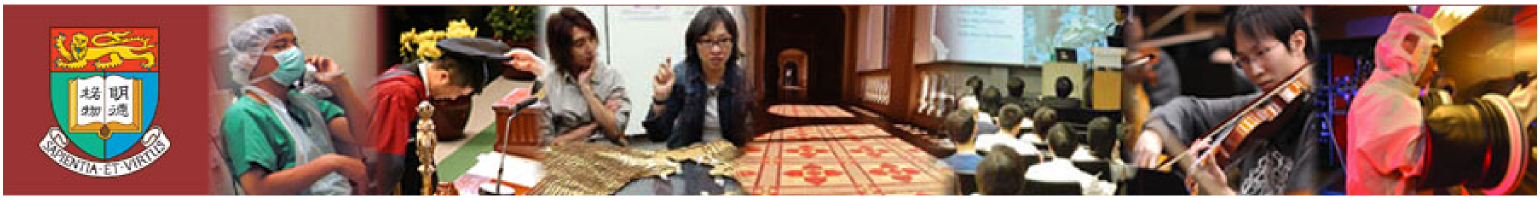

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EXPONENTIAL STABILIZATION OF LAMINATED BEAMS WITH STRUCTURAL DAMPING AND BOUNDARY FEEDBACK CONTROLS*

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Abstract. We study the boundary stabilization of laminated beams with structural damping which describes the slip occurring at the interface of two-layered objects. By using an invertible matrix function with an eigenvalue parameter and an asymptotic technique for the first order matrix differential equation, we find out an explicit asymptotic formula for the matrix fundamental solutions and then carry out the asymptotic analyses for the eigenpairs. Furthermore, we prove that there is a sequence of generalized eigenfunctions that forms a Riesz basis in the state Hilbert space, and hence the spectrum determined growth condition holds. Furthermore, exponential stability of the closed-loop system can be deduced from the eigenvalue expressions. In particular, the semigroup generated by the system operator is a C_0-group due to the fact that the three asymptotes of the spectrum are parallel to the imaginary axis.

Key words. Riesz basis, laminated beams, exponential stability

AMS subject classifications. 93C20, 93D15, 35B35, 35P10

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1. Introduction.

The vibration suppression of the laminated beams due to the demand for advanced performance has been one of the main research topics in smart materials and structures. These composite laminates usually have superior structural properties such as adaptability, and the design of their piezoelectric materials can be used as both actuators and sensors. The detailed physical background can be found in [10] and the references therein. In [4], Hansen and Spies derived three mathematical models for two-layered beams with structural damping due to the interfacial slip. Our interest in this paper is to study the first model in [4] which is closely related to the Timoshenko beam theory. The equations for this beam model are

$$\begin{cases} mw_{tt} + (G(\psi - w_x))_x = 0, & 0 < x < 1,\ t \geq 0, \\ I_m(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (D(3s_x - \psi_x))_x = 0, & 0 < x < 1,\ t \geq 0, \\ I_m s_{tt} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta I_m s_t - (Ds_x)_x = 0, & 0 < x < 1,\ t \geq 0, \end{cases} \tag{1.1}$$

where $w(x,t)$ denotes the transverse displacement, $\psi(x,t)$ represents the rotation angle and $s(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, respectively, and $m > 0$ is the density of the beams, $G, I_m, D, \gamma > 0$ are the shear stiffness, mass moment of inertia, flexural rigidity, and adhesive stiffness of the beams together with $\beta > 0$ as the adhesive damping parameter. Moreover, $\sqrt{G/m}$ and $\sqrt{D/I_m}$ are two wave speeds and we always assume

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that they are different in the present paper (see [7]). We refer to [4] for the detailed derivation of the mathematical model and its physical parameters. It is easy to find that if the slip s is assumed to be identically zero, then the first two equations of system (1.1) can be reduced exactly to the Timoshenko beam system. The third equation in (1.1) describes the dynamics of the slip. For convenience, if we introduce another variable ξ of the effective rotation angle by

$$\xi = 3s - \psi, \tag{1.2}$$

then (1.1) changes to

$$\begin{cases} mw_{tt} + (G(3s - \xi - w_x))_x = 0, & 0 < x < 1,\ t \geq 0, \\ I_m\xi_{tt} - G(3s - \xi - w_x) - (D\xi_x)_x = 0, & 0 < x < 1,\ t \geq 0, \\ I_m s_{tt} + G(3s - \xi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta I_m s_t - (Ds_x)_x = 0, & 0 < x < 1,\ t \geq 0. \end{cases} \tag{1.3}$$

For system (1.3), we impose the cantilever boundary conditions, which can be easily obtained from the principle of virtual work (see [4]),

$$\begin{cases} w(0,t) = 0, \quad \xi(0,t) = 0, \quad s(0,t) = 0, \\ \xi_x(1,t) = u_2(t), \quad s_x(1,t) = 0, \quad 3s(1,t) - \xi(1,t) - w_x(1,t) = u_1(t), \end{cases} \tag{1.4}$$

where $u_1(t)$ and $u_2(t)$ are boundary control forces, and the initial conditions (for $0 < x < 1$)

$$(w, \xi, s)\Big|_{t=0} = (w_0, \xi_0, s_0) \quad \text{and} \quad (w_t, \xi_t, s_t)\Big|_{t=0} = (w_1, \xi_1, s_1). \tag{1.5}$$

We point out that due to the action of the slip s, the uncontrolled system (1.3) with boundary conditions (1.4) ($u_1 = u_2 \equiv 0$) in [4] can achieve the asymptotic stability but it does not reach the exponential stability (see Corollary 2.3 and Note 2.1).

In this paper, the following boundary feedback controls are proposed to exponentially stabilize systems (1.3) and (1.4):

$$u_2(t) = -k_2\xi_t(1,t), \quad u_1(t) = k_1 w_t(1,t), \tag{1.6}$$

where k_1 and k_2 are positive constant feedback gains. Then the boundary conditions become

$$\begin{cases} w(0,t) = 0, \quad \xi(0,t) = 0, \quad s(0,t) = 0, \\ \xi_x(1,t) = -k_2\xi_t(1,t), \quad s_x(1,t) = 0, \quad 3s(1,t) - \xi(1,t) - w_x(1,t) = k_1 w_t(1,t), \end{cases} \tag{1.7}$$

and the closed-loop system has both internal damping and boundary controls.

