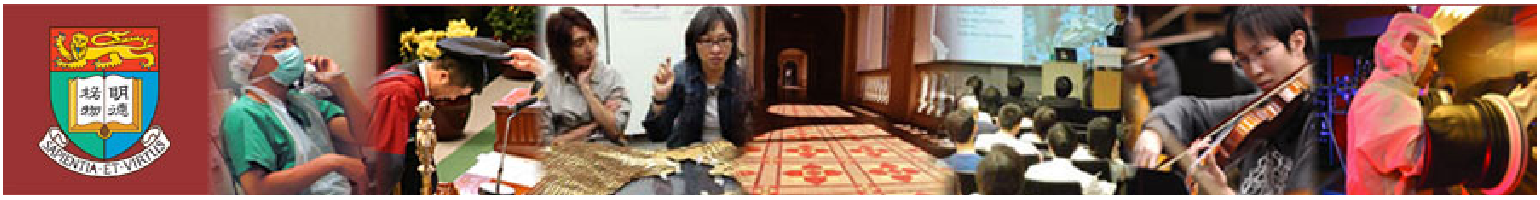

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Using the Learning Study Grounded on the Variation Theory to Improve Students' Mathematical Understanding

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Abstract

This paper illustrates how teachers make use of a learning theory, the variation theory, as well as their own professional expertise and collaboration to help students improve their mathematical understanding. A learning study (cf. Pang & Marton, 2003) involves a group of teachers who undertake theoretically grounded collaborative action research on their own practice. Unlike design experiments, a learning study emphasises teachers' involvement in and ownership of the innovative practices that echo the spirit of the lesson study. The primary role of the researcher(s) in a learning study is to have a professional dialogue with the teachers and to provide professional support when necessary. Furthermore, the major focus of a learning study is on the objects of learning, i.e., on what students are expected to learn, rather than on the teaching arrangements. According to the variation theory, to help students appropriate certain objects of learning, certain patterns of variation and invariance that are co-constituted by the learners and the teacher are necessary. To exemplify this, this paper presents two learning studies conducted in mathematics in Hong Kong. The results show that there was a marked improvement in students' mathematical understanding after learning studies grounded in the variation theory were introduced.

Introduction

A good number of scholars argue that one of the main functions of educational research is to improve educational practice (e.g., Hargreaves, 1996; Mortimore, 2000). Paradoxically, however, research is rarely found to lead to improvements in classroom practice. A yawning gap thus seems to exist between educational research and what is going on in the school and the classroom (e.g., Kaestle, 1993; Kennedy, 1999).

Similar dilemmas can also be found in the field of research on learning. Some researchers argue that research on student learning should take a more pragmatic approach and focus on helping teachers to improve their classroom instruction, rather than on developing a theory of cognition or learning (e.g., Cobb, 1995), whilst other researchers advocate more theoretically orientated educational research (e.g., diSessa, 1991).

However, it is encouraging to note that increasing numbers of researchers regard educational research and its application in education as a synergistic enterprise and aim to promote an environment that allows research and practice to occur in unison (e.g., Schoenfeld, 1999). As Schoenfeld (1999) puts it, 'We can choose to explore theoretical issues in contexts that really matter; and, when we work on important problems we can try to frame them so that our work helps us make progress on fundamental issues' (p. 14).

Along this direction, 'design experiments'[1] (see, for instance, Kelly, 2003; Cobb et al., 2003; Shavelson et al., 2003; Barab & Squire, 2004; Collins, Joseph & Bielaczyc, 2004) have gained increasing importance in the field of educational research. The basic premise of design experiments is for researchers to construct innovative learning environments, sometimes in collaboration with practitioners, to improve learning and teaching and to carry out research by systematically evaluating the results generated from this design to advance a theory of learning and instruction. The primary aim of such experiments is to 'engineer innovative educational environments and simultaneously conduct experimental studies of those innovations' (Brown, 1992, p. 141).

According to Collins (1992), design experiments are premised on the design science of education, which attempts to compare different learning environment designs grounded in particular theoretical approaches and evaluate how these different instructional designs contribute to student learning. Through systematic intervention and unbiased observations, as espoused by Brown (1992), design experiments can offer insight into the operation of some of the major variables that have an impact on student learning.

Design experiments are usually theoretically grounded or driven, with the prime objective being to uncover the principles of learning and to organise the learning environment in a way that is not fully captured by existing theories. These principles are integrally related to efforts to improve practices, which can then become the hypotheses and explanatory principles for further research (Collins, 1992). As put forward by Collins et al. (2004), a design experiment addresses the need: (1) to answer theoretical questions about the nature of learning in context; (2) to identify approaches to study learning phenomena in authentic situations; (3) to broaden the measures of learning; and (4) to generate research findings by undertaking formative evaluation. Also according to Collins (1999), a design experiment is a kind of intervention research that combines both the instrumental and theory-orientated functions of undertaking research.

Increasing numbers of projects today adopt this approach with a view to understanding the process of teaching and learning in situations in which innovative, theory-grounded practices are used and simultaneously changing and reforming educational practices to make a difference (e.g., Brown & Campione, 1996; Scardamalia, Bereiter, & Lamon, 1994).

One noteworthy element of design experiments is that the innovative instructional designs generated usually come from the research team itself, rather than from the teachers. Although, in many research teams, teachers are involved in the process of developing the design by contributing practical ideas and classroom teaching experience, the ownership of the entire design lies largely in the hands of the researchers. The major role of teachers is in facilitating the implementation of the innovative learning environments or designs, and the teachers may see themselves as vehicles to help the research team to collect data to address the research questions.

[1]We make no distinction between 'design experiments', 'design-based research', and 'design studies'.

In light of this, a new research approach, namely the 'learning study', has been developed (see, e.g., Pang & Marton, 2003, 2005). Similar to design experiments, the aims of conducting a learning study are to develop innovative learning environments, to conduct research studies of theoretically grounded innovations and to pool teachers' valuable experiences into one or a series of research lessons to improve teaching and learning.

However, as Marton and Pang (2006) argued, the basic difference between a learning study and a design experiment is that

> in a design experiment, the theory is in the first place in the hands of the researchers themselves, as is the design. In a learning study, teachers are expected to use the theory as a tool and a resource, and to set up the design themselves. They are also expected to use the theory in the implementation of the design, and in their interaction with the students to help them make sense and use of what the students say and then to choose how to respond. So, as with a lesson study, the teachers are the designers. The researcher's role is merely to assist the teachers in the appropriation of the theory to be used (Marton & Pang, 2006, p.196)

In a learning study, teachers work together to find ways of making it possible for their students to appropriate a specific object of learning.[2] An object of learning is a specific capability that students are expected to develop during a single lesson or over a longer period of time. Examples include understanding the concept of price, using calculus to handle a mechanics problem and developing intercultural competence.

