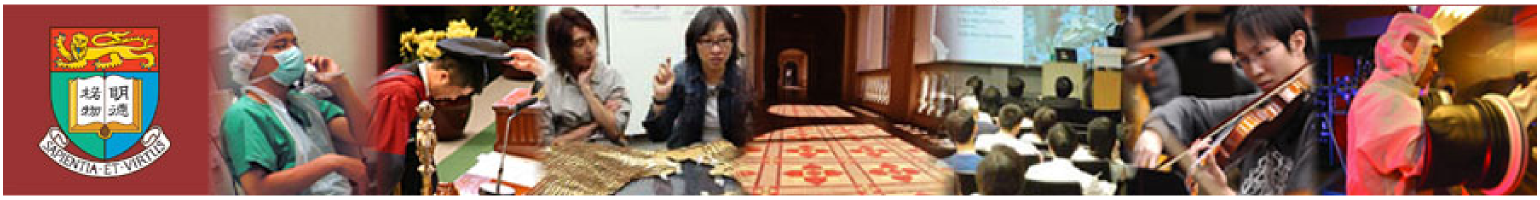

Title	**Properties of a class of C0 semigroups on Banach spaces and their applications**
Author(s)	**Xu, GQ; Yung, SP**
Citation	**Journal Of Mathematical Analysis And Applications, 2007, v. 328 n. 1, p. 245-256**
Issued Date	**2007**
URL	**http://hdl.handle.net/10722/54300**
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Properties of a class of C_0 semigroups on Banach spaces and their applications *

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Abstract

In this paper we investigate some properties of a class of C_0 semigroups on Banach spaces. Suppose that the spectrum of the infinitesimal generator is discrete and there is only finitely many eigenvalues in each vertical strip, we show that such semigroups can be expanded by their generalized eigenvectors under certain conditions. As a consequence, we assert that the semigroups with these conditions is eventually differentiable provided that the system of generalized eigenvectors is complete. As an example, we apply our results to a one-dimensional wave control system.

Key Words: C_0 semigroups, expansion, compactness, differentiability, wave equation.

AMS subject classification. 35C20 35P10 47D06

1 Introduction

Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. In practice, there are many problems in which we need to discuss properties of the semigroups, such as differentiability, compactness and generalized eigenvectors expansion. Since very frequently we merely have information about the infinitesimal generator $\mathcal{A}$, so it is important to know how to deduce these properties from those of $\mathcal{A}$. One of the successful ones is the Hille-Yosida Theorem, which makes use of the information from the resolvent $R(\lambda;\mathcal{A})$ of $\mathcal{A}$. Others such as those in [2], [3], [4], [5] and [6], which tried to provide an answer to the statement put up by Pazy in [1]: "so far there are no known necessary and sufficient conditions, in terms of $\mathcal{A}$ or the resolvent $R(\lambda;\mathcal{A})$, which assure the continuous for $t>0$ of $T(t)$ in the uniform operator topology". Again, these answers more or less rely on information of $R(\lambda;\mathcal{A})$. For example, in the case that X is a Hilbert space, You in [2] showed that a semigroup is norm continuous

*This work is supported by the National Science Foundation of China grant NSFC-60474017, by a grant of Liu Hui Center for Applied Mathematics of Nankai University and Tianjin University and by the Hong Kong CERG grant HKU 7059/06P.

if and only if $\lim_{\tau\to\infty} ||R(\omega + i\tau; \mathcal{A})|| = 0$ for sufficiently large ω. Indeed, in a lot of cases, proving some properties for the resolvents is not hard, for instance proving that the k-power of the resolvent, $R^k(\lambda, \mathcal{A})$, is compact for some integer k is possible for transport operators (see [7]). However, finding an explicit expression for the resolvents is usually not easy. So using other type of information in the investigation become viable. It is along this line of thoughts that we try to use spectral information of $\mathcal{A}$ to investigate properties of the semigroups.