Our goal is to show that the closed-loop system (1.3) with (1.7) is exponentially stable in the state Hilbert space. This will follow from proving the following three aspects: (i) the closed-loop system is dissipative in the state space and the system operator has compact resolvents; (ii) there exist three asymptotes of frequencies for the system which are parallel to the imaginary axis from the left side; (iii) the generalized eigenfunctions of the system form a Riesz basis in the state space and hence the spectrum determined growth condition, and the exponential stability holds for the system. Among these, (i) is easy to verify while (ii) and (iii) are very difficult to solve. Our interests in this paper are mainly concentrated on the asymptotically spectral analysis and the proof of Riesz basis for the system.

Now let us briefly outline the contents of this paper. In the next section, the well-poseness of the system will be established. Asymptotic estimates of the eigenvalues for the system will be given in section 3. This is the foundation that we shall use to investigate the exponential stability and basis property for the system. Section 4 is devoted to the asymptotic expansion of the corresponding eigenfunctions. Finally, in the last section, we obtain a more profound result, namely, the existence of a sequence of the generalized eigenfunctions of the system that forms a Riesz basis in the state Hilbert space. Consequently, the spectrum determined growth condition and the exponential stability are concluded. Furthermore, the semigroup generated by the system operator is actually a C_0-group based on the spectrum distribution of the system.

2. Well-posedness of the system. We start our investigation by formulating the problem on the state Hilbert space. Let

$$\mathcal{H} := \left(H_E^1(0,1) \times L^2(0,1)\right)^3 \tag{2.1}$$

with

$$H_E^i(0,1) := \{f \in H^i(0,1) \mid f(0) = 0\} \quad \text{for } \ i = 1,2, \tag{2.2}$$

where $H^i(0,1)$ $(i = 1,2)$ denote the usual Sobolev spaces. The inner product in $\mathcal{H}$ is defined by

$$\begin{aligned}\langle Y_1, Y_2\rangle_{\mathcal{H}} : &= m\langle z_1, z_2\rangle_{L^2} + G\langle 3s_1 - \xi_1 - w_1', 3s_2 - \xi_2 - w_2'\rangle_{L^2} + I_m\langle \varphi_1, \varphi_2\rangle_{L^2} \\ &\quad + D\langle \xi_1', \xi_2'\rangle_{L^2} + 3I_m\langle h_1, h_2\rangle_{L^2} + 3D\langle s_1', s_2'\rangle_{L^2} + 4\gamma\langle s_1, s_2\rangle_{L^2},\end{aligned} \tag{2.3}$$

where $Y_i := [w_i, z_i, \xi_i, \varphi_i, s_i, h_i]^\top \in \mathcal{H}$ with $i = 1,2$, in which the superscript $\top$ denotes the transpose of a vector or a matrix, $\langle \cdot, \ \cdot\rangle_{L^2}$ is the inner product on $L^2(0,1)$, and the prime represents the differentiation with respect to x. In view of system (1.3) and (1.7), we define a linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ in Hilbert space $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} w \\ z \\ \xi \\ \varphi \\ s \\ h \end{bmatrix} := \begin{bmatrix} z \\ \frac{G}{m}(\xi' + w'' - 3s') \\ \varphi \\ \frac{G}{I_m}(3s - \xi - w') + \frac{D}{I_m}\xi'' \\ h \\ \frac{G}{I_m}(\xi + w' - 3s) - \frac{4}{3}\frac{\gamma}{I_m}s - \frac{4}{3}\beta h + \frac{D}{I_m}s'' \end{bmatrix} \tag{2.4}$$

with

$$\mathcal{D}(\mathcal{A}) : = \left\{ [w, z, \xi, \varphi, s, h]^\top \in \mathcal{H} \;\middle|\; \begin{array}{l} w \in H_E^2(0,1),\ \xi \in H_E^2(0,1),\ s \in H_E^2(0,1), \\ z \in H_E^1(0,1),\ \varphi \in H_E^1(0,1),\ h \in H_E^1(0,1), \\ \xi'(1) = -k_2\varphi(1),\ s'(1) = 0, \\ 3s(1) - \xi(1) - w'(1) = k_1 z(1) \end{array} \right\}. \tag{2.5}$$

If we set $Y := [w, w_t, \xi, \xi_t, s, s_t]^\top$, then the closed-loop system (1.3), (1.5), and (1.7) can be formulated into an abstract evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{d}{dt}Y(t) = \mathcal{A}Y(t), \quad t > 0, \\ Y(0) := [w_0,\ w_1,\ \xi_0,\ \xi_1,\ s_0,\ s_1]^\top. \end{cases} \tag{2.6}$$

THEOREM 2.1. *Let $\mathcal{A}$ be defined by* (2.4) *and* (2.5). *Then $\mathcal{A}$ is dissipative in $\mathcal{H}$. In addition, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$. Therefore, $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$ and the spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues only.*

