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# AN EQUIVALENCE FORM OF THE BRUNN-MINKOWSKI INEQUALITY FOR VOLUME DIFFERENCES

Chang-Jian Zhao* and Wing-Sum Cheung**

Abstract. In this paper, we establish an equivalence form of the Brunn-Minkowski inequality for volume differences. As an application, we obtain a general and strengthened form of the dual Kneser-Süss inequality.

## 1. Introduction

If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then there is convex body $K\dot{+}L$ such that

$$S(K\dot{+}L,\cdot) = S(K,\cdot) + S(K,\cdot),$$

where $S(K,\cdot)$ denotes the surface area measure of $K$. This is a Minkowski's existence theorem; see [3] or [9]. The operation $\dot{+}$ is called *Blaschke addition.*

**Theorem A** (The Kneser-Süss inequality [9])**.** *If $K$ and $L$ are convex bodies in $\mathbb{R}^n$, then*

$$V(K\dot{+}L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \tag{1}$$

*with equality if and only if $K$ and $L$ are homothetic.*

The *volume differences function* of convex bodies $K$ and $L$ was defined by Leng [5]:

$$Dv(K,D) = V(K) - V(D), \; D \subset K.$$

In [5], Leng established the following Brunn-Minkowski inequality for volume differences.

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**Theorem B.** *If* $K, L$, *and* $D$ *are convex bodies in* $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ *is a homothetic copy of* $D$, *then*

$$Dv(K + L, D + D')^{1/n} \geq Dv(K, D)^{1/n} + Dv(L, D')^{1/n} \tag{2}$$

*with equality if and only if* $K$ *and* $L$ *are homothetic and* $(V(K), V(D)) = \mu(V(L), V(D'))$, *where* $\mu$ *is a constant.*

If $p \geq 1$ and $K$ and $L$ contain the origin in their interiors, a convex body $K +_p L$ can be defined by

$$h(K +_p L, u)^p = h(K, u)^p + h(L, u)^p$$

for $u \in S^{n-1}$. The operation $+_p$ is called the *p-Minkowski addition.* Firey [2] proved the following inequality.

**Theorem C$_1$.** *If* $K$ *and* $L$ *are convex bodies in* $\mathbb{R}^n$ *containing the origin in their interiors,* $p \geq 1$, *and* $0 \leq i < n$, *then*

$$W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}. \tag{3}$$

*Furthermore, when* $p > 1$, *the equality holds if and only if* $K$ *and* $L$ *are dilates of each other.*

Firey's ideas were transformed into a remarkable extension of the Brunn-Minkowski theory, called the *Brunn-Minkowski-Firey theory*, by Lutwak [6], [7]. Lutwak found the appropriate $p$-analog $S_p(K, \cdot)$, $p \geq 1$, of the surface area measure of a convex body $K$ in $\mathbb{R}^n$ containing the origin in its interior. In [6], Lutwak generalized Firey's inequality (3). He also generalized Minkowski's existence theorem, deduced the existence of a convex body $K +_p L$ for which

$$S_p(K \dot{+}_p L, \cdot) = S_p(K, \cdot) + S_p(K, \cdot),$$

(when $K$ and $L$ are origin-symmetric convex bodies), and proved the following result.

**Theorem C$_2$** (Lutwak's $p$-surface area measure inequality)**.** *If* $K$ *and* $L$ *are origin-symmetric convex bodies in* $\mathbb{R}^n$, *and* $n \neq p \geq 1$, *then*

$$V(K \dot{+}_p L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n}. \tag{4}$$

*Furthermore, when* $p > 1$, *the equality holds if and only if* $K$ *and* $L$ *are dilates of each other.*

In [8], Lutwak established the following dual Brunn-Minkowsi inequality.

**Theorem D.** *If* $K, L$ are star bodies in $\mathbb{R}^n$, *then*

$$V(K \check{+} L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \tag{5}$$

*with equality if and only if* $K$ *and* $L$ *are dilates of each other.*

The aim of this paper is to extend Kneser-Süss inequality (Theorem A) to the context of volume differences, which is in turn proved to be equivalent to Leng's result (Theorem B). We then extend Lutwak's $p$-surface area measure inequality (Theorem C$_2$) to the context of volume differences. Finally, a general dual Brunn-Minkowski inequality which strengthens Lutwak's result (Theorem D) is also given.