An object of learning has two constituent parts: the direct and indirect objects of learning. The former is defined in terms of content, that is, supply and demand, calculus, the Mandarin language, etc., and the latter refers to the specific capability that students are expected to develop, for example, being able to calculate, being able to pronounce words, being able to discern the object of learning in novel situations, etc.

Teachers may work together by themselves or together with a research team to conduct a learning study. The first and foremost task is to choose a particular object of learning. The object of learning is usually chosen from amongst those capabilities that are central to the curriculum and/or are known to cause consistent difficulties for students. After agreeing upon the object of learning, the group starts to plan the lesson and design the learning environment, with a focus on making it possible for students to appropriate the particular object of learning chosen. The resources that the group members may draw upon include their own experiences in dealing with this object of learning, previous research (which can be pure academic research or teachers' action research and case sharing), the particular theory to which they subscribe to identify the necessary conditions for appropriating the object of learning, and their understanding of students' prior knowledge and experience or an examination of the degree to which the object of

[2]It should be noted that the critical difference between our concept of the object of learning and the commonly-used concept of a learning objective is that whereas the latter refers to the intended goal, aim or target of the learning to be achieved, the former also comprises the conditions and the outcome of learning. The idea is to describe the goal, conditions and outcome of learning in commensurable terms.

learning and/or its prerequisites have been mastered by the students before the learning study begins.

The students' initial level of capability to appropriate the object of learning is called *the lived object of learning (1)*. Teachers who subscribe to the variation theory of learning (which is elaborated on in the following section) focus on the way in which the specific object of learning is dealt with in terms of the varying aspects of the direct object of learning and those that are kept invariant in the instructional design. This is a fundamental principle derived from the variation theory. The teachers subsequently consider what kind of teaching arrangements and learning resources are most conducive to the constitution and realisation of their designed pattern of variation and invariance. The finalised instructional design, usually in the form of lesson plans and materials, is called *the intended object of learning*.

One of the teachers in the group then carries out the lesson while the other group members observe. The lesson is usually video-taped, and students' learning outcomes are assessed by means of written questions and interviews. The lesson is then analyzed in terms of whether the object of learning was made attainable through the actual patterns of variation and invariance that were co-constituted by the teacher and the students. This is *the enacted object of learning* that has a real impact on student learning, i.e., the object of learning that students possibly experience within the learning environment.

Based on students' answers to the post-lesson written and interview questions, *the lived object of learning (2)*, i.e., the object of learning that students experienced and appropriated after the lesson, can be characterised. The group of teachers then compares the post-lesson results with the pre-lesson student capability, that is, they compare the lived objects of learning (1) and (2). The degree of improvement or deterioration in student learning outcomes can then be related to what has taken place in the learning environment, especially when the enacted and lived objects of learning are described in commensurable terms, i.e., in terms of variation and invariance, based on the analytical framework derived from the variation theory of learning.

The group then has a debriefing session to reflect on the lesson, and it may come up with suggestions as to how to further improve the lesson design or its implementation. If there are changes, then another teacher subsequently carries out the revised lesson design, following the same procedure as that previously described.

This process continues until all of the group members have gone through the learning study cycle, and the study concludes by documenting what they have done and learned. In the past couple of years, this model has been widely used in Hong Kong and Sweden (see, for instance, Lo, Marton, Pang & Pong, 2004; Holmqvist, 2006).

The Variation Theory of Learning

Marton and Booth (1997) argued that a person's way of experiencing a phenomenon is related to his or her structure of awareness. This can be defined in terms of the critical aspects of the phenomenon that the person simultaneously discerns and focuses upon. In other words, if two people simultaneously focus on different critical aspects of the same phenomenon, then they will come up with two different ways of experiencing it.

Learning, according to the variation theory (see Marton & Booth, 1997; Marton & Tsui, 2004; Marton & Pang, 2006), is defined as a change in the way a person experiences a particular phenomenon and is associated with a change in discernment in that person's structure of awareness. This means that there is a change in the critical aspect(s) of the phenomenon that the learner simultaneously focuses on after learning has taken place; the learner is able to discern critical aspects that he or she could not before.

The necessary condition for discernment is 'experienced variation'. According to the variation theory, for a person to be able to discern a particular aspect of a phenomenon, he or she needs to experience variation in that aspect or the 'dimension of variation'. If all other aspects of the phenomenon are kept invariant, and only this particular aspect varies, then the person will be able to discern the varying aspect. According to Marton and Pang (2006), 'the discernment of a feature amounts to experiencing a difference between two things or between two parts of the same thing. This is because we cannot discern quality X without simultaneously experiencing a mutually exclusive quality ~X' (p. 199). For instance, one would not be able to discern the aspect of gender if there were only females in the world. Similarly, if everything in the universe were static in nature, then the aspect of motion could not be discerned. Any aspect of a phenomenon can be a dimension of variation as long as there are potentially varying values along that dimension. Whether a person can discern a particular aspect is a function of the variation in that aspect that he or she experiences.

According to Marton and Booth (1997), to help learners appropriate a particular object of learning, a certain pattern of variation and invariance in the learning environment must be implied. Therefore, the professional role of a teacher is to design a learning environment that enables students to discern the critical aspects of the object of learning, with the systematic and conscious use of variation as a pedagogical tool. Teachers may make use of the following principles when designing the patterns of variation and invariance (see Marton & Tsui, 2004; Marton & Pang, 2006).

1. *The principle of contrast*. To discern quality X, a mutually exclusive quality ~X needs to be experienced simultaneously. For instance, to understand what a fraction is, students need to be presented with a non-example of a fraction, such as a whole number.

2. *The principle of separation*. To discern a dimension of variation that can take on different values, the other dimensions of variation need to be kept invariant or varying at a different rate. For instance, if teachers want students to understand the relationship of a numerator to the value of a fraction, then they may keep the denominator invariant but vary the numerator. In this way, students' attention will be drawn to the numerator, which has been separated from the other critical aspects that affect the value of the fraction.

3. *The principle of generalisation*. To discern a certain value, X_1, in one of the dimensions of variation X from other values in other dimensions of the variation, X_1 needs to remain invariant while the other dimensions vary. For instance, to help students to generalise the concept of 1/2,

teachers may give all kinds of examples that involve 1/2, say half of a pizza, half of an apple, half of an hour, etc.

4. *The principle of fusion*. To experience the simultaneity of two dimensions of variation, these two dimensions need to vary simultaneously and be experienced by the learner. For instance, to enable students to understand the two critical aspects of numerator and denominator in determining the value of a fraction, teachers may vary both the numerator and the denominator at the same time, systematically, such as 1/2, 2/3, 3/4, 4/5, etc.