Proof. Since for any $[w, z, \xi, \varphi, s, h]^\top \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}
&\left\langle \mathcal{A}[w, z, \xi, \varphi, s, h]^\top,\ [w, z, \xi, \varphi, s, h]^\top \right\rangle_{\mathcal{H}} \\
&= \Big\langle \Big[z,\ \frac{G}{m}(\xi' + w'' - 3s'),\ \varphi,\ \frac{G}{I_m}(3s - \xi - w') + \frac{D}{I_m}\xi'',\ h, \\
&\frac{G}{I_m}(\xi + w' - 3s) - \frac{4}{3}\frac{\gamma}{I_m}s - \frac{4}{3}\beta h + \frac{D}{I_m}s''\Big]^\top,\ [w, z, \xi, \varphi, s, h]^\top \Big\rangle_{\mathcal{H}} \\
&= G\langle \xi' + w'' - 3s',\ z\rangle_{L^2} + G\langle 3h - \varphi - z',\ 3s - \xi - w'\rangle_{L^2} \\
&+ \langle G(3s - \xi - w') + D\xi'',\ \varphi\rangle_{L^2} + D\langle \varphi', \xi'\rangle_{L^2} + 3D\langle h', s'\rangle_{L^2} + 4\gamma\langle h, s\rangle_{L^2} \\
&+ \langle G(\xi + w' - 3s) - \frac{4}{3}\gamma s - \frac{4}{3}I_m\beta h + Ds'',\ 3h\rangle_{L^2} \\
&= G\big[\xi(x) + w'(x) - 3s(x)\big]\overline{z(x)}\Big|_0^1 + D\xi'(x)\overline{\varphi(x)}\Big|_0^1 + 3Ds'(x)\overline{h(x)}\Big|_0^1 \\
&- G\langle 3s - \xi - w',\ 3h - \varphi - z'\rangle_{L^2} + G\langle 3h - \varphi - z',\ 3s - \xi - w'\rangle_{L^2} \\
&- D\langle \xi',\ \varphi'\rangle_{L^2} + D\langle \varphi', \xi'\rangle_{L^2} + 3D\langle h', s'\rangle_{L^2} + 4\gamma\langle h, s\rangle_{L^2} \\
&- 3D\langle s',\ h'\rangle_{L^2} - 4\gamma\langle s, h\rangle_{L^2} - 4\beta I_m\langle h, h\rangle_{L^2} \\
&= -k_1G|z(1)|^2 - k_2D|\varphi(1)|^2 - G\langle 3s - \xi - w',\ 3h - \varphi - z'\rangle_{L^2} \\
&+ G\langle 3h - \varphi - z',\ 3s - \xi - w'\rangle_{L^2} - D\langle \xi',\ \varphi'\rangle_{L^2} + D\langle \varphi', \xi'\rangle_{L^2} + 3D\langle h', s'\rangle_{L^2} \\
&+ 4\gamma\langle h, s\rangle_{L^2} - 3D\langle s',\ h'\rangle_{L^2} - 4\gamma\langle s, h\rangle_{L^2} - 4\beta I_m\langle h, h\rangle_{L^2},
\end{aligned}$$

it follows that

$$\mathrm{Re}\left\langle \mathcal{A}[w, z, \xi, \varphi, s, h]^\top, [w, z, \xi, \varphi, s, h]^\top \right\rangle_{\mathcal{H}} = -k_1G|z(1)|^2 - k_2D|\varphi(1)|^2 - 4\beta I_m\|h\|_{L^2}^2 \le 0.$$

Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$. We accomplish the proof by showing that $0 \in \rho(\mathcal{A})$ because from Theorem 4.6 of [6], if $\mathcal{A}^{-1}$ exists, $\mathcal{A}$ must be densely defined in $\mathcal{H}$. Therefore, the Lumer–Phillips theorem can be applied to conclude that $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$.

To do so, for each $F := [u_1, u_2, \eta_1, \eta_2, v_1, v_2]^\top \in \mathcal{H}$, we seek $Y := [w, z, \xi, \varphi, s, h]^\top \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}Y = F$$

which yields

$$
\begin{cases} z = u_1, \quad G(\xi' + w'' - 3s') = mu_2, \\ \varphi = \eta_1, \quad G(3s - \xi - w') + D\xi'' = I_m\eta_2, \\ h = v_1, \quad 3G(\xi + w' - 3s) - 4\gamma s - 4\beta I_m h + 3Ds'' = 3I_m v_2, \\ \xi'(1) = -k_2\varphi(1), \quad s'(1) = 0, \\ 3s(1) - \xi(1) - w'(1) = k_1 z(1), \quad w(0) = \xi(0) = s(0) = 0. \end{cases} \tag{2.7}
$$

From the first equation of (2.7), we have

$$
G\big(\xi(x) + w'(x) - 3s(x)\big) = Gw'(0) + m\int_0^x u_2(r)dr. \tag{2.8}
$$

By eliminating the term $G\big(\xi(x) + w'(x) - 3s(x)\big)$ from the second and the third equations of (2.7), it follows that

$$
D\xi''(x) = I_m\eta_2(x) + Gw'(0) + m\int_0^x u_2(r)dr \tag{2.9}
$$

and

$$
3Ds''(x) - 4\gamma s(x) = 3I_m v_2(x) + 4\beta I_m v_1(x) - 3\left[Gw'(0) + m\int_0^x u_2(r)dr\right]. \tag{2.10}
$$

A simple computation of (2.9), yields

$$
\xi(x) = -k_2\eta_1(1)x - \frac{G}{D}w'(0)\left(x - \frac{x^2}{2}\right) - \widehat{\xi}(x), \tag{2.11}
$$

where

$$
\widehat{\xi}(x) := \frac{I_m}{D}\int_0^1 K_1(x,r)\eta_2(r)dr + \frac{m}{D}\int_0^1 K_2(x,r)u_2(r)dr \tag{2.12}
$$

and

$$
K_1(x,r) := \begin{cases} r, & 0 \le r < x, \\ x, & x \le r \le 1, \end{cases} \qquad K_2(x,r) := \begin{cases} x - \frac{x^2}{2} - \frac{r^2}{2}, & 0 \le r \le x, \\ x(1-r), & x \le r \le 1. \end{cases}
$$

Similarly, it follows from (2.10) that

(2.13)

$$
s(x) = a\sinh(bx) + \frac{G}{D}w'(0)\frac{1 - \cosh(bx)}{b^2} + \frac{4\beta I_m}{3Db}\int_0^x \sinh(b(x-r))v_1(r)dr + \widehat{s}(x),
$$

where a will be given later in (2.18), and

$$
b := \sqrt{\frac{4\gamma}{3D}}, \quad \widehat{s}(x) := \frac{1}{b}\int_0^x \sinh(b(x-r))\left[\frac{I_m}{D}v_2(r) - \frac{m}{D}\int_0^r u_2(t)dt\right]dr. \tag{2.14}
$$

Substitute (2.11) and (2.13) into (2.8), and integrate from 0 to x respect to x, to obtain