Now suppose that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of all isolated eigenvalues of finite algebraic multiplicity, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda_k\}_{k=1}^r, r \le \infty$. Then the operator $\mathcal{A}$ has the form

$$\mathcal{A} \approx \sum_{k=1}^{r} [\lambda_k E(\lambda_k, \mathcal{A}) + D_k] + \mathcal{A}_0 \tag{1.1}$$

where $E(\lambda_k, A)$ is the Riesz projection and D_k is a nilpotent operator, and for each $x \in X$,

$$T(t)x \approx \sum_{k=1}^{r} e^{\lambda_k t} P_k(t, x) + R(t, x), \tag{1.2}$$

where $P_k(t, x)$ is a X-valued polynomial in t and $R(t, x)$ is the remainder term. Clearly, if (1.2) holds as an equality, then we can deduce the desirable properties of the semigroups from the spectral information. However, in the case that $r = \infty$, the series in (1.2) could diverge. Therefore, for such class of semigroups, it is important to study the summability of the Riesz projections of $\mathcal{A}$. That is we need to know whether the sequence $\sum_{i=1}^{n} E(\lambda_i; \mathcal{A})$ of Riesz projections of $\mathcal{A}$ is uniformly bounded in a Banach space. It is the topic that we shall study in this paper.

In fact, many researchers have worked on this problem, for instance see Lang and Locker [8] [9], Verduyn Lunel [10], Guo [11], Rao [12], Shubov [13] and Xu [14], etc. In [8], [9], [11]–[14], the partial sum $\sum_{i=1}^{n} E(\lambda_i; \mathcal{A})$ is shown to be uniformly bounded and unconditionally convergent in the strong topology. However, we observe that the case that $\sup_n ||E(\lambda_n; \mathcal{A})|| = \infty$ is not included there and we shall address this in this paper.

Assume that $\mathcal{A}$ has a discrete spectrum. When $\mathcal{A}$ is a generator, its spectra are located in a left-half plane and fall into one the two cases: (1) there exists a vertical strip in which there are infinite many eigenvalues of $\mathcal{A}$; (2) there are only finitely many number of eigenvalues of $\mathcal{A}$ in any vertical strip. The first case (1) has been studied in [15]. In this paper, we are going to address the second case (2). We shall make use of the information of $\mathcal{A}$ from three different portions: distribution of the eigenvalues, Riesz projections and their actions on each root subspace $E(\lambda_k, \mathcal{A})X$. Our results will include the case that $\sup_n ||E(\lambda_n; \mathcal{A})|| = \infty$. Under these information, we shall deduce summability of Riesz projections, compactness and differentiability of the semigroups.

The organization of this paper is as follows. In section 2, we shall prove our main theorem and its corollary. In section 3, we shall illustrate our result by an example that comes from a controlled one dimensional wave equation.

2 Main result and its proof

In this section we shall prove the main result of this paper as well as its corollary. We begin with some basic notations.

Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Assume that $\mathcal{A}$ has a discrete spectrum, that is, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda_n; n \in \mathbb{N}\}$. For each $\lambda_n \in \sigma(\mathcal{A})$, denote by $E(\lambda_n; \mathcal{A})$ its Riesz projection on X. We define the $T(t)-$invariant spectral-subspace of X by

$$Sp(\mathcal{A}) := \overline{\text{span}\left\{\sum_{j=1}^{m} E(\lambda_j, \mathcal{A})x \;\middle|\; x \in X; \forall\, m \in \mathbb{N}\right\}},$$

and another $T(t)-$invariant subspace by

$$\mathcal{M}_\infty := \{x \in X \mid E(\lambda; \mathcal{A})x = 0, \forall \lambda \in \sigma(\mathcal{A})\}.$$

Clearly. $Sp(\mathcal{A}) \cap \mathcal{M}_\infty = \{0\}$, and $\overline{Sp(\mathcal{A}) + \mathcal{M}_\infty} \subseteq X$.

For each $\lambda_n \in \sigma(\mathcal{A})$, we denote the algebraic multiplicity of λ_n by m_n, and define operators

$$D_n := (\mathcal{A} - \lambda_n)E(\lambda_n, \mathcal{A}) \qquad \text{and} \qquad D_n^0 = E(\lambda_n, \mathcal{A}).$$

Then for each $n \in \mathbb{N}$, D_n is a bounded linear operator with the property that

$$D_n^k = (\mathcal{A} - \lambda_n)^k E(\lambda_n, \mathcal{A}) \qquad \text{and} \qquad D_n^{m_n} = 0.$$

Now we state our main result of this paper.

Theorem 2.1 *Let $T(t)$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that $\mathcal{A}$ satisfies the following conditions:*

(c1). there exist positive constants M_1 , ρ_1 and ρ_3 such that

$$\sum_{k=0}^{m_n} \frac{t^k \|D_n^k\|}{k!} \leq M_1 e^{-\rho_1 \Re\lambda_n} e^{\rho_3 t}, \quad \forall n \in \mathbb{N}, \quad t \geq 0. \tag{2.1}$$

(c2). there exists a $\tau_0 > 0$ such that the series $\sum\limits_{n=1}^{\infty} e^{\Re\lambda_n \tau_0}$ converges.