With the use of a learning study that is grounded in the variation theory, teachers are given the opportunity to develop a theory-driven learning environment design with a special focus on the pattern of variation and invariance to be constituted, i.e., which aspects of the direct object of learning are to be varied and which are to be kept invariant. Therefore, there is a greater likelihood of improving students' understanding and thus bringing about learning. To demonstrate the efficacy of learning studies, two such studies conducted by two groups of teachers in Hong Kong are described below. Both of these learning studies, one of which took place in a secondary school and the other in a primary school, aimed to enhance students' mathematical understanding. The former study took on the concept of 'slope', whilst the latter dealt with the concept of 'fractions'.

The Learning Study One

Learning Study One was conducted by a teacher-researcher with three classes of 141 Secondary Three students in Hong Kong. Two mathematics teachers and the teacher-researcher formed the learning study group, and they worked together to develop a joint lesson plan for two lessons to achieve the object of learning based on their understanding of the variation theory, the students' pre-lesson level of understandings and their past experience of handling the topic. The student learning results were then compared with those of another class taught by a teacher who had opted not to join the learning study group, but had developed his own lesson plan. This can, in a sense, be considered a comparison group.

1. Choosing the object of learning

The learning study group agreed upon an object of learning, which was to help students develop the capability to understand the mathematical concept of slope. This particular object of learning was chosen for a variety of reasons. First, 'Introduction to the slope of a straight line' is one of the mathematical topics that students need to learn under the Hong Kong mathematics curriculum. Second, it is a mathematical concept that students have consistently been found to have difficulty with. In their preparatory meetings, the teachers identified four common misconceptions amongst their students, as follows.

a. Some students think the formula to find the slope is (y2 – y1)/(x1 – x2), (y1 – y2)/(x2 – x1) or (x1 – x2)/(y1 – y2).
b. Some students perceive slope to be the angle of inclination with the x-axis.

c. If two straight lines are parallel in the Cartesian coordinate system, then some students think that the longer one will have a steeper slope than the shorter one.
d. Some students conceive that straight lines with greater values of negative slope are steeper.

They also shared four of the major difficulties that their students had encountered, as follows.

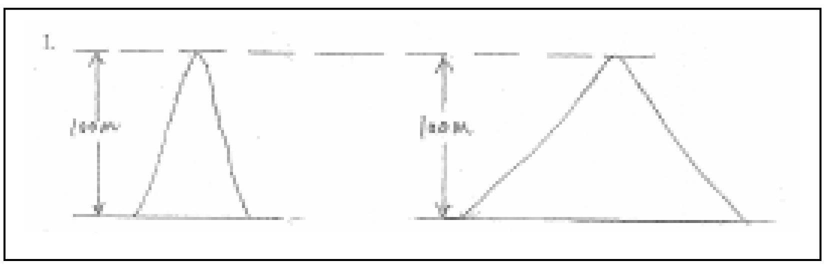
a. Some students are weak at the manipulation of slopes, especially negative slopes.
b. Some students do not understand what slope really means.
c. Some students have difficulty in solving mathematical problems in which the angle of inclination is larger than 90°.
d. Some students do not understand the relationship between $y1 – y2/x1 – x2$ and $\tan \theta$.

2. Ascertaining student pre-understanding

To ascertain their understanding and experience of the object of learning before instruction, a pre-test was conducted with all of the students involved in the study.

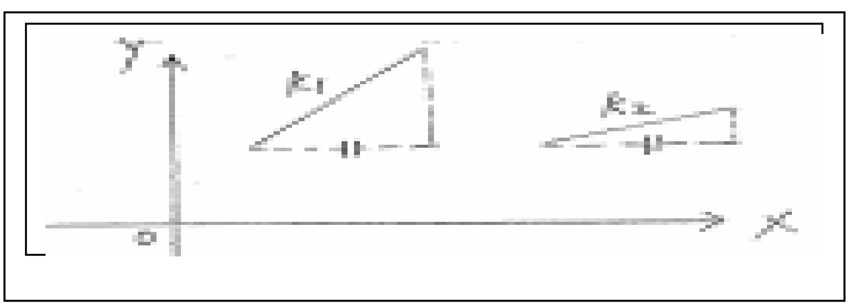
In question 1, as shown in Figure 1, the students were presented with two hills that had the same vertical distance but different horizontal distances. They were asked to say which hill would be more difficult to climb and to explain why. This question was designed to assess whether students were able to discern that horizontal distance is one of the critical aspects of understanding slope.

Figure 1



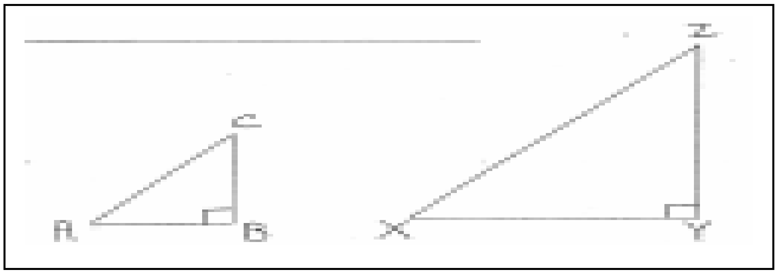
In question 2, as shown in Figure 2, the students were given a diagram with two right-angled triangles that had the same horizontal distance, but different vertical distances. They were invited to explain which hypotenuse of the two triangles was steeper. This was to determine whether students could discern vertical distance as one of the critical aspects of understanding slope.

Figure 2



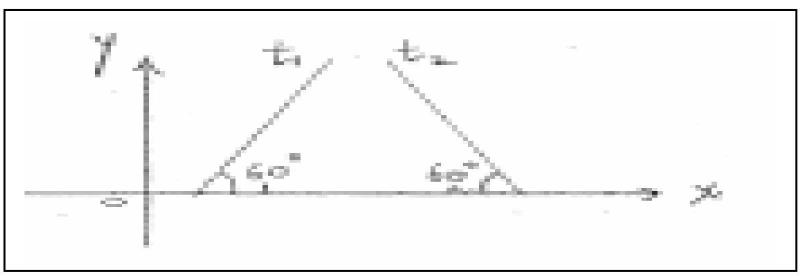
Question 3 was more difficult than questions 1 and 2. The students were presented with two similar right-angled triangles that had different vertical and horizontal distances (see Figure 3). They were asked to identify which of the two triangles had the steeper hypotenuse. This served to ascertain whether they could simultaneously discern both vertical and horizontal distance as critical aspects of understanding slope.