(2.15)

$$
\begin{aligned} w(x) = 3a\int_0^x \sinh(br)dr + w'(0)\left[\frac{3G}{Db^2}\int_0^x (1 - \cosh(br))dr + \frac{G}{D}\left(\frac{x^2}{2} - \frac{x^3}{6}\right) + x\right] \\ + \frac{4\beta I_m}{bD}\int_0^x (x-r)\sinh(b(x-r))v_1(r)dr - \frac{1}{2}\xi'(1)x^2 + \widehat{w}(x), \end{aligned}
$$

where $\widehat{w}(x)$ is given by

$$\widehat{w}(x) := 3\int_0^x \widehat{s}(r)dr - \int_0^x \widehat{\xi}(r)dr + \frac{m}{G}\int_0^x (x-r)u_2(r)dr. \tag{2.16}$$

Using the boundary conditions $s'(1) = 0$ in (2.13) and $3s(1) - \xi(1) - w'(1) = k_1 u_1(1)$ in (2.8), respectively, we obtain that

$$\begin{cases} ab\cosh b - \dfrac{G}{D}w'(0)\dfrac{\sinh b}{b} + \dfrac{4\beta I_m}{3D}\displaystyle\int_0^1 \cosh(b(1-r))v_1(r)dr + \widehat{s}'(1) = 0, \\ Gw'(0) + m\displaystyle\int_0^1 u_2(r)dr = -k_1 G u_1(1). \end{cases} \tag{2.17}$$

Thus, a and $w'(0)$ in (2.13) and (2.15), respectively, can be obtained as follows:

$$\begin{cases} a = \dfrac{G}{D}w'(0)\dfrac{\sinh b}{b^2\cosh b} - \dfrac{4\beta I_m}{3Db\cosh b}\displaystyle\int_0^1 \cosh(b(1-r))v_1(r)dr - \dfrac{\widehat{s}'(1)}{b\cosh b}, \\ w'(0) = -\dfrac{m}{G}\displaystyle\int_0^1 u_2(r)dr - k_1 u_1(1). \end{cases} \tag{2.18}$$

Hence, there is a solution $Y = [w, z, \xi, \varphi, s, h]^\top \in \mathcal{D}(\mathcal{A})$ so that $\mathcal{A}Y = F$, which in turn implies that $\mathcal{A}^{-1}$ exists. Finally, by the Sobolev embedding theorem, we can claim that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$ and thus the spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues only (see [5]). The proof is complete. ∎

As a consequence of Theorem 2.1, we have the following corollary.

COROLLARY 2.2. *Let $\mathcal{A}$ be defined by (2.4) and (2.5), and let $T(t)$ be a C_0-semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. Then $T(t)$ is asymptotically stable in $\mathcal{H}$, i.e.,*

$$\lim_{t\to\infty} \|T(t)Y\| = 0 \quad \forall\, Y \in \mathcal{H}.$$

Proof. Since $T(t)$ is a C_0-semigroup of contractions on $\mathcal{H}$, the proof will be accomplished by showing that there is no eigenvalue on the imaginary axis (see [3, p. 130]). Assume that $\lambda = i\tau,\ \tau \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$ and $Y := [w, z, \xi, \varphi, s, h]^\top \in \mathcal{D}(\mathcal{A})$ is an eigenfunction associated with λ. Then we have

$$z = i\tau w, \quad \varphi = i\tau\xi, \quad h = i\tau s,$$

and

$$\mathrm{Re}\langle \mathcal{A}Y,\ Y\rangle_{\mathcal{H}} = -k_1 G|z(1)|^2 - k_2 D|\varphi(1)|^2 - 4\beta I_m\|h\|_{L^2} \equiv 0.$$

Thus, it follows that

$$h(x) = i\tau s(x) \equiv 0, \quad z(1) = i\tau w(1) \equiv 0, \quad \varphi(1) = i\tau\xi(1) \equiv 0$$

and functions w and ξ satisfy the following equations:

$$\begin{cases} m\tau^2 w(x) + G\big(\xi'(x) + w''(x)\big) = 0, \quad 0 < x < 1, \\ I_m\tau^2\xi(x) - G\big(\xi(x) + w'(x)\big) + D\xi''(x) = 0, \quad 0 < x < 1, \\ \xi(x) + w'(x) = 0, \quad 0 < x < 1, \\ w(0) = \xi(0) = w(1) = \xi(1) = 0, \quad \xi'(1) = -k_2\varphi(1) = 0,\ w'(1) = -\xi(1) - k_1 z(1) = 0. \end{cases} \tag{2.19}$$

By a direct computation, we obtain that (2.19) has a unique trivial solution only. Thus $w(x) = \xi(x) \equiv 0$ and hence $Y \equiv 0$, which contradicts that Y is an eigenfunction. Therefore, no eigenvalue exists on the imaginary axis. The proof is complete. ◻

COROLLARY 2.3. *If $k_1 = k_2 \equiv 0$ in (1.6), that is, there is no control imposed on system (1.3), then there is no eigenvalue on the imaginary axis. So the uncontrolled system (1.3) with its boundary conditions (1.4) is also asymptotically stable.*

Proof. Similar to the proof of Corollary 2.2, if $k_1 = k_2 \equiv 0$ and assume that $\lambda = i\tau,\ \tau \in \mathbb{R}$ is an eigenvalue with $Y := [w, z, \xi, \varphi, s, h]^\top$ being an eigenfunction, then it follows that $s \equiv 0$ and the functions w and ξ satisfy the following equations:

$$(2.20)\qquad \begin{cases} m\tau^2 w(x) = 0, \quad 0 < x < 1, \\ I_m \tau^2 \xi(x) + D\xi''(x) = 0, \quad 0 < x < 1, \\ w(0) = \xi(0) = \xi'(1) = 0, \quad w'(1) = -\xi(1). \end{cases}$$

Therefore, one has $w = \xi \equiv 0$ and hence $Y \equiv 0$. The proof is complete. ◻

Note 2.1. We should note here that if $k_1 = k_2 \equiv 0$ in (1.6), then system (1.3) with its boundary conditions (1.4) cannot achieve the exponential stability. This is because of the fact that from the asymptotes of the system given later in (3.28), if $k_1 = k_2 \equiv 0$, then the eigenvalues of the first and second branches are very close to the imaginary axis as their moduli go to the infinity.