Then we can define two family of operators parametrized on $[\tau_0 + \rho_1, \infty)$,

$$T_1(t) : X \to Sp(\mathcal{A}) \qquad \text{and} \qquad T_2(t) : X \to \mathcal{M}_\infty,$$

such that

1). $T_1(t)$ is a compact operator, $T_1(t)$ and $T_2(t)$ are strongly continuous;

2). $T_j(t)T(s) = T(s)T_j(t) = T_j(t+s)$, for $t \geq \tau_0 + \rho_1,\ s \geq 0,\ j = 1, 2$;

3). $T(t)$ has a decomposition

$$T(t) = T_1(t) + T_2(t), \quad t \geq \tau_0 + \rho_1.$$

In addition, if the following condition on the spectrum of $\mathcal{A}$ holds:

(c3). there exist constants $M_2 > 0$ and $\rho_2 > 0$ such that

$$|\Im\lambda_n| \leq M_2 e^{-\rho_2 \Re\lambda_n},$$

then, for each $x \in X$, $T_1(t)x$ is differentiable in $(\tau_0 + \rho_1 + \rho_2, \infty)$.

Remark 2.1 *In theorem 2.1, condition (c1) is a condition on the action of $\mathcal{A}$ on each root subspace. If we take $t = 0$, then we condition (c1) is just*

$$\|E(\lambda_n, \mathcal{A})\| \leq M_1 e^{-\rho_1 \Re\lambda_n}.$$

Therefore, condition (c1) includes the case that $\sup\limits_n ||E(\lambda_n, \mathcal{A})|| = \infty$. *Also, it requires that* $\|E(\lambda_n; \mathcal{A})\|$ *grows not faster than* $e^{-\rho_1 \Re\lambda_n}$ *as* $\Re\lambda_n \to -\infty$.

Conditions (c2) and (c3) are requirements on the spectral distribution of $\mathcal{A}$. Condition (c3) is also a spectral condition for the differentiable semigroup (see, [1] and [6]). It is equivalent to the condition

$$|\lambda_n| \leq M_2 e^{-\rho_2 \Re\lambda_n}, \quad \forall n \in \mathbb{N}.$$

The proof of Theorem 2.1 Let $T(t)$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that conditions (c1) and (c2) are fulfilled. For each $\lambda_n \in \sigma(\mathcal{A})$ and any $x \in X$, we have

$$T(t)E(\lambda_n, \mathcal{A})x = e^{\lambda_n t} \sum_{k=0}^{m_n} \frac{t^k}{k!}(\mathcal{A} - \lambda_n)^k E(\lambda_n, \mathcal{A})x = e^{\lambda_n t} \sum_{k=0}^{m_n} \frac{t^k}{k!} D_n^k x.$$

Since

$$\left\| \sum_{k=0}^{m_n} \frac{t^k}{k!} D_n^k x \right\| \leq \sum_{k=0}^{m_n} \frac{t^k \|D_n^k\|}{k!} \|x\|,$$

so condition (c1) ensures that there exist positive constants M_1, ρ_1 and ρ_3 such that

$$\|T(t)E(\lambda_n, \mathcal{A})\| \leq M_1 e^{\Re\lambda_n (t-\rho_1)} e^{\rho_3 t}, \quad \forall n \in \mathbb{N}, \quad t \geq 0.$$

Then, as $t \geq \tau_0 + \rho_1$, it holds that

$$\sum_{n=1}^{\infty} \|T(t)E(\lambda_n, \mathcal{A})\| \leq M_1 e^{\rho_3 t} \sum_{n=1}^{\infty} e^{\Re\lambda_n (t-\rho_1)}.$$

Since X is a Banach space, the series $\sum\limits_{n=1}^{\infty} T(t)E(\lambda_n, \mathcal{A})$ converges in the operator norm. Note that for each n, $T(t)E(\lambda_n, \mathcal{A})$ is compact, so $\sum\limits_{n=1}^{\infty} T(t)E(\lambda_n, \mathcal{A})$ is also compact. Now we define the operator $T_1(t)$ for each $t \geq \tau_0 + \rho_1$ by

$$T_1(t) := \sum_{n=1}^{\infty} T(t)E(\lambda_n, \mathcal{A}) = \sum_{n=1}^{\infty} E(\lambda_n, \mathcal{A})T(t). \tag{2.2}$$

Evidently, $T_1(t)$ is compact for $t \geq \tau_0 + \rho_1$ and $T_1(t)x \in Sp(\mathcal{A})$.