Figure 3



Finally, in question 4, there were two straight lines drawn in the Cartesian coordinate system, one with an angle of inclination of 60º with the x-axis and the other with an angle of inclination of 120° (refer to Figure 4). The students were required to explain which straight line was steeper. This was aimed at finding out whether the students grasped the concept of negative slope.

Figure 4



The following table summarises the pattern of variation and invariance as contained in the pre-test questions.

	Question 1	Question 2	Question 3	Question 4
Vertical distance	Invariant	Varied	Varied	Varied
Horizontal distance	Varied	Invariant	Varied	Varied
Angle of inclination	Varied	Varied	Invariant	Varied

Four qualitatively different ways of experiencing or understanding slope were identified. The first two ways indicated that students had managed to discern either horizontal distance or vertical distance as the critical aspect of understanding slope. The third more advanced question demonstrated that students could discern the critical aspects of both horizontal and vertical distance simultaneously. The most sophisticated, and desirable, way of understanding slope showed that students were able to take into account horizontal distance, vertical distance and the angle of inclination in a simultaneous manner.

According to the results of this pre-test, only 4.3% and 8.5% of all of the students participating in the study could discern horizontal distance and vertical distance, respectively, as the critical aspect of understanding slope. Only one out of 141 students (0.7%) could discern both of these critical aspects simultaneously by correctly explaining why the slopes of the hypotenuses of the similar triangles in Question 3 were equal. Finally, none of the students managed to explain why the two straight lines in Question 4 had the same steepness. This seemed to suggest that the object of learning that was chosen was quite a difficult one for the students concerned.

3. Planning the lessons – the intended object of learning

Based on these findings, the teachers sat down together and developed a lesson plan to teach the mathematical concept of slope, based on their own teaching experience and intuition. The major difference between the learning study group and the comparison group was that the former made *explicit* use of the variation theory when planning the lessons, whereas the latter followed no explicit theory.

The first lesson plan of the learning study group was for a 40-minute lesson, the structure of which was as follows.

Part 1: Introduction (5 minutes)

The teacher first uses the principle of ***contrast*** to allow the students to experience what is meant by vertical and what is not. The teacher shows photos of the Two International Finance Centres in the Central district of Hong Kong and concludes with the remark that many buildings are vertical in shape. The teacher then asks the students to give examples of some buildings around the world that are not vertical. They are expected to answer ‘the Campanile di Pisa in Italy’ because it is world famous and well known locally. The teacher then puts the following question to the students: ‘Imagine you had visited the Campanile di Pisa. One of your friends who had not visited it asks you how steep the tower is. How would you describe its steepness to her?’ After a brief discussion amongst the students, the teacher then suggests that the concept of slope could be applied to describe steepness.

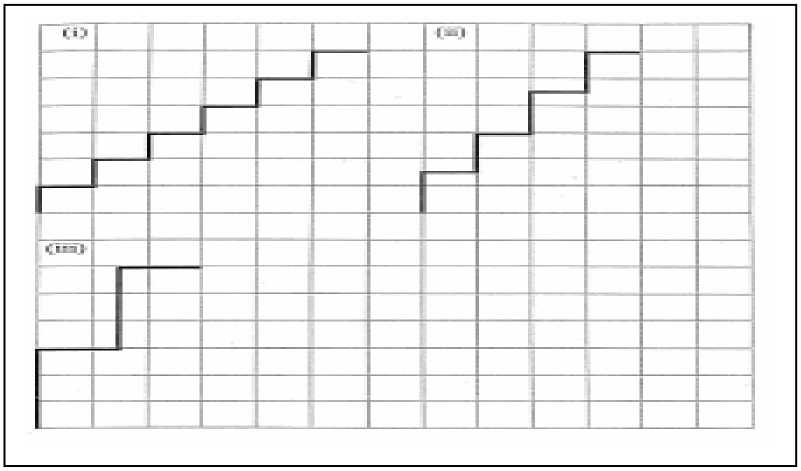
Part 2: Development

I. Group discussion and presentation by students (15 minutes)

The teacher makes use of four worksheets to enable students to discern the critical aspects of slope. Each worksheet shows a diagram of staircases. The students are asked to determine and explain which staircase is the steepest and to identify the factors that affect the steepness of staircases.

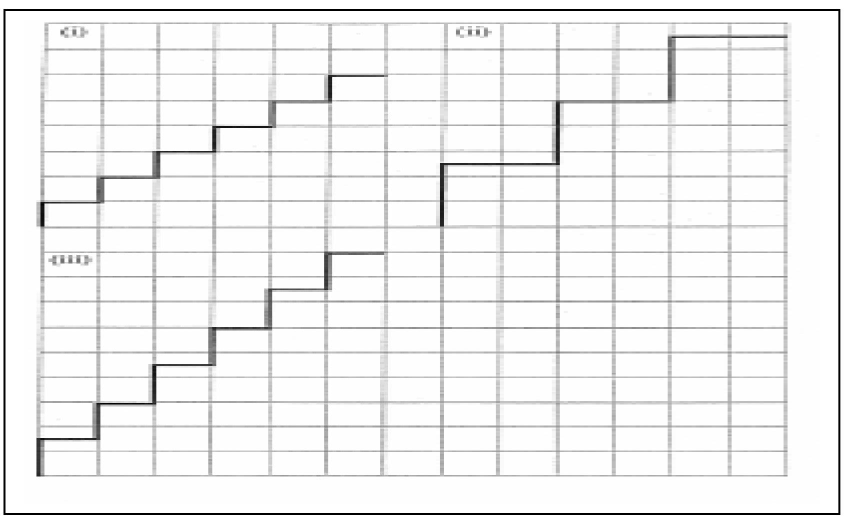
The teacher first divides students into groups and ask them to discuss the two cases in worksheets 1 and 2 in which the teacher has applied the principle of ***separation***. In worksheet 1, the vertical distance of each staircase is kept invariant, but the horizontal distance varies (see Figure 5). Therefore, the students should be able to experience variation in the dimension of horizontal distance and thus be able to discern it as a critical aspect in the steepness of a staircase.