## 2. Definitions and preliminaries

The setting of this paper is $n$-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors and the letter $B$ for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. We denote by $V(K)$ the $n$-dimensional volume of a convex body $K$. Let $h_K : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, i.e., $h_K(u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}$, where $u \cdot x$ denotes the usual inner product of $u$ and $x$ in $\mathbb{R}^n$.

Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = \mathrm{Max}\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body. Let $\varphi^n$ denote the set of star bodies in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ on $S^{n-1}$.

### 2.1. Mixed volume and dual mixed volume

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and $\lambda_i$ $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by

$$V(\sum_{i=1}^{r} \lambda_i K_i) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1 \cdots i_n}), \tag{6}$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V(K_{i_1 \cdots i_n})$, which is called the *mixed volume* of $K_i, \ldots, K_{i_n}$, depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (6). If $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \cdots K_n)$ is usually written as $V_i(K, L)$.

From (6), we easily get: If $K, L, M \in \mathcal{K}^n$ and $\alpha, \mu \geq 0$, then

$$V_1(M, \alpha K + \mu L) = \alpha V_1(M, K) + \mu V_1(M, L). \tag{7}$$

Further, from (6) it follows immediately that

$$\lim_{\varepsilon \to 0} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} = n V_1(K, L). \tag{8}$$

If $K_1, \ldots, K_n \in \varphi^n$, then the dual mixed volume of $K_1, \ldots, K_n$ is written as $\tilde{V}(K_1, \ldots, K_n)$. If $K_1 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = L$, then $\tilde{V}(K_1, \ldots, K_n)$ is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume

$\tilde{V}(K, B)$ is written as $\tilde{W}_i(K)$ and is called the *i-th dual Quermassintegral* of $K$.

### 2.2. The Blaschke addition and the radial Blaschke addition

If $K$, $L$ and $\alpha, \mu \geq 0$, then the Theorem of Fenchel-Jessen-Alexandrov tells that there exists a convex body, unique up to translation, which we denote by $\alpha \cdot K \dot{+} \mu \cdot L$, such that

$$S(\alpha \cdot K \dot{+} \mu \cdot L, \cdot) = \alpha S(K, \cdot) + \mu S(L, \cdot).$$

This addition is called *Blaschke addition.*

The following result will be used later: If $K, L, M \in \mathcal{K}^n$ and $\alpha, \mu \geq 0$, then

$$V_1(\alpha K \dot{+} \mu L, M) = \alpha V_1(K, M) + \mu V_1(L, M). \tag{9}$$

As an aside, we note that corresponding to (8) one has for $K, L \in \mathcal{K}^n$,

$$\lim_{\varepsilon \to 0} \frac{V(L \dot{+} \varepsilon K) - V(L)}{\varepsilon} = \frac{n}{n-1} V_1(K, L). \tag{10}$$

See Goikkman [4].

If $K, L \in \varphi^n$ and $\alpha, \mu \geq 0$, then the radial Blaschke linear combination, $\alpha \cdot K \check{+} \mu \cdot L$, is the star body whose radial function is given by

$$\rho(\alpha \cdot K \check{+} \mu \cdot L, \cdot)^{n-1} = \alpha \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{11}$$

We shall call the addition *radial Blaschke addition.*

## 3. Lemmas

The following well-known results will be required to prove our main Theorems.

**Lemma 1** (Bellman's inequality)**.** *Let* $a = \{a_1, \ldots, a_n\}$ *and* $b = \{b_1, \ldots, b_n\}$ *be two sequences of positive real numbers and* $p > 1$ *such that* $a_1^p - \sum_{i=2}^n a_i^p > 0$ *and* $b_1^p - \sum_{i=2}^n b_i^p > 0$, *then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p} \tag{12}$$

*with equality if and only if* $a = \upsilon b$ *where* $\upsilon$ *is a constant.*

**Lemma 2** (Minkowski's inequality for integrals)**.** *If* $f_j \geq 0 (j = 1, \ldots, m)$, $p > 1$, *then*

$$\left(\int_{S^{n-1}} \left(\sum_{j=1}^m f_j(u)\right)^p dS(u)\right)^{1/p} \leq \sum_{j=1}^m \left(\int_{S^{n-1}} f_j^p(u) dS(u)\right)^{1/p}, \tag{13}$$

*with equality if and only if* $f_j$ *are effectively proportional.*

*This inequality is reversed if* $0 < p < 1$ *or* $p < 0$.