For each $x \in X$ and $t \geq \tau_0 + \rho_1$,$s \geq 0$, we have

$$\begin{aligned} T(s)T_1(t)x &= \sum_{n=1}^{\infty} T(s)T(t)E(\lambda_n, \mathcal{A})x = \sum_{n=1}^{\infty} T(t)E(\lambda_n, \mathcal{A})T(s)x = T_1(t)T(s)x \\ &= \sum_{n=1}^{\infty} T(s+t)E(\lambda_n, \mathcal{A})x = T_1(t+s)x. \end{aligned}$$

Hence

$$\|T_1(t+h)x - T_1(t)x\| = \|[T(t+h-(\tau_0+\rho_1)) - T(t-(\tau_0+\rho_1))]T_1(\tau_0+\rho_1)x\|, \quad t+h > \tau_0+\rho_1, t > \tau_0+\rho_1.$$

So,

$$\lim_{h\to 0}\|T_1(t+h)x - T_1(t)x\| = 0.$$

Now we define $T_2(t)$ for $t \in [\tau_0+\rho_1, \infty)$ by

$$T_2(t) := T(t) - T_1(t).$$

Clearly, $\{T_2(t)\}_{t\geq \tau_0+\rho_1}$ are bounded linear operators on X and strongly continuous with respect to t. Furthermore,

$$T_2(t)T(s) = T(t)T(s) - T_1(t)T(s) = T(t+s) - T_1(t+s) = T_2(t+s), \quad t \geq \tau_0+\rho_1,\ s\geq 0,$$

and

$$E(\lambda_n, \mathcal{A})T_2(t) = E(\lambda_n, \mathcal{A})T(t) - E(\lambda_n, \mathcal{A})T_1(t) = 0.$$

So $T_2(t)x \in \mathcal{M}_\infty$.

We now further suppose that the (c3) holds. For $x \in X$, $t \geq \tau_0 + \rho_1 + \rho_2$, we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}\lambda_n e^{\lambda_n t}\sum_{k=0}^{m_n}\frac{t^k}{k!}D_n^k x\right\| &\leq \sum_{n=1}^{\infty}|\lambda_n| e^{\Re\lambda_n t}\sum_{k=0}^{m_n}\frac{t^k\|D_n^k\|}{k!}\|x\| \\ \leq\quad M_1 e^{\rho_3 t}\sum_{n=1}^{\infty}|\lambda_n| e^{\Re\lambda_n(t-\rho_1)}\|x\| &\leq M_1(1/\rho_2 + M_2)e^{\rho_3 t}\sum_{n=1}^{\infty} e^{\Re\lambda_n(t-\rho_1-\rho_2)}\|x\|.\end{aligned}$$

So the series

$$\sum_{n=1}^{\infty}\mathcal{A}T(t)E(\lambda_n, \mathcal{A})x = \sum_{n=1}^{\infty}\left[\lambda_n e^{\lambda_n t}\sum_{k=0}^{m_n}\frac{t^k D_n^k}{k!} + e^{\lambda_n t}\sum_{k=0}^{m_n-1}\frac{t^k D_n^{k+1}}{k!}\right]x$$

converges absolutely for $t \geq \tau_0 + \rho_1 + \rho_2$. Since $\mathcal{A}$ is a closed linear operator, it holds that

$$\mathcal{A}\sum_{n=1}^{\infty}T(t)E(\lambda_n, \mathcal{A})x = \sum_{n=1}^{\infty}\mathcal{A}T(t)E(\lambda_n, \mathcal{A})x.$$

Therefore for $t > \tau_0 + \rho_1 + \rho_2$, $T_1(t)x$ is differentiable and

$$\frac{dT_1(t)x}{dt} = \sum_{n=1}^{\infty}\mathcal{A}T(t)E(\lambda_n, \mathcal{A})x = \mathcal{A}T_1(t)x.$$

The proof is then complete. □

Corollary 2.1 *Let $T(t)$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that conditions (c1)–(c3) in Theorem 2.1 hold. In addition, if one of the following conditions is fulfilled:*

1) the generalized eigenvectors of $\mathcal{A}$ are complete in X;

2) the restriction of the resolvent of $\mathcal{A}$ to $\mathcal{M}_\infty$ is an entire function with values in X of finite exponential type h;

then $T(t)$ is a differentiable semigroup for $t > \tau_1$, where

$$\tau_1 := \max\{\tau_0 + \rho_1 + \rho_2, \tau_0 + \rho_1 + h\}. \tag{2.3}$$

Proof Let $\mathcal{A}$ be the generator of $T(t)$. We can assume without loss of generality that $T(t)$ is exponential stable.