Figure 5



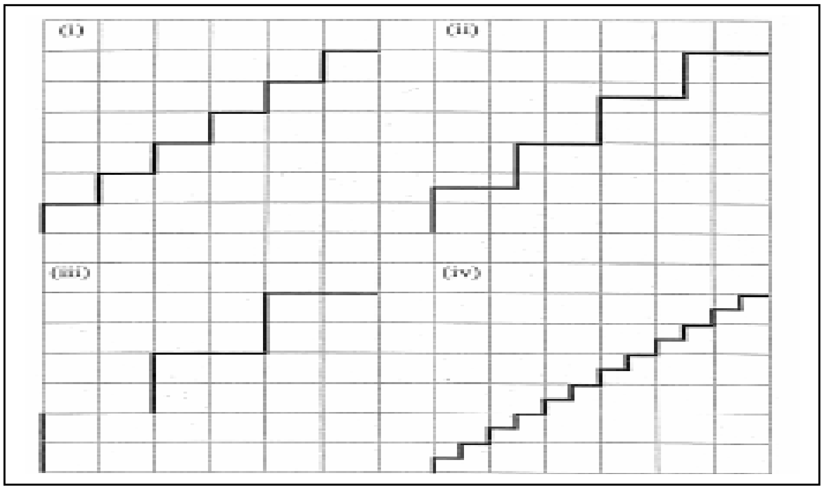
In worksheet 2, the horizontal distance is kept invariant, but the vertical distance varies (see Figure 6). It is hoped that students will be able to experience variation in the dimension of vertical distance and also to discern it as a critical aspect.

Figure 6



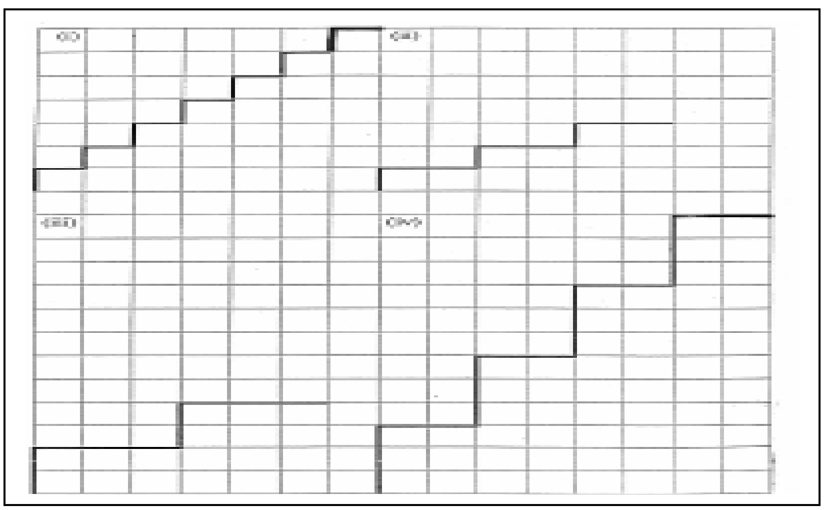
The teacher then invites the students to discuss worksheets 3 and 4. In worksheet 3, both the horizontal and vertical distances of each staircase are kept invariant, but the number of steps varies (see Figure 7). Students should then be able to understand that, although some staircases have different numbers of steps, they can still have the same steepness. The understanding of steepness can thus be generalised.

Figure 7



Finally, the principle of ***fusion*** is used in worksheet 4. Both the horizontal and vertical distances of the staircases vary at the same time (see Figure 8). In this way, students should be able to experience simultaneous variation in both aspects and thus be able to discern both critical aspects of steepness simultaneously.

Figure 8



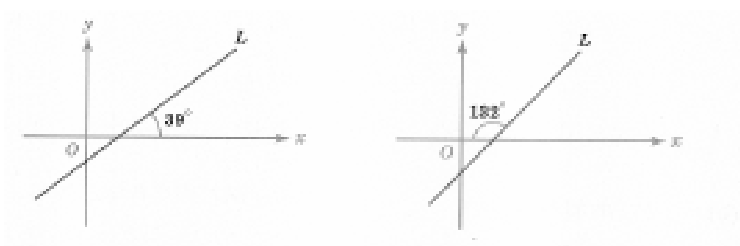
(II) Definition and application of the slope of a straight line (20 minutes)

The teacher introduces the three definitions of the slope of a straight line. First, he or she introduces the general definition of slope as the ‘vertical distance over the horizontal distance’. By using the rectangular coordinates system, the teacher explains the second definition of slope by saying that the slope of a straight line passing through two points (x1, y1) and (x2, y2) is equal to (y1 - y2)/(x1 - x2).

Students then attempt three problems in which variation is introduced in the dimension of the angle of inclination. The problems are to find the slope of the following three pairs of coordinates: A (0, 0), B (2, 4); C (1, -2), D (-1/2, 4); E (14, 7), F (-8, 7). The solutions to the three problems are that the slopes are positive, negative and zero, respectively. To highlight the inter-relationships amongst the three critical aspects, the teacher asks the students about the relationship between the vertical distance, the horizontal distance and the angle of inclination in a right-angled triangle.

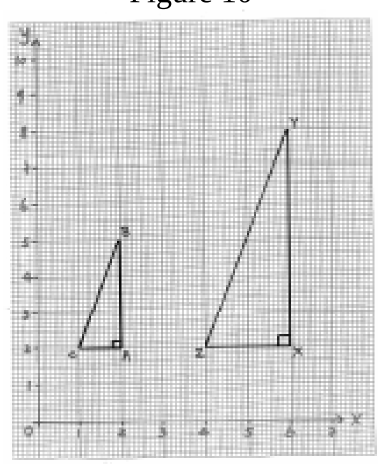
At this juncture, the teacher introduces the third definition of slope, i.e., that the slope of a straight line = tan θ, where θ is the angle of inclination of the straight line with the x-axis in the rectangular coordinates system, and gives two examples. In the first example (see Figure 9), a straight line with a given angle of inclination with the x-axis is shown, and the teacher demonstrates how to find the slope of the given straight line by using the definition: slope = tan θ; and in the second example, the slope of a straight line is given, and the teacher demonstrates how to find the angle of inclination by using slope = tan θ.

Figure 9



Subsequently, the teacher draws two similar right-angled triangles on the rectangular coordinates system (see Figure 10). Triangles ABC and XYZ are given, and their adjacent sides, i.e., AC and XZ, are parallel to the x-axis. The teacher asks: 'Which hypotenuse of the triangles has a steeper slope? Explain.' In this case, both the horizontal and vertical distances vary to the same extent, but the tan θ is kept invariant. This allows the students to understand that the slope will be the same if both the horizontal and vertical distances vary at the same rate in the same direction.

Figure 10



4. Implementing, evaluating and revising the lesson

This lesson plan was carried out by Teacher A in Class 1. During the debriefing session, the group determined that there had been insufficient time to carry out the lesson plan and thus suggested the following modifications. First, the time allotted would be doubled from 40 minutes to 80 minutes. Second, to enable the students to better grasp the concept of slope, a photograph of a real staircase and a ramp for disabled people in the school would be shown during the introduction part. This would enable them to experience the phenomenon of slope in a more authentic context. Furthermore, as both the staircase and the ramp have the same vertical distance, but a different horizontal distance, the teacher can make use of this variation to allow students to discern horizontal distance as a critical aspect of slope.