Let us now formulate the eigenvalue problem for the operator $\mathcal{A}$. If $\lambda \in \sigma(\mathcal{A})$ and $Y_\lambda := [w, z, \xi, \varphi, s, h]^\top \in \mathcal{D}(\mathcal{A})$ is a corresponding eigenfunction, then it is routine to verify that $\mathcal{A}Y_\lambda = \lambda Y_\lambda$ implies that $z = \lambda w,\ \varphi = \lambda\xi,\ h = \lambda s$ with w, ξ as well as s satisfying the following characteristic equations, for $0 < x < 1$:

$$(2.21)\qquad \begin{cases} m\lambda^2 w(x) + G(3s' - \xi' - w'')(x) = 0, \\ I_m\lambda^2\xi(x) - G(3s - \xi - w')(x) - D\xi''(x) = 0, \\ I_m\lambda^2 s(x) + G(3s - \xi - w')(x) + \frac{4}{3}\gamma s(x) + \frac{4}{3}\beta\lambda I_m s(x) - Ds''(x) = 0, \\ w(0) = 0, \quad \xi(0) = 0, \quad s(0) = 0, \\ \xi'(1) = -\lambda k_2 \xi(1), \quad s'(1) = 0, \quad 3s(1) - \xi(1) - w'(1) = \lambda k_1 w(1). \end{cases}$$

For brevity in notation, from now on, we define

$$(2.22)\qquad r_1 := \sqrt{\frac{m}{G}}, \quad r_2 := \sqrt{\frac{I_m}{D}}, \quad d_1 := \frac{G}{D}, \quad d_2 := \frac{\gamma}{D}, \quad d_3 := 3d_1 + \frac{4}{3}d_2.$$

(2.21) then becomes

$$(2.23)\qquad \begin{cases} r_1^2\lambda^2 w(x) + 3s'(x) - \xi'(x) - w''(x) = 0, \\ r_2^2\lambda^2\xi(x) - 3d_1 s(x) + d_1\xi(x) + d_1 w'(x) - \xi''(x) = 0, \\ r_2^2\lambda^2 s(x) + d_3 s(x) - d_1\xi(x) - d_1 w'(x) + \frac{4}{3}\beta r_2^2\lambda s(x) - s''(x) = 0, \\ w(0) = 0, \quad \xi(0) = 0, \quad s(0) = 0, \\ \xi'(1) = -\lambda k_2\xi(1), \quad s'(1) = 0, \quad 3s(1) - \xi(1) - w'(1) = \lambda k_1 w(1). \end{cases}$$

Clearly, (2.23) is a coupled system of ordinary differential equations. In order to solve these equations, we shall use the matrix operator pencil method (see [8]). Let

$$(2.24)\qquad w_1 := w, \quad w_2 := w', \quad \xi_1 := \xi, \quad \xi_2 := \xi', \quad s_1 := s, \quad s_2 := s'$$

and

$$\Phi := [w_1,\ w_2,\ \xi_1,\ \xi_2,\ s_1,\ s_2]^\top. \tag{2.25}$$

Then (2.23) becomes

$$\begin{cases} T^D(x,\lambda)\Phi(x) := \Phi'(x) + M(\lambda)\Phi(x) = 0, \\ T^R(x,\lambda)\Phi(x) := W^0(\lambda)\Phi(0) + W^1(\lambda)\Phi(1) = 0, \end{cases} \tag{2.26}$$

where

(2.27)

$$W^0(\lambda) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ & & O_{3\times 6} & & & \end{bmatrix}, \quad W^1(\lambda) := \begin{bmatrix} & & O_{3\times 6} & & & \\ 0 & 0 & \lambda k_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \lambda k_1 & 1 & 1 & 0 & -3 & 0 \end{bmatrix},$$

and

$$M(\lambda) := D_0 - \lambda D_1 - \lambda^2 D_2 \tag{2.28}$$

with $D_0,\ D_1$, and D_2 being three matrices defined by

$$D_0 := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -d_1 & -d_1 & 0 & 3d_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & d_1 & d_1 & 0 & -d_3 & 0 \end{bmatrix}, \quad D_1 := \begin{bmatrix} O_{4\times 4} & O_{4\times 2} \\ O_{2\times 4} & D_{11} \end{bmatrix}, \tag{2.29}$$

$$D_2 := \begin{bmatrix} r_1^2 D_{21} & O_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & r_2^2 D_{21} & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} & r_2^2 D_{21} \end{bmatrix}, \quad D_{11} := \begin{bmatrix} 0 & 0 \\ \frac{4}{3}\beta r_2^2 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.30}$$

THEOREM 2.4. *The characteristic equation* (2.21) *is equivalent to the first order linear system* (2.26). *Also* $\lambda \in \sigma(\mathcal{A})$ *if and only if* (2.26) *has a nontrivial solution.*

3. Asymptotic behavior of eigenfrequencies. In this section, we are looking for the asymptotic expressions for the eigenvalues of $\mathcal{A}$. It will be accomplished by expanding the characteristic determinant $\Delta(\lambda)$ of (2.26) via an asymptotic expression of the fundamental matrix solution, which can be obtained by modifying a standard technique of Birkhoff–Langer (see [1]) and later of Tretter (see [8] or [9]) for tackling the matrix operator pencils. A key step is an invertible matrix transformation which is very powerful and universal in the sense that it can be applied to a lot of other coupled problems.