If the system of generalized eigenvectors of $\mathcal{A}$ is complete in X, i.e., $Sp(\mathcal{A}) = X$, the desired result follows immediately from Theorem 2.1. Now we suppose instead that the second condition 2) is verified. From Theorem 2.1, we have a decomposition

$$T(t)x = T_1(t)x + T_2(t)x, \quad t \geq \tau_0 + \rho_1,$$

with $T_1(t)x \in Sp(\mathcal{A})$ and $T_2(t)x \in \mathcal{M}_\infty$. If $\mathcal{M}_\infty = \{0\}$, then $T_2(t) = 0$, so the desired result follows. If $\mathcal{M}_\infty \neq \{0\}$, for each $x \in \mathcal{M}_\infty$, we have $T(t)x \in \mathcal{M}_\infty$, and $R(\lambda, \mathcal{A})x$ is an entire function of finite exponential type h, which is independent of x. From the theory of semigroups of linear bounded operators, we have

$$R(\lambda, \mathcal{A})z = \int_0^\infty e^{-\lambda t} T(t) z dt, \quad z \in X, \quad \Re\lambda \geq 0 > \omega_0(\mathcal{A}).$$

So

$$R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t) x dt, \quad x \in \mathcal{M}_\infty, \quad \Re\lambda \geq 0 > \omega_0(\mathcal{A}).$$

Therefore, for each $f \in X^*$, $\langle R(\lambda, \mathcal{A})x, f\rangle$ is in $H^2(\mathbb{C}_+)$. Note that $\langle R(\lambda, \mathcal{A})x, f\rangle$ is an entire function of exponential type at most h. The Wiener-Paley Theorem says that there is a function $T_{x,f}(t) \in L^2[0, h]$ such that

$$\langle R(\lambda, \mathcal{A})x, f\rangle = \int_0^h e^{-\lambda t} T_{x,f}(t) dt, \quad \forall \lambda \in \mathbb{C}.$$

The uniqueness theorem on the Fourier transform ensures that $T_{x,f}(t) = \langle T(t)x, f\rangle$. Therefore,

$$T(t)x = 0, \quad \forall x \in \mathcal{M}_\infty, \quad t > h.$$

Now for any $x \in X$, we have

$$T(\tau_0 + \rho_1)x = T_1(\tau_0 + \rho_1)x + T_2(\tau_0 + \rho_1)x,$$

and for $t > \tau_1 := \max\{\tau_0 + \rho_1 + \rho_2, \tau_0 + \rho_1 + h\}$, $0 < \varepsilon < t - \tau_1$,

$$T(t)x = T(t - h - \varepsilon - \tau_0 - \rho_1)T(h + \varepsilon)T(\tau_0 + \rho_1)x = T(t - h - \varepsilon)T_1(h + \varepsilon)x = T_1(t)x.$$

This together with Theorem 2.1 asserts that $T(t)x$ is differentiable for $t \in (\tau_1, \infty)$. The proof is then complete. □

3 Application

In this section we shall give an example that comes from control theory to illustrate our result. In control theory, to show that a system satisfies the spectrum determined growth condition is an interesting but difficult problem. If the system operator generates an eventually norm continuous semigroup, then the spectrum determined growth condition holds. However, verifying this condition is difficult because the semigroup usually may not have a workable expression. So using the spectrum of the system operator to verify this property becomes an attractive alternative. The authors in literatures [11]–[14] achieved this aim by proving the Riesz basis property of the eigenvectors of the system operator. In our example below, the eigenvectors of the system operator fail to form a basis.