**Lemma 3.** *If $K, L$, and $D$ are convex bodies in $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ is a homothetic copy of $D$, then*

$$Dv(K \dotplus L, D \dotplus D')^{(n-1)/n} \geq Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n} \tag{14}$$

*with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

*Proof.* We will prove the lemma using the method of Leng [5].

Applying the Knesser-Süss inequality (1), we obtain

$$V(K \dotplus L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n} \tag{15}$$

with equality if and only if $K$ and $L$ are homothetic, and

$$V(D \dotplus D')^{(n-1)/n} = V(D)^{(n-1)/n} + V(D')^{(n-1)/n}. \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned} Dv(K \dotplus L, D \dotplus D') \geq {} & [V(K)^{(n-1)/n} + V(L)^{(n-1)/n}]^{n/(n-1)} \\ & - [V(D)^{(n-1)/n} + V(D')^{(n-1)/n}]^{n/(n-1)}. \end{aligned} \tag{17}$$

From (17) and applying inequality (12), we have

$$Dv(K \dotplus L, D \dotplus D')^{(n-1)/n} \geq (V(K) - V(D))^{(n-1)/n} + (V(L) - V(D'))^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant. □

*Remark 1.* In the special case where $D$ and $D'$ are single points, inequality (14) becomes the classical Kneser-Süss Inequality.

## 4. Main results

We next observe that Lemma 3 is actually equivalent to Leng's result (Theorem B).

**Theorem 1.** *If $K, L$, and $D$ are convex bodies in $\mathbb{R}^n$, $D \subset K$, and $D' \subset L$ is a homothetic copy of $D$, then*

$$\begin{aligned} & Dv(K \dotplus L, D \dotplus D')^{(n-1)/n} \geq Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n} \\ & \qquad \Leftrightarrow Dv(K + L, D + D')^{1/n} \geq Dv(K,D)^{1/n} + Dv(L,D')^{1/n}, \end{aligned} \tag{18}$$

*where the conditions of equality are also equivalent.*

*Proof.* (⇒) Suppose that

$$Dv(K \dotplus L, D \dotplus D')^{(n-1)/n} \geq Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.

From (10), we obtain

$$\begin{aligned} &\frac{n}{n-1}(V_1(K,L)-V_1(D,D')) \\ (19)\qquad &= \lim_{\varepsilon\to 0}\frac{Dv(L\dot{+}\varepsilon K, D'\dot{+}\varepsilon D)+Dv(D',L)}{\varepsilon} \\ &\geq \lim_{\varepsilon\to 0}\frac{(Dv(L,D')^{(n-1)/n}+\varepsilon Dv(K,D)^{(n-1)/n})^{n/(n-1)}+Dv(D',L)}{\varepsilon}, \end{aligned}$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where $\mu$ is a constant.

On the other hand, from (19) and in view of L'Hôpital's rule, we have

$$\begin{aligned} &V_1(K,L)-V_1(D,D') \\ (20)\qquad &\geq \lim_{\varepsilon\to 0}(Dv(L,D')^{(n-1)/n}+\varepsilon Dv(K,D)^{(n-1)/n})^{1/(n-1)}Dv(K,D)^{(n-1)/n} \\ &= Dv(L,D')^{1/n}Dv(K,D)^{(n-1)/n}. \end{aligned}$$

Suppose that $M,N\in\mathcal{K}^n$ and $N\subset M$, from (7) and (20), it follows that

$$\begin{aligned} &V_1(M,K+L)-V_1(N,D+D') \\ (21)\qquad &= (V_1(M,K)-V_1(N,D))+(V_1(M,L)-V_1(N,D')) \\ &\geq (Dv(K,D)^{1/n}+Dv(L,D')^{1/n})Dv(M,N)^{(n-1)/n}. \end{aligned}$$

If we take $M=K+L$ and $N=D+D'$ in (21), in view of $V(K,\ldots,K)=V(K)$, we have

$$Dv(K+L,D+D')^{1/n}\geq Dv(K,D)^{1/n}+Dv(K,D)^{1/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where $\mu$ is a constant.