To begin, we shall diagonalize the leading term $\lambda^2 D_2$ in (2.28). For each $0 \neq \lambda \in \mathbb{C}$, define an invertible matrix in λ by

$$P(\lambda) := \begin{bmatrix} P_1(\lambda) & & \\ & P_2(\lambda) & \\ & & P_2(\lambda) \end{bmatrix}, \quad P_1(\lambda) := \begin{bmatrix} r_1\lambda & r_1\lambda \\ r_1^2\lambda^2 & -r_1^2\lambda^2 \end{bmatrix}, \tag{3.1}$$

$$P_2(\lambda) := \begin{bmatrix} r_2\lambda & r_2\lambda \\ r_2^2\lambda^2 & -r_2^2\lambda^2 \end{bmatrix}.$$

For any $\lambda \neq 0$, a simple computation shows that

$$P^{-1}(\lambda) := \begin{bmatrix} P_1^{-1}(\lambda) & & \\ & P_2^{-1}(\lambda) & \\ & & P_2^{-1}(\lambda) \end{bmatrix} \tag{3.2}$$

with

$$P_1^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_1\lambda} & \frac{1}{2r_1^2\lambda^2} \\ \frac{1}{2r_1\lambda} & \frac{-1}{2r_1^2\lambda^2} \end{bmatrix}, \quad P_2^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_2\lambda} & \frac{1}{2r_2^2\lambda^2} \\ \frac{1}{2r_2\lambda} & \frac{-1}{2r_2^2\lambda^2} \end{bmatrix}.$$

So matrix $P(\lambda)$ is a polynomial of degree 2 in λ. Define

$$\Psi(x) := P^{-1}(\lambda)\Phi(x), \quad \widehat{T}^D(x,\lambda) := P(\lambda)^{-1}T^D(x,\lambda)P(\lambda). \tag{3.3}$$

Then we have

$$\widehat{T}^D(x,\lambda)\Psi(x) = \Psi'(x) - \widehat{M}(\lambda)\Psi(x) = 0, \tag{3.4}$$

where

$$\begin{aligned}
\widehat{M}(\lambda) &= -P(\lambda)^{-1}M(\lambda)P(\lambda) \\
&= -P(\lambda)^{-1}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -r_1^2\lambda^2 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -d_1 & -d_1 - r_2^2\lambda^2 & 0 & 3d_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & d_1 & d_1 & 0 & -d_3 - r_2^2(\frac{4}{3}\beta\lambda + \lambda^2) & 0 \end{bmatrix} P(\lambda) \\
&= -\begin{bmatrix} -\frac{1}{2} & \frac{-1}{2r_1\lambda} & 0 & \frac{1}{2r_1^2\lambda^2} & 0 & \frac{-3}{2r_1^2\lambda^2} \\ \frac{1}{2} & \frac{-1}{2r_1\lambda} & 0 & \frac{-1}{2r_1^2\lambda^2} & 0 & \frac{3}{2r_1^2\lambda^2} \\ 0 & \frac{-d_1}{2r_2^2\lambda^2} & \frac{-d_1}{2r_2^2\lambda^2} - \frac{1}{2} & \frac{-1}{2r_2\lambda} & \frac{3d_1}{2r_2^2\lambda^2} & 0 \\ 0 & \frac{d_1}{2r_2^2\lambda^2} & \frac{d_1}{2r_2^2\lambda^2} + \frac{1}{2} & \frac{-1}{2r_2\lambda} & \frac{-3d_1}{2r_2^2\lambda_2^2} & 0 \\ 0 & \frac{d_1}{2r_2^2\lambda^2} & \frac{d_1}{2r_2^2\lambda^2} & 0 & -\frac{1}{2} - \frac{2}{3}\frac{\beta}{\lambda} - \frac{d_3}{2r_2^2\lambda^2} & \frac{-1}{2r_2\lambda} \\ 0 & \frac{-d_1}{2r_2^2\lambda^2} & \frac{-d_1}{2r_2^2\lambda^2} & 0 & \frac{1}{2} + \frac{2}{3}\frac{\beta}{\lambda} + \frac{d_3}{2r_2^2\lambda^2} & \frac{-1}{2r_2\lambda} \end{bmatrix} P(\lambda) \\
&= -\begin{bmatrix} -r_1\lambda & 0 & \frac{1}{2}d_4 & -\frac{1}{2}d_4 & -\frac{3}{2}d_4 & \frac{3}{2}d_4 \\ 0 & r_1\lambda & -\frac{1}{2}d_4 & \frac{1}{2}d_4 & \frac{3}{2}d_4 & -\frac{3}{2}d_4 \\ -\frac{1}{2}d_6 & \frac{1}{2}d_6 & -r_2\lambda - \frac{d_7}{2\lambda} & -\frac{d_7}{2\lambda} & \frac{3d_7}{2\lambda} & \frac{3d_7}{2\lambda} \\ \frac{1}{2}d_6 & -\frac{1}{2}d_6 & \frac{d_7}{2\lambda} & r_2\lambda + \frac{d_7}{2\lambda} & -\frac{3d_7}{2\lambda} & -\frac{3d_7}{2\lambda} \\ \frac{1}{2}d_6 & -\frac{1}{2}d_6 & \frac{d_7}{2\lambda} & \frac{d_7}{2\lambda} & -r_2\lambda - \frac{2}{3}\beta r_2 - \frac{d_8}{2\lambda} & -\frac{2}{3}\beta r_2 - \frac{d_8}{2\lambda} \\ -\frac{1}{2}d_6 & \frac{1}{2}d_6 & -\frac{d_7}{2\lambda} & -\frac{d_7}{2\lambda} & \frac{2}{3}\beta r_2 + \frac{d_8}{2\lambda} & r_2\lambda + \frac{2}{3}\beta r_2 + \frac{d_8}{2\lambda} \end{bmatrix}
\end{aligned}$$

with

$$d_4 := \frac{r_2^2}{r_1^2}, \quad d_5 := \frac{r_1^2}{r_2^2}, \quad d_6 := d_1 d_5, \quad d_7 := \frac{d_1}{r_2}, \quad d_8 := \frac{d_3}{r_2}. \tag{3.5}$$