Consider a controlled wave equation:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad 0 < x < 1, \quad t > 0 \\ w_x(0,t) = k_1 w_t(0,t) + \alpha w(0,t), \\ w_x(1,t) = -k_2 w_t(1,t) - \beta w(1,t) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases} \tag{3.1}$$

where

$$\alpha \geq 0, \beta \geq 0, k_1 \geq 0, k_2 \geq 0 \quad \text{and} \quad k_1 + k_2 \neq 0, \quad \alpha + \beta \neq 0. \tag{3.2}$$

Let the state space be

$$\mathcal{H} := H^1(0,1) \times L^2(0,1),$$

where $H^k(0,1)$ is the usual Sobolev space order k and $H^1(0,1)$ is equipped with the inner product

$$(u,v)_{H^1} := \int_0^1 u'(x)\overline{v'(x)}dx + \alpha u(0)\overline{v(0)} + \beta u(1)\overline{v(1)}.$$

In $\mathcal{H}$, for any $F = (f_1, f_2), G = (g_1, g_2) \in \mathcal{H}$, the inner product is defined by

$$< F, G >_{\mathcal{H}} := \int_0^1 f_1'(x)\overline{g_1'(x)}dx + \alpha f_1(0)\overline{g_1(0)} + \beta f_1(1)\overline{g_1(1)} + \int_0^1 f_2(x)\overline{g_2(x)}dx.$$

Here we use the notation $u'(x) = \frac{du}{dx} = u_x(x)$ for the sake of convenience. Clearly, $\mathcal{H}$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$D(\mathcal{A}) := \{(u,v) \in H^2(0,1) \times H^1(0,1) \mid u'(0) = \alpha u(0) + k_1 v(0), \quad u'(1) = -\beta u(1) - k_2 v(1)\} \tag{3.3}$$

$$\mathcal{A}(u,v) := (v, u''), \quad (u,v) \in D(\mathcal{A}). \tag{3.4}$$

With the help of these notations, we can rewrite the system (3.1) into an evolutionary equation in $\mathcal{H}$:

$$\begin{cases} \frac{d}{dt}W(t) = \mathcal{A}W(t), \quad t > 0, \\ W(0) = W_0. \end{cases} \tag{3.5}$$

where $W(t) := (w(x,t), w_t(x,t))$ and $W_0 := (w_0(x), w_1(x))$.

It is easy to prove the following result.

Theorem 3.1 *Let $\mathcal{A}$ be defined by (3.3) and (3.4). Then $\mathcal{A}$ has compact resolvents and generates a C_0 semigroup of contractions on $\mathcal{H}$.*

In [16], the author has discussed the case that $k_1 \neq 1$ and $k_2 \neq 1$. Here we shall discuss the case that $k_1 = 1$ and $k_2 \geq 0$.

Theorem 3.2 *Let $\mathcal{A}$ be defined by (3.3) and (3.4). For $k_1 = 1$ and $k_2 \geq 0$, we have*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \Gamma(\lambda) = 0\}$$

where

$$\Gamma(\lambda) := [(1+k_2)\lambda + \beta][2\lambda + \alpha]e^{\lambda} + \alpha[(1-k_2)\lambda - \beta]e^{-\lambda}. \tag{3.6}$$

The proof is a direct verification and we omit the details.

From theorem 3.2 we can see that each eigenvalue of $\mathcal{A}$ is a zero of $\Gamma(\lambda)$:

$$\begin{aligned}\Gamma(\lambda) &= [(1+k_2)\lambda + \beta][2\lambda + \alpha]e^{\lambda} + \alpha[(1-k_2)\lambda - \beta]e^{-\lambda} \\ &= [2\lambda^2(1+k_2) + (\alpha(1+k_2) + 2\beta)\lambda + \alpha\beta]e^{\lambda} - [\alpha(k_2-1)\lambda + \alpha\beta]e^{-\lambda}.\end{aligned}$$

So $\Gamma(\lambda) = 0$ implies

$$e^{2\lambda} = \frac{\alpha(k_2-1)\lambda + \alpha\beta}{2\lambda^2(1+k_2) + [\alpha(1+k_2) + 2\beta]\lambda + \alpha\beta},$$

which means that $\Re\lambda \to -\infty$ as $|\lambda| \to \infty$. Thus, we see that there exist positive constants $D_j, j = 1, 2$, such that as $k_2 \neq 1$,

$$|\lambda| \leq D_1 e^{-2\Re\lambda}, \tag{3.7}$$

and as $k_2 = 1$,

$$|\lambda^2| \leq D_2 e^{-2\Re\lambda}. \tag{3.8}$$

Let $\sigma(\mathcal{A}) = \{\lambda_n, n \in \mathbb{N}\}$. For each $\lambda_n \in \sigma(\mathcal{A})$, we now find its Riesz projection $E(\lambda_n; \mathcal{A})$. It is easy to see that the adjoint operator of $\mathcal{A}$, $\mathcal{A}^*$, is given by

$$\mathcal{A}^*(u, v) = -(v, u''), \quad (u, v) \in D(\mathcal{A}^*) \tag{3.9}$$

where

$$D(\mathcal{A}^*) = \{(u, v) \in H^2(0,1) \times H^1(0,1) \mid u'(0) = -v(0) + \alpha u(0); u'(1) = k_2 v(1) - \beta u(1)\}. \tag{3.10}$$