($\Leftarrow$) Suppose that

$$Dv(K+L,D+D')^{1/n}\geq Dv(K,D)^{1/n}+Dv(L,D')^{1/n},$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where $\mu$ is a constant.

From (8), we have

$$\begin{aligned} &n(V_1(K,L)-V_1(D,D')) \\ (22)\qquad &= \lim_{\varepsilon\to 0}\frac{Dv(K+\varepsilon L,D+\varepsilon D')+Dv(D,K)}{\varepsilon} \\ &\geq \lim_{\varepsilon\to 0}\frac{(Dv(K,D)^{1/n}+\varepsilon Dv(L,D')^{1/n})^n+Dv(D,K)}{\varepsilon}, \end{aligned}$$

with equality if and only if $K$ and $L$ are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where $\mu$ is a constant.

On the other hand, from (22) and in view of L'Hôpital's rule, we have

$$\begin{aligned} &V_1(K,L) - V_1(D,D') \\ (23) \qquad &\geq \lim_{\varepsilon\to 0}(Dv(K,D)^{1/n} + \varepsilon Dv(L,D')^{1/n})^{n-1} Dv(L,D')^{1/n} \\ &= Dv(K,D)^{(n-1)/n} Dv(L,D')^{1/n}. \end{aligned}$$

From (9) and (23), for any $M, N \in \mathcal{K}^n$ and $N \subset M$, we have

$$\begin{aligned} &V_1(K\dot{+}L, M) - V_1(D\dot{+}D', N) \\ (24) \qquad &= (V_1(K,M) - V_1(D,N)) + (V_1(L,M) - V_1(D',N)) \\ &\geq (Dv(K,D)^{(n-1)/n} + Dv(L,D')^{(n-1)/n}) Dv(M,N)^{1/n}. \end{aligned}$$

If we take $M = K\dot{+}L$ and $N = D\dot{+}D'$ in (24), and in view of $V(K,\ldots,K) = V(K)$, we obtain inequality (14). □

*Remark 2.* In the special case where $D$ and $D'$ are single points, Theorem 1 gives the following important result.

**Corollary 1.** *The Knesser-Süss inequality is equivalent to the Bunn-Minkowski inequality, namely, for $K, L \in \mathcal{K}^n$,*

$$\begin{aligned} &V(K\ddot{+}L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n} \\ \Leftrightarrow \quad &V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \end{aligned}$$

*with equality if and only if $K$ and $L$ are homothetic.*

Similarly, from the Lutwak's $p$-surface area measure inequality (4) and the Bellman's inequality, we can get the following result which is a general form of (4).

**Theorem 2.** *If $K, L$, and $D$ are origin-symmetric convex bodies in $\mathbb{R}^n$, $D \subset K$,* and *$D' \subset L$ is a homothetic copy of $D$, then for $n \neq p \geq 1$,*

$$(25) \qquad Dv(K\dot{+}_pL, D\dot{+}_pD')^{(n-p)/n} \geq Dv(K,D)^{(n-p)/n} + Dv(L,D')^{(n-p)/n}.$$

*Furthermore, when $p > 1$, the equality holds if and only if $K$ and $L$ are dilates of each other and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant.*

*Remark 3.* Note that the Knesser-Süss inequality (14) for volume differences corresponds to the case $p = 1$ in (25). On the other hand, if $D$ and $D'$ are single points, (25) reduces to the classical Knesser-Süss inequality.

Finally, the following is a general and strengthened form of Lutwak's dual Brunn-Minkowski inequality.