All of the lessons were videotaped and transcribed verbatim. After the lessons, a post-test that was identical to the pre-test was conducted with all of the students. Inter-group and intra-group comparisons were conducted to examine the relationship between the differences in the enacted object of learning and the lived object of learning (i.e., student learning outcomes).

5. Reporting and disseminating the results

Based on a comparison of the pre-test and post-test results (see Table 1), it seemed that students demonstrated quite different lived objects of learning for the first two questions. Students in the learning study group performed better than did their counterparts in the lesson study group in both questions 1 and 2. A total of 40.4% and 44.2% of the students in the learning study group were found to have discerned horizontal and vertical distance as the critical aspects of understanding slope in questions 1 and 2, respectively, whereas only 10.8% and 13.5% of those in the comparison group were able to do so. However, for questions 3 and 4, it appeared that there were no significant differences between the two groups with regard to learning outcomes. Most of the students in both groups were still unable to discern the critical aspects of horizontal distance, vertical distance and the angle of inclination simultaneously when considering slope in question 3, whereas, in question 4, students in both groups were found to be rather weak at handling mathematical problems with negative slope.

Table 1: Comparison of Pre-test and Post-test Results between the Learning Study Group and the Comparison Group

	The Learning Study Group (107 students)		The Comparison Group (34 students)	
	Pre-test	Post-test	Pre-test	Post-test
Question 1	4.8%	40.4%	2.7%	10.8%
Question 2	11.5%	44.2%	0.0%	13.5%
Question 3	0.9%	9.6%	0.0%	8.1%
Question 4	0.0%	8.7%	0.0%	8.1%

Learning Study Two

Learning Study Two was conducted by a teacher-researcher in a Hong Kong primary school that has adopted English as the medium of instruction. A total of four classes of Grade Four students were involved in the study, with two classes belonging to the learning study group and two belonging to the comparison group.

Four teachers were involved in the study. Two teachers formed a learning study group in which they worked collaboratively to develop a joint lesson plan based on their understanding of the variation theory and their past experience of teaching the concept in question. The other two teachers participated in the study in a more passive way. They just did what they normally did with their classes and administered the pre-test and post-test to their students. These two teachers constituted the comparison group.

With regard to the student participants, one of the two classes in the comparison group comprised students who were native English speakers, whereas the learning study group was made up purely of native Cantonese speakers. Both of the classes in the learning study group contained students who were recognised as having learning difficulties or special needs.

1. Choosing the object of learning

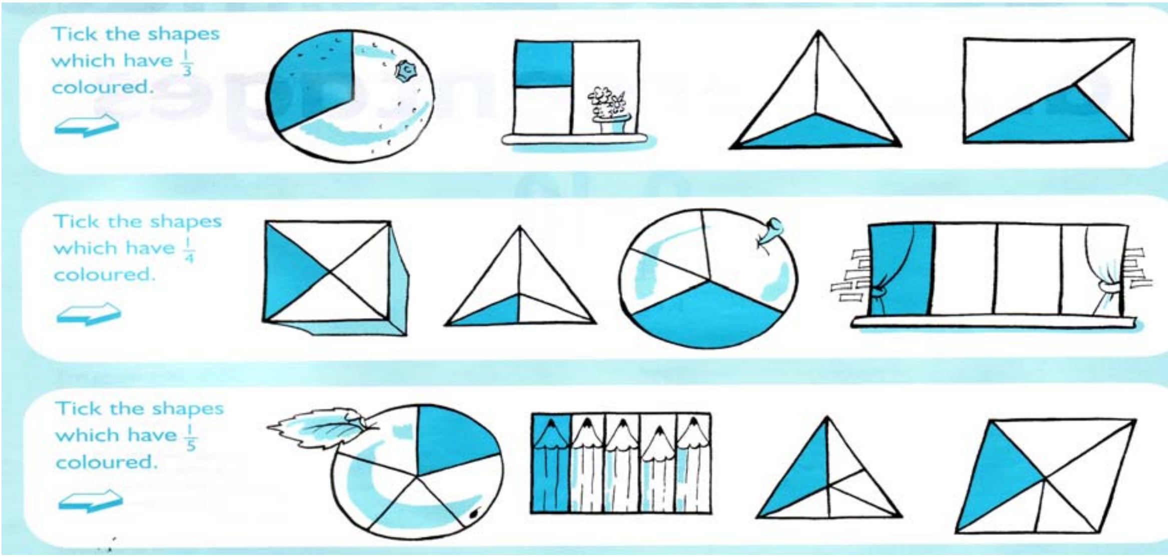
The first step in conducting a learning study is to choose an object of learning. In this study, the learning study group agreed that the object of learning would be to develop students' capability to understand the concept of an 'equal share' within the larger mathematical topic of fractions. This topic was selected because the teachers believed that the topic of fractions is one that students consistently have difficulties with. They also believed that it is important that students learn this concept because fractions form the foundation of proportional thinking, and fractions and/or fraction-related concepts are used in everyday life. The notion of 'equal shares' was considered to be fundamental to helping students to develop a good understanding of the concept of fractions.

2. Ascertaining student pre-understanding

A pre-test was designed to determine how the students experienced the concept of 'equal shares' and how well they grasped it. This test contained questions of a more quantitative nature, in which the answers were relatively black and white, and others of a more qualitative nature, which afforded students with an opportunity to elaborate on the way they understood the concept. In all of the questions except for Question 3, the concept of 'equal share' for fractions was at the core of the tasks.

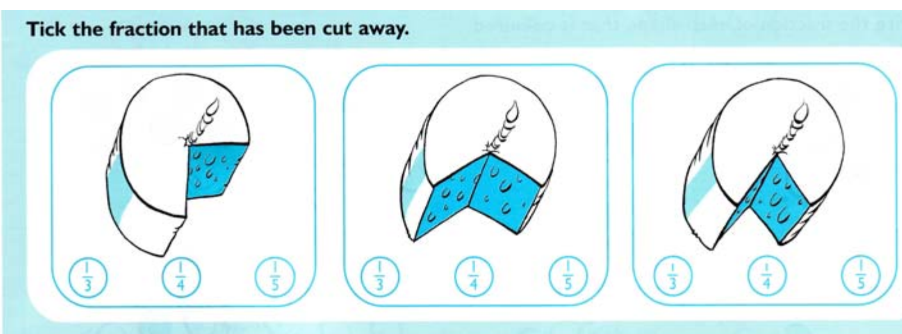
For question 1 (see Figure 11), students were asked to tick shapes that had 1/3, 1/4 and 1/5 coloured in. In each case, a circle, a triangle, a square and a rectangle were used. Two of the answers were correct and two were incorrect, but the students were not aware of this. In all cases, the shapes were divided into the number of parts that the question pertained to, but they varied in terms of whether the parts were divided equally or not. The invariants were the shapes used and the numerator in each case. The variants were the fraction required, the order of the shapes and the way in which they were cut up.