For each $\lambda_n \in \sigma(\mathcal{A})$, the corresponding eigenvector of $\mathcal{A}$ is

$$\Phi(\lambda_n) = \left(\lambda_n^{-1}\left[e^{\lambda_n x} + \frac{\alpha}{(2\lambda_n + \alpha)}e^{-\lambda_n x}\right], \left[e^{\lambda_n x} + \frac{\alpha}{(2\lambda_n + \alpha)}e^{-\lambda_n x}\right]\right) \tag{3.11}$$

and the eigenvector of $\mathcal{A}^*$ corresponding to $\overline{\lambda_n}$ is

$$\Psi(\overline{\lambda_n}) = \xi_n\left(\overline{\lambda_n^{-1}}\left[e^{\overline{\lambda_n} x} + \frac{\alpha}{(2\overline{\lambda_n} + \alpha)}e^{-\overline{\lambda_n} x}\right], -\left[e^{\overline{\lambda_n} x} + \frac{\alpha}{(2\overline{\lambda_n} + \alpha)}e^{-\overline{\lambda_n} x}\right]\right), \tag{3.12}$$

where

$$\xi_n^{-1} = -\frac{4\alpha}{(2\lambda_n + \alpha)} + \frac{\alpha}{\lambda_n^2}\left[1 + \frac{\alpha}{(2\lambda_n + \alpha)}\right]^2 + \frac{\beta}{\lambda_n^2}\left[e^{\lambda_n} + \frac{\alpha}{(2\lambda_n + \alpha)}e^{-\lambda_n}\right]^2. \tag{3.13}$$

Evidently,

$$\|\Phi(\lambda_n)\|^2 = |\xi_n|^{-2}\|\Psi(\overline{\lambda_n})\|^2 = \frac{1 - e^{2\Re\lambda_n}}{-2\Re\lambda_n} + \left|\frac{\alpha}{(2\lambda_n + \alpha)}\right|^2 \frac{e^{-2\Re\lambda_n} - 1}{-2\Re\lambda_n}$$

and

$$< \Phi(\lambda_n), \Psi(\overline{\lambda_n}) >_{\mathcal{H}} = 1, \quad < \Phi(\lambda_m), \Psi(\overline{\lambda_n}) >_{\mathcal{H}} = 0, \quad m \neq n.$$

These imply that each eigenvalue of $\mathcal{A}$ is simple.

For any $F \in \mathcal{H}$, we have

$$E(\lambda_n, \mathcal{A})F =< F, \Psi(\overline{\lambda_n}) >_H \Phi(\lambda_n)$$

and

$$\|E(\lambda_n; \mathcal{A})\| = \|\Psi(\overline{\lambda_n})\| \|\Phi(\lambda_n)\| = |\xi_n| \|\Phi(\lambda_n)\|^2. \tag{3.14}$$

Since

$$\lim_{n\to\infty} \left|\lambda_n^{-4} e^{-2\lambda_n} \xi_n\right| = \frac{4}{\beta\alpha^2},$$

$$\lim_{n\to\infty} |\lambda_n^2 \Re\lambda_n e^{2\Re\lambda_n}| \|\Phi(\lambda_n)\|^2 = \frac{\alpha^2}{8},$$

we have

$$\lim_{n\to\infty} |\xi_n| \|\Phi(\lambda_n)\|^2 |\lambda_n^{-2} \Re\lambda_n| = \frac{1}{2\beta}. \tag{3.15}$$

This implies that $\sup_n \|E(\lambda_n, \mathcal{A})\| = \infty$ and so the eigenvectors fail to be a basis for $\mathcal{H}$ when $\alpha > 0$ and $\beta > 0$.

Combining (3.14), (3.15) and (3.7) (or (3.8)) yield

$$\|E(\lambda_n; \mathcal{A})\| \approx \frac{|\lambda_n^2|}{2|\Re\lambda_n|\beta} \le \begin{cases} M_1 e^{-4\Re\lambda_n}, & \text{as} \quad k_2 \neq 1, \\ M_1 e^{-2\Re\lambda_n}, & \text{as} \quad k_2 = 1, \end{cases} \tag{3.16}$$

where $M_1 > 0$ is a positive constant.

