \forall z \in V_2.$$



Moreover, if V is sufficiently small we can use *Remarks 1 and 2* above to conclude that

$$|M_D^E(z)| \geq \frac{k}{(r(z))^{n+1}} \quad \forall z \in \mathcal{N}_p$$

where $\mathcal{N}_p = \{\text{the axis } \text{Re}(z_1)\} \cap V$, k is a suitable constant.

Finally we have to observe that all of our processes described in this section are *uniform* in the following sense. We can choose a sufficiently small neighborhood $T \subset \partial D$ of p in such a way that all the *sizes* of domains of comparison and constants can be unchanged so that all the above arguments will remain valid for all $q \in T$. Then we further refine our $V = \cup_{q \in T} \{\mathcal{N}_q\}$. This completes the whole proof. \square

Remark . More precise boundary estimates for both Eisenman-Kobayashi and Carathéodory measures were carried out in [16] following the original work of Graham [4,5] in the case of metrics. The localization lemma in the case of the Kobayashi metric was first used by Royden [12] and Graham [4,5]. Some other related results can be found in [13, 2, 6, 3].

Proof of the Corollary. Applying our theorem, we let $m = n$ and $s = 1$. \square

4. Proof of our main statement. Take the neighborhood N as in the assumption of our theorem. Since D is complete hyperbolic in the sense of Kobayashi, consequently $\ln |M_D^E|$ is a continuous function on

$$\{z_0 + \rho z_1 : \rho \in \mathbf{C}, |\rho| \leq 1\} \subset N \cap D.$$

We can always find a real valued function h , defined and continuous on $|\rho| \leq 1$, harmonic in $|\rho| < 1$, and equal to $(1/m) \ln |M_D^E|$ on $|\rho| = 1$, that is,

$$h(\rho) = (1/m) \ln |M_D^E(z_0 + \rho z_1)| \quad \forall |\rho| = 1.$$

Let h^* be a harmonic conjugate of h , set $g = h + ih^*$. Then g is continuous on $|\rho| \leq 1$ and holomorphic in $|\rho| < 1$. Next, let b be any vector in \mathbf{C}^n with $\|b\| = 1$ and λ_0 any real number satisfying $0 < \lambda_0 < 1$. Consider the analytic disk

$$\Sigma_\lambda : \rho \rightarrow z_0 + \rho z_1 + \lambda e^{-g(\rho)} b,$$

in \mathbf{C}^n where $|\rho| \leq 1$ and λ fixed, $0 \leq \lambda \leq \lambda_0$.

Claim. $\cup_{0 \leq \lambda \leq \lambda_0} \partial \Sigma_\lambda \subset \subset D$.

Since for all $z \in \partial \Sigma_\lambda$, z is the image of some ρ with $|\rho| = 1$,

$$\begin{aligned} \|z - (z_0 + \rho z_1)\| &= \|\lambda e^{-g(\rho)} b\| \\ &= \lambda e^{-h(\rho)} \leq \lambda_0 e^{-(1/m) \ln |M_D^E(z_0 + \rho z_1)|} \\ &= \lambda_0 \cdot |M_D^E(z_0 + \rho z_1)|^{-1/m}. \end{aligned}$$

By our assumption, since $z_0 + \rho z_1 \in N \cap D$, we have

$$|M_D^E(z_0 + \rho z_1)| \geq \frac{k}{(r(z_0 + \rho z_1))^{m+s}}.$$

Hence,

$$\frac{1}{|M_D^E(z_0 + \rho z_1)|^{1/m}} \leq r(z_0 + \rho z_1) \cdot \left[\frac{(r(z_0 + \rho z_1))^s}{k} \right]^{1/m}.$$

Moreover, we can choose $U \subset N$ to be sufficiently small so that the inequality

$$\left[\frac{(r(z_0 + \rho z_1))^s}{k} \right]^{1/m} < 1$$

also holds. Therefore, we obtain

$$\|z - (z_0 + \rho z_1)\| < \lambda_0 \cdot r(z_0 + \rho z_1).$$

However, λ_0 lies between zero and one; this yields

$$\|z - (z_0 + \rho z_1)\| < r(z_0 + \rho z_1),$$

which is independent of z and λ . Hence, the inequality holds for all $z \in \cup \partial \Sigma_\lambda$, that is, $\cup \partial \Sigma_\lambda$ is bounded. Furthermore, for all points $z \in \cup \partial \Sigma_\lambda$ and $w \in \partial D$,

$$\|z - w\| \geq \|(z_0 + \rho z_1) - w\| - \|z - (z_0 + \rho z_1)\|,$$

thus

$$r(z) \geq r(z_0 + \rho z_1) - \|z - (z_0 + \rho z_1)\| > 0,$$

hence

$$\bigcup_{0 \leq \lambda \leq \lambda_0} \partial \Sigma_\lambda \subset \subset D.$$

This verifies our claim. Applying *Kontinuitätssatz*, we therefore obtain

$$z_0 + \rho z_1 + \lambda e^{-g(\rho)} b \in D \quad \forall 0 \leq \lambda < 1, \quad |\rho| \leq 1.$$

Hence,

$$z_0 + \lambda e^{-g(0)} b e^{i\theta} \in D \quad \forall 0 \leq \lambda < 1, \quad 0 \leq \theta \leq 2\pi,$$

where b is in arbitrary direction. That is, the ball $B(z_0, \|\lambda e^{-g(0)} b\|) \subset D$, for all $0 \leq \lambda < 1$. It implies that

$$B(z_0, \|e^{-g(0)} b\|) \subset D;$$

consequently,

$$B(z_0, e^{-h(0)}) \subset D.$$

Hence, the function $f : B_n^{e^{-h(0)}} \rightarrow D$ (recall that b is in arbitrary direction) such that $f(z) = z_0 + z$ is well defined and holomorphic. Note that $f(0) = z_0$ and $\det(Jf(0)) = 1$, hence

$$|M_D^E(z_0)| = \inf \left\{ \frac{1}{R^{2n}} : \exists f \in \text{Hol}(B_n^R, D), \right. \\ \left. \text{such that } f(0) = z_0, \det(Jf(0)) = 1 \right\} \\ \leq \frac{1}{e^{-2n \cdot h(0)}} = e^{2n \cdot h(0)}.$$

Therefore,

$$\ln |M_D^E(z_0)| \leq 2n \cdot h(0) = 2n \cdot \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) d\theta \\ = \frac{n}{m\pi} \int_0^{2\pi} \ln |M_D^E(z_0 + e^{i\theta} z_1)| d\theta,$$

which is exactly what we want to show. \square

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