It is seen from the above that $\widehat{M}(\lambda)$ can be written as

$$\widehat{M}(\lambda) := \lambda \widehat{M}_1 + \widehat{M}_0 + \lambda^{-1} \widehat{M}_{-1}, \tag{3.6}$$

where

$$\widehat{M}_1 := \operatorname{diag}[r_1, -r_1, r_2, -r_2, r_2, -r_2] \tag{3.7}$$

and

$$\widehat{M}_0 := \begin{bmatrix} O_{2\times 2} & \frac{1}{2}d_4\widehat{M}_{01} & -\frac{3}{2}d_4\widehat{M}_{01} \\ -\frac{1}{2}d_6\widehat{M}_{01} & O_{2\times 2} & O_{2\times 2} \\ \frac{1}{2}d_6\widehat{M}_{01} & O_{2\times 2} & \frac{2}{3}\beta r_2\widehat{M}_{02} \end{bmatrix}, \quad \widehat{M}_{-1} := \begin{bmatrix} O_{2\times 2} & O_{2\times 4} \\ O_{4\times 2} & \widehat{M}_{-11} \end{bmatrix} \tag{3.8}$$

with

$$\widehat{M}_{01} := \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \widehat{M}_{02} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \widehat{M}_{-11} := \begin{bmatrix} \frac{1}{2}d_7\widehat{M}_{02} & -\frac{3}{2}d_7\widehat{M}_{02} \\ -\frac{1}{2}d_7\widehat{M}_{02} & \frac{1}{2}d_8\widehat{M}_{02} \end{bmatrix}.$$

On the basis of these transformations, we are now in a position to find an asymptotic expression for the fundamental matrix solution of system (3.4).

THEOREM 3.1. *Let $0 \neq \lambda \in \mathbb{C}$, and let $\widehat{M}(\lambda)$ be given by* (3.6) *and assume that $r_1 \neq r_2$. For $x \in [0,1]$, set*

$$E(x,\lambda) := \operatorname{diag}\left[e^{r_1\lambda x},\ e^{-r_1\lambda x},\ e^{r_2\lambda x},\ e^{-r_2\lambda x},\ e^{r_2\lambda x},\ e^{-r_2\lambda x}\right]. \tag{3.9}$$

Then there exists a fundamental matrix solution $\widehat{\Psi}(x,\lambda)$ for system (3.4)*, which satisfies*

$$\Psi'(x) = \widehat{M}(\lambda)\Psi(x) \tag{3.10}$$

such that for large enough $|\lambda|$,

$$\widehat{\Psi}(x,\lambda) = \left(\widehat{\Psi}_0(x) + \frac{\widetilde{\Theta}(x,\lambda)}{\lambda}\right) E(x,\lambda), \tag{3.11}$$

where

$$\widehat{\Psi}_0(x) := \operatorname{diag}[1,\ 1,\ 1,\ 1,\ e_1(x),\ e_2(x)] \tag{3.12}$$

and

$$\widetilde{\Theta}(x,\lambda) := \widehat{\Psi}_1(x) + \lambda^{-1}\widehat{\Psi}_2(x) + \cdots \tag{3.13}$$

with all entries uniformly bounded in $[0,1]$*. Here,*

$$e_1(x) := e^{\frac{2}{3}\beta r_2 x} \quad \text{and} \quad e_2(x) := e^{-\frac{2}{3}\beta r_2 x}. \tag{3.14}$$

Proof. Since $\widehat{M}_1$ given by (3.7) is a diagonal matrix, it follows that $E(x,\lambda)$ given by (3.9) is a fundamental matrix solution to (3.10) which involves only the leading order terms, that is, to say

$$E'(x,\lambda) = \lambda\widehat{M}_1 E(x,\lambda).$$

Now we look for a fundamental matrix solution of (3.10) in the form of

$$\widehat{\Psi}(x,\lambda) = \left(\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \cdots + \lambda^{-n}\widehat{\Psi}_n(x) + \cdots\right) E(x,\lambda).$$

The left-hand side of (3.10) is

$$\begin{aligned}\widehat{\Psi}'(x,\lambda) = &\left(\widehat{\Psi}'_0(x) + \lambda^{-1}\widehat{\Psi}'_1(x) + \cdots + \lambda^{-n}\widehat{\Psi}'_n(x) + \cdots\right) E(x,\lambda)\\ &+\lambda\left(\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \cdots + \lambda^{-n}\widehat{\Psi}_n(x) + \cdots\right) \widehat{M_1} E(x,\lambda).\end{aligned}$$

Compare it with the right-hand side of (3.10),

$$\big(\lambda\widehat{M_1} + \widehat{M_0} + \lambda^{-1}\widehat{M_{-1}}\big)\big(\widehat{\Psi}_0(x) + \lambda^{-1}\widehat{\Psi}_1(x) + \cdots + \lambda^{-n}\widehat{\Psi}_n(x) + \cdots\big)E(x,\lambda),$$

to give, according to the coefficients of $\lambda^1, \lambda^0, \lambda^{-1}, \ldots, \lambda^{-n}, \ldots$, that

$$\begin{gathered}\widehat{\Psi}_0(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_0(x) = 0,\\ \widehat{\Psi}'_0(x) - \widehat{M_0}\widehat{\Psi}_0(x) + \widehat{\Psi}_1(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_1(x) = 0,\\ \widehat{\Psi}'_1(x) - \widehat{M_0}\widehat{\Psi}_1(x) - \widehat{M_{-1}}\widehat{\Psi}_0(x) + \widehat{\Psi}_2(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_2(x) = 0,\\ \vdots\\ \widehat{\Psi}'_n(x) - \widehat{M_0}\widehat{\Psi}_n(x) - \widehat{M_{-1}}\widehat{\Psi}_{n-1}(x) + \widehat{\Psi}_{n+1}(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_{n+1}(x) = 0,\\ \vdots\end{gathered}$$

Using the arguments in [8, p. 135] (or [1]), we conclude that there is an asymptotic fundamental matrix solution $\widehat{\Psi}(x,\lambda)$ for system (3.10). It remains to show that the leading order term $\widehat{\Psi}_0(x)$ is given by (3.12). Indeed, since $\widehat{\Psi}_0(x)$ can be determined by the matrix equations

$$\widehat{\Psi}_0(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_0(x) = 0 \tag{3.15}$$

and

$$\widehat{\Psi}'_0(x) - \widehat{M_0}\widehat{\Psi}_0(x) + \widehat{\Psi}_1(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_1(x) = 0, \tag{3.16}$$

where $\widehat{M_1}$ and $\widehat{M_0}$ are given in (3.7), (3.8), respectively, it follows that if $\widehat{\Psi}_0$ is known, then one can deduce the leading order term $\widehat{\Psi}_1$ of $\widetilde{\Theta}(x,\rho)$ in (3.13) from (3.16) and

$$\widehat{\Psi}'_1(x) - \widehat{M_0}\widehat{\Psi}_1(x) - \widehat{M_{-1}}\widehat{\Psi}_0(x) + \widehat{\Psi}_2(x)\widehat{M_1} - \widehat{M_1}\widehat{\Psi}_2(x) = 0$$

with $\widehat{M_{-1}}$ being given in (3.8). Similarly, we obtain all the terms $\widehat{\Psi}_1, \widehat{\Psi}_2, \ldots, \widehat{\Psi}_n, \ldots$ of $\widetilde{\Theta}(x,\lambda)$ in (3.13). So, the proof will be accomplished if we would find the leading order term $\widehat{\Psi}_0$ in (3.11).