Note that $\Gamma(\lambda)$ is an entire function of finite exponential type 1 and $\lambda_n, n \in \mathbb{N}$, are zeros of $\Gamma(\lambda)$. So we have

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{1+\varepsilon}} < \infty, \quad \forall \varepsilon > 0.$$

Obviously, when $k_2 \neq 1$ and $\tau_0 > 2$, we have

$$\sum_{n=1}^{\infty} e^{\Re\lambda_n \tau_0} \le D_1 \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{\tau_0/2}} < \infty, \tag{3.17}$$

and when $k_2 = 1$ and $\tau_0 > 1$, the series

$$\sum_{n=1}^{\infty} e^{\Re\lambda_n \tau_0} \le D_2 \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{\tau_0}} < \infty \tag{3.18}$$

converges.

Therefore, we have the following result.

Theorem 3.3 *Let $\mathcal{A}$ be defined as (3.3) and (3.4) and $T(t)$ be the semigroups generated by $\mathcal{A}$. Then for $k_1 = 1,\ k_2 \ge 0$ and $\alpha\beta > 0$, $T(t)$ is differentiable for $t > 8$.*

Proof Let $\mathcal{A}$ be defined as (3.3) and (3.4) and $T(t)$ be the semigroup generated by $\mathcal{A}$. From (3.16) we see that condition (c1) is satisfied with

$$\rho_1 = \begin{cases} 4, & \text{if} \quad k_2 \neq 1 \\ 2, & \text{if} \quad k_2 = 1. \end{cases}$$

Equations (3.7) and (3.8) lead to

$$\rho_2 = \begin{cases} 2, & \text{if} \quad k_2 \neq 1 \\ 1, & \text{if} \quad k_2 = 1. \end{cases}$$

Also, (3.17) and (3.18) show that for any small $\varepsilon > 0$, we can take

$$\tau_0 = \begin{cases} 2+\varepsilon, & \text{if} \quad k_2 \neq 1 \\ 1+\varepsilon, & \text{if} \quad k_2 = 1. \end{cases}$$

For $\lambda \in \rho(\mathcal{A})$, we can get from the resolvent equation $(\lambda I - \mathcal{A})(f,g) = (u,v)$ that

$$R(\lambda, \mathcal{A}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f(x) \\ \lambda f(x) - u(x) \end{bmatrix}$$

with

$$f(x) = e^{\lambda x} u(0) - \left[\frac{(\alpha+2\lambda)}{\alpha} e^{\lambda x} - e^{-\lambda x}\right] b(\lambda) - \frac{1}{\lambda}\int_0^x \sinh\lambda(x-s)[\lambda u(s) + v(s)]ds,$$

and

$$\begin{aligned}\Gamma(\lambda)b(\lambda) \;&=\; [\beta+\lambda(1+k_2)]e^{\lambda}u(0) - \alpha k_2 u(1) - \frac{\alpha(\beta+\lambda k_2)}{\lambda}\int_0^1 \sinh\lambda(1-s)[\lambda u(s)+v(s)]ds \\ &\quad -\alpha\int_0^1 \cosh\lambda(1-s)[\lambda u(s)+v(s)]ds.\end{aligned}$$

Note that $\Gamma(\lambda)b(\lambda)$ is an entire function of finite exponential type 1 and

$$|f(x)| \leq e^{|\lambda|}|u(0)| + \left[\frac{(\alpha+2|\lambda|)}{\alpha} + 1\right] e^{|\lambda|}|b(\lambda)| + \frac{2e^{|\lambda|}}{|\lambda|}\int_0^1 |\lambda u(s)+v(s)|ds.$$

So $R(\lambda, \mathcal{A})(u,v)^T$ is a meromorphic function of finite exponential type at most $h = 2$. According to Corollary 2.1, the semigroup $T(t)$ is differentiable for $t > \tau_1 = 8 + \varepsilon$. So the proof is complete because $\varepsilon > 0$ is arbitrary. □

As a consequence of the spectral mapping theorem for differential semigroups, we have the following result.

Corollary 3.1 *System (3.5) satisfies the spectrum determined growth assumption.*

Remark 3.1 *In this example, we have only used the distribution information of the spectrum of $\mathcal{A}$ which is determined by the equation $\Gamma(\lambda) = 0$. More detailed description can be found in the recent paper [17].*

Acknowledgment

The authors would like to thank the referees for their helpful comments and suggestions.

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