Figure 11



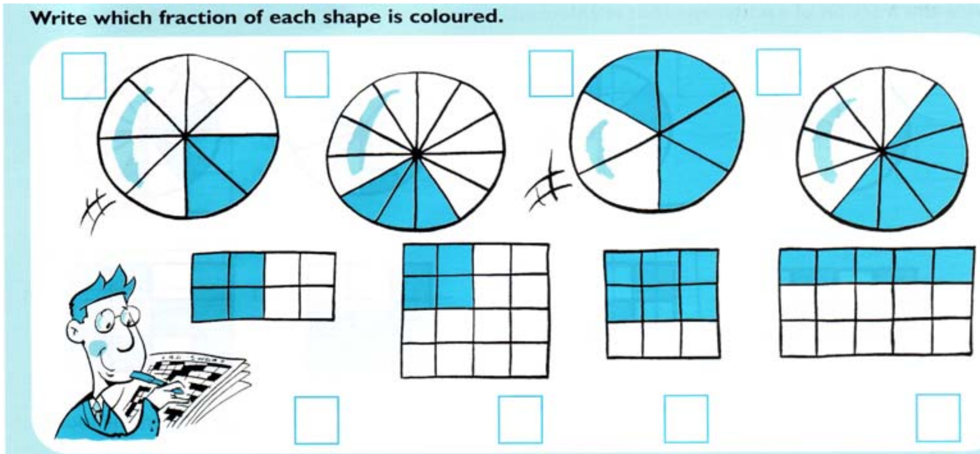
For question 2 (see Figure 12), students had to identify what fraction – 1/3, 1/4 or 1/5 – had been removed from a shape. The shape was kept identical, with only the amount removed varying. In all three cases, the students were presented with the same answer choices, although in each case the answer was different.

Figure 12



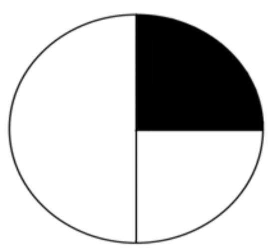
For question 3 (see Figure 13), students had to identify what fraction of a shape was coloured and were presented with either a circle or a rectangle that had been cut into a different number of pieces with varying amounts shaded in. The number of variants increased, with the key invariant being that all of the shares in all of the shapes were 'equal' in size. This question was included to investigate students' performance when the variant of equal and unequal shares was completely removed.

Figure 13



Question 4 (see Figure 14) was much more qualitative in nature. It contained a circle, one half of which was left blank and the other half of which was divided into two equal parts. One of these parts was shaded. The students were asked to state which portion was shaded and then to explain their answer. It was the quality of their explanations that was deemed to be the most important piece of information here, irrespective of whether their answers were correct. In questions 1-3 (especially 1 and 2), it was possible to simply guess correctly, as each question involved making a choice from amongst the options given. For question 4, however, although the first part was of this nature, the second was definitely not. It was here that the students were allowed to demonstrate the differences in their ways of understanding or seeing the notion of the 'equal shares' of a fraction.

Figure 14



3. Planning the lesson

To help the students to discern the critical aspect of the denominator of understanding fractions, the principle of ***separation*** is used. The teacher begins by asking the students the meaning of 1/2 and then varying the denominator by asking them to examine the fractions 1/3 and 1/4 while keeping the numerator invariant. The students thus understand intuitively that the number of pieces into which the whole is cut changes in accordance with the change in the denominator. Their focal awareness is then directed to the relationship between the denominator of a fraction and the meaning of a fraction.

Next, the teacher makes use of the principle of ***generalisation*** to help the students to generalise this idea to other occasions. He or she first demonstrates the meaning of 1/2 by cutting a pizza into two equal halves and then demonstrates halves of other items.

To help students discern the critical aspects of ‘equal shares’ in fractions, the principle of ***contrast*** is used. With the pizza remaining the same (invariant), the teacher cuts it into two unequal pieces and invites the students to contrast this with the previous scenario in which it was cut into two equal halves. It is hoped that students will then be able to discern that fractions carry the notion of equal shares. To further generalise this notion, the teacher then applies the principle of ***generalisation*** and uses other examples to demonstrate the splitting of two unequal pieces from a whole.

After the students have learned to discern the notion of equal shares, the teacher returns to the definitions of 1/3 and 1/4 presented earlier in the lesson to see whether the notion of ‘equal shares’ is present.

Finally, the teacher employs the principle of ***fusion*** by varying both the number of pieces and whether they are divided into equal parts through some well-chosen examples.

To wrap up the lesson and consolidate understanding, students are asked to write down what they have learnt about fractions in the lesson.

4. Implementing, evaluating and revising the lesson

The teachers in the learning study group implemented the lesson plan faithfully in their classes. The planned pattern of variation and invariance was delivered in accordance with their own style of teaching, and no revisions of the lesson design were made.

5. Reporting and disseminating the results

The lived objects of learning, i.e., the student learning outcome data, can be categorised into two areas: questions 1, 2 and 3 had very clear-cut answers and were marked according to how many correct answers the students got. Based on the results of these questions (see Table 2), the students in the learning study group performed better than did their counterparts in the comparison group.

Table 2: Comparison of Pre-test and Post-test Results between the Learning Study Group and the Comparison Group for Questions 1, 2 and 3

	The Learning Study Group (n = 28)			The Comparison Group (n = 31)		
	Pre-test (mean score)	Post-test (mean score)	% Gain	Pre-test (mean score)	Post-test (mean score)	% Gain
Q.1(max: 12)	8.78	10.46	19.13	8.74	10.03	14.76
Q.2 (max: 3)	2.11	2.46	16.59	1.84	1.94	5.40
Q.3 (max: 8)	6.82	7.57	11.00	6.93	7.55	8.95

Because of the qualitative nature of question 4, students' answers were analysed in accordance with the following categories, which differed according to the extent to which the critical aspects of the question were identified and explained.