**Theorem 3.** *If $K, L \in \varphi^n$, $\alpha \in [0,1]$, then for $i < 1$,*

$$\begin{aligned} &\tilde{W}_i(K\check{+}L)^{(n-1)/(n-i)} \\ (26) \qquad &\leq \tilde{W}_i(\alpha K\check{+}(1-\alpha)L)^{(n-1)/(n-i)} + \tilde{W}_i((1-\alpha)K\check{+}\alpha L)^{(n-1)/(n-i)} \\ &\leq \tilde{W}_i(K)^{(n-1)/(n-i)} + \tilde{W}_i(L)^{(n-1)/(n-i)}, \end{aligned}$$

*with equality if and only if $K$ and $L$ are dilates of each other.*

*These inequalities are reversed for $i > n$ or $1 < i < n$.*

*Proof.* Noting that $\tilde{W}_i(K) = \int_{S^{n-1}} \rho(K)^{n-i} dS(u)$, and from (11), (13), we have for $i < 1$,

$$\begin{aligned}
&\tilde{W}_i(K \check{+} L)^{(n-1)/(n-i)} \\
=&\left(\frac{1}{n}\int_{S^{n-1}} \rho(K\check{+}L,u)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
=&\left(\frac{1}{n}\int_{S^{n-1}} \left(\rho(K,u)^{n-1}+\rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)} \\
\le&\left(\frac{1}{n}\int_{S^{n-1}} \left(\alpha\rho(K,u)^{n-1}+(1-\alpha)\rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)} \\
&+\left(\frac{1}{n}\int_{S^{n-1}} \left((1-\alpha)\rho(K,u)^{n-1}+\alpha\rho(L,u)^{n-1}\right)^{(n-i)/(n-1)} dS(u)\right)^{(n-1)/(n-i)} \\
=&\left(\frac{1}{n}\int_{S^{n-1}} \left(\rho(\alpha\cdot K\check{+}(1-\alpha)\cdot L,u)\right)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
&+\left(\frac{1}{n}\int_{S^{n-1}} \left(\rho((1-\alpha)\cdot K\check{+}\alpha\cdot L,u)\right)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
=&\tilde{W}_i(\alpha\cdot K\check{+}(1-\alpha)\cdot L)^{(n-1)/(n-i)}+\tilde{W}_i((1-\alpha)\cdot K\check{+}\alpha\cdot L)^{(n-1)/(n-i)}.
\end{aligned}$$

On the other hand, for $i < 1$,

$$\begin{aligned}
&\tilde{W}_i(\alpha\cdot K\check{+}(1-\alpha)\cdot L)^{(n-1)/(n-i)} \\
=&\left(\frac{1}{n}\int_{S^{n-1}} \rho(\alpha\cdot K\check{+}(1-\alpha)L)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
\le&\ \alpha\left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
&+(1-\alpha)\left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-i} dS(u)\right)^{(n-1)/(n-i)} \\
=&\ \alpha\tilde{W}_i(K)^{(n-1)/(n-i)}+(1-\alpha)\tilde{W}_i(L)^{(n-1)/(n-i)}.
\end{aligned}$$

Similarly, we get

$$\tilde{W}_i((1-\alpha)\cdot K\check{+}\alpha\cdot L)^{(n-1)/(n-i)} \le (1-\alpha)\tilde{W}_i(K)^{(n-1)/(n-i)}+\alpha\tilde{W}_i(L)^{(n-1)/(n-i)}.$$

Hence,

$$\begin{aligned}
&\tilde{W}_i(\alpha\cdot K\check{+}(1-\alpha)\cdot L)^{(n-1)/(n-i)}+\tilde{W}_i((1-\alpha)\cdot K\check{+}\alpha\cdot L)^{(n-1)/(n-i)} \\
\le&\ W_i(K)^{(n-1)/(n-i)}+W_i(L)^{(n-1)/(n-i)},
\end{aligned}$$

with equality if and only if $K$ and $L$ are dilates of each other.

The cases of $i > n$ and $1 < i < n$ are obtained analogously. □

*Remark 4.* Taking $i = 0$, inequality (26) becomes the following strengthened form of the dual Knesser-Süss inequality.

**Corollary 2.** *If* $K, L \in \varphi^n$, $\alpha \in [0,1]$, *then*

(27)
$$\begin{aligned} V(K\check{+}L)^{(n-1)/n} &\leq V(\alpha K\check{+}(1-\alpha)L)^{(n-1)/n} + V((1-\alpha)K\check{+}\alpha L)^{(n-1)/n} \\ &\leq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \end{aligned}$$

*with equality if and only if $K$ and $L$ are dilates of each other.*

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