A The answer is incorrect, and the explanation given appears to be unrelated to the answer given or does not make sense.
Example: "1/3 because the case could keep them warm in the case"

B The answer for deriving the fraction is related simply to the number of pieces rather than to the size of the pieces.
Example: "1/3 because there are 3 boxes and one of them is shaded. So it is 1/3"

C Either an incorrect answer is given, but the student notes a difference in the size of the pieces;
Or a correct answer is given, but the explanation does not mention the size of the pieces involved.
Example: "I think it is 1/3 but I am not sure because one part is half"

D A correct answer is given, and the explanation is related to size, but appears to be incomplete.
Example: "1/3 (if it is not equal) because there are three boxes and one is shaded. 1/4 (if you draw a line) because if it is shared equally"

E A correct answer is given, and the student explains the size or idea of a 1/2 of a 1/2 being a 1/4 or adds a line to the diagram to explain his or her answer.
Example: "1/4 of this shape is shaded. I know this case because there are 3 pieces but they are not equal, so if you add a line, it becomes 1/4"

"The shaded fraction is 1/4. I know that this is the case because the shaded part in that circle is half of 1/2, so it has to be 1/4"

Table 3: Comparison of Pre-test and Post-test Results between the Learning Study Group and the Comparison Group for Question 4

Category	The Learning Study Group (n = 28)		The Comparison Group (n = 31)	
	Pre-test (%)	Post-test (%)	Pre-test (%)	Post-test (%)
A	14.29	0	12.90	3.23
B	21.43	14.29	38.71	41.94
C	32.14	17.86	12.90	12.90
D	7.14	10.71	12.90	9.68
E	25.00	57.14	22.58	29.03

As can be seen from the data, the percentage of students in the learning study group who managed to give the correct answer and explain it correctly using the concept of ‘equal shares’ was much higher than that for the comparison group. Regarding this question, in which the sophistication of the students’ explanations was analysed and categorised, there was a pronounced difference in the ‘way of understanding’ that the students in the learning study group possessed by the end of the study as compared to their counterparts in the comparison group: 57.14% of the students in the learning study were able to appropriate the object of learning and demonstrate a desirable way of understanding (Category E) it, whilst only 29.03% of the comparison group could do so. Overall, based on the results of the quantitative and qualitative questions, it would appear that the lesson had a significant effect on how students both saw and experienced fractions.

Discussion and Conclusion

Methodologically, the two learning studies presented here, as well as the term ‘learning study’, are the result of grafting Japanese lesson study (see Stigler & Hilbert, 1999) into the richly branching tree of design experiments. The fact that the teachers developed the teaching design and implemented it is derived from the lesson study tradition. Also, the comparatively narrow nature of the educational goal – which was something that the students were supposed to learn during one lesson or during a limited sequence of lessons – is also in accordance with the idea of the lesson study.

As espoused by Marton and Pang (2006), if the design of teaching is theory-driven, then the theory must be embodied in the teachers. Due to the interactive nature of teaching and to the fact that successful teaching requires continuous monitoring of this interaction, a teacher’s actions when teaching cannot be predicted, much less prescribed. Teachers need to obtain information from students prior to their next move at every point in the lesson. Teaching requires that the learners and the teacher are in sync with one another. Thus, investigating the way in which a certain theory works in educational practice presupposes that the practitioner sees his or her practice in terms of that theory. A learning study is thus a design experiment in which the teachers are the chief designers. It is important that teachers have a sense of ownership in the instructional design so that they will be more involved in it. At the same time, a learning study is a lesson study that is theory-based, systematically evaluated and grounded in teachers’ expertise and collaboration. Research into a theory of learning and instruction using this approach is a new kind of educational inquiry.

In this connection, the findings of the two learning studies presented here seem to suggest that collaboration between the teachers in a learning study that is grounded in the variation theory of learning is quite effective in improving students’ mathematical understanding, when compared with the results for the comparison groups. The variation theory seems to be a powerful pedagogical tool for enhancing student learning. However, we are not attempting to argue that a learning study is generally superior to the normal classroom teaching of a particular teacher. Rather, what we would argue is that a learning study offers potential gain over a comparison group in the sense that it offers an additional theoretical component, which may open up certain possibilities to improve student learning (Pang & Marton, 2003), and over an individual teacher in the sense that teachers can learn from one another and be exposed to and share a wider range of repertoires and strategies in the learning study setting.

According to Pang (2006), learning is considered to have taken place when people have developed a more considered and sophisticated way of understanding the phenomenon in question. This means that they can discern those critical aspects of the phenomenon that they had taken for granted or not noticed before and see the phenomenon in a new light. In the same vein, teachers can be seen as having experienced professional learning when they have managed to develop a new way of experiencing and seeing their own practice. As reflected in the two teacher-researchers' comments above, by participating in a learning study, teachers' sensitivity towards student learning and their own teaching, towards students' different ways of understanding an object of learning and their own different ways of handling it, and towards the potential of a learning theory are all simultaneously enhanced, thus empowering them to see their own professional practice from both a learning- and learner-centred perspective.

This is evidenced by the following reflective notes made by the two teachers who led the two learning studies reported above.

> *In the present study, teachers in the learning study group actually learned professionally from the learning study. They pooled their experience together to discuss how best to handle a difficult mathematical topic and focused themselves on the object of learning. They learned a new way of implementing mathematics lessons. After the study, the teachers gained precious experience of learning from one another and reflected on their own practice (the teacher-researcher of Learning Study One).*

> *By all accounts, this was simply a single lesson intervention that represented a first attempt at conducting a research lesson. There are things that could be refined, improved and maybe added to enhance the learning experience even further. Either way, I feel this will be a useful lesson to teach every year. It has inspired me to look at how the idea of variation can be further applied across the board throughout the various topics within mathematics as well as in other subjects. Naturally, things take time, but even if one highly structured lesson with systematic variation could be designed for each module, based on the results yielded in this experiment, the overall effect in terms of students' ability to perceive critical factors and differences will improve monumentally. As a result of this, I will be doing an even better job than I used to and hopefully equipping my fellow professionals with a piece of something that I myself am only beginning to understand more completely (the teacher-researcher of Learning Study Two).*

Overall, the two learning studies presented lends support to the central tenet of learning study: the pupils learn better with regard to the object of learning to be dealt with; the teachers learn professionally to improve their practice; and the researcher(s) learn more about the theory of learning and instruction. Through a learning study, teachers can make use of a learning theory, the variation theory in this case, as well as their own professional expertise and collaboration to help students improve their mathematical understanding. A learning study involves a group of teachers who undertake theoretically grounded collaborative action research on their own practice. Unlike design experiments, a learning study emphasises teachers' involvement in and ownership of the innovative practices that echo the spirit of the lesson study. The primary role of the researcher(s) in a learning study is to have a professional dialogue with the teachers and to provide professional support when necessary. Furthermore, the major focus of a learning study is on the objects of learning, i.e., on what students are expected to learn, rather than on the teaching arrangements. According to the variation theory, to help students appropriate certain objects of learning, certain patterns of variation and invariance that are co-constituted by the learners and the teacher are necessary.

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