Let us denote by $c_{ij}(x)$ the (i,j)-entry of the matrix $\widehat{\Psi}_0(x)$ with $i,j = 1,2,\ldots,6$. Since $\widehat{M_1}$ is diagonal, it follows from (3.15) and $r_1 \neq r_2$ that the entries $c_{ij}(x)$ of $\widehat{\Psi}_0$ satisfy

$$\begin{cases} c_{ij}(x) = 0 & \text{if} \quad 1 \le i \le 2,\ 1 \le j \le 6,\ i \neq j,\\ c_{ij}(x) = 0 & \text{if} \quad 3 \le i \le 4,\ 1 \le j \le 6,\ i \neq j,\ j \neq i+2,\\ c_{ij}(x) = 0 & \text{if} \quad 5 \le i \le 6,\ 1 \le j \le 6,\ i \neq j,\ j \neq i-2,\end{cases}$$

and the entries $c_{ii}(x)$ $(i = 1, 2, \dots, 6)$, $c_{35}(x)$, $c_{53}(x)$, $c_{46}(x)$, and $c_{64}(x)$ can be found by substituting them into (3.16) to obtain

$$\begin{cases} c'_{ii}(x) = 0 \quad \text{for } \ i = 1, 2, 3, 4, \\ c'_{55}(x) = \frac{2}{3}\beta r_2 c_{55}(x), \quad c'_{66}(x) = -\frac{2}{3}\beta r_2 c_{66}(x), \\ c'_{35}(x) = 0, \quad c'_{53}(x) = \frac{2}{3}\beta r_2 c_{53}(x), \\ c'_{46}(x) = 0, \quad c'_{64}(x) = -\frac{2}{3}\beta r_2 c_{64}(x). \end{cases} \tag{3.17}$$

(3.12) then follows from $\widehat{\Psi}_0(0) = I$. The proof is complete. ☐

By virtue of transformation for $\widehat{\Psi}(x, \lambda)$ in (3.3), we have immediately the following corollary, which shows the relationship between (2.26) and (3.4).

COROLLARY 3.2. *Let* $0 \neq \lambda \in \mathbb{C}$, *let* $r_1 \neq r_2$, *and let* $\widehat{\Psi}(x, \lambda)$ *given by* (3.11) *be a fundamental matrix solution of system* (3.4). *Then*

$$\widehat{\Phi}(x, \lambda) := P(\lambda)\widehat{\Psi}(x, \lambda) \tag{3.18}$$

is a fundamental matrix solution for the first order linear system (2.26).

We are now ready to estimate the asymptotic eigenfrequencies of the system. Note that the eigenvalues of the first order linear system in (2.26) are given by the zeros of the characteristic determinant

$$\Delta(\lambda) := \det\left(T^R\widehat{\Phi}(x, \lambda)\right), \quad \lambda \in \mathbb{C}, \tag{3.19}$$

where operator T^R is given in (2.26) and $\widehat{\Phi}(x, \lambda)$ is any fundamental matrix of $T^D(x, \lambda)\Phi(x) = 0$ (see [8]). We shall derive the asymptotic expansion of eigenfrequencies by substituting (3.11) and (3.18) into (3.19), together with the boundary conditions in (2.26). In fact, since

$$T^R\widehat{\Phi}(x, \lambda) = W^0(\lambda)P(\lambda)\widehat{\Psi}(0, \lambda) + W^1(\lambda)P(\lambda)\widehat{\Psi}(1, \lambda), \tag{3.20}$$

using (2.27) and (3.1), a simple computation gives

$$W^0(\lambda)P(\lambda) = \begin{bmatrix} r_1\lambda & r_1\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2\lambda & r_2\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2\lambda & r_2\lambda \\ & & & O_{3\times 6} & & \end{bmatrix}$$

and

$$W^1(\lambda)P(\lambda) = \begin{bmatrix} & & & O_{3\times 6} & & \\ 0 & 0 & r_2r_3\lambda^2 & r_2r_4\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2^2\lambda^2 & -r_2^2\lambda^2 \\ r_1r_5\lambda^2 & r_1r_6\lambda^2 & r_2\lambda & r_2\lambda & -3r_2\lambda & -3r_2\lambda \end{bmatrix},$$

where

$$r_3 := k_2 + r_2, \quad r_4 := k_2 - r_2, \quad r_5 := k_1 + r_1, \quad r_6 := k_1 - r_1. \tag{3.21}$$

Once again for notational simplicity, set

$$[a]_1 := a + \mathcal{O}(\lambda^{-1}).$$

Since $\widehat{\Psi}_0(0) = I$ and $E(0,\lambda) = I$, a direct computation yields

$$W^0(\lambda)P(\lambda)\widehat{\Psi}(0,\lambda) = \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda[r_2]_1 & \lambda[r_2]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda[r_2]_1 & \lambda[r_2]_1 \\ & & & O_{3\times 6} & & \end{bmatrix}$$

and

$$W^1(\lambda)P(\lambda)\widehat{\Psi}(1,\lambda) =$$

$$\begin{bmatrix} & & & O_{3\times 6} & & \\ 0 & 0 & \lambda^2 E_3[r_2r_3]_1 & \lambda^2 E_4[r_2r_4]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & [r_2^2]_1\lambda^2 E_3E_5 & -[r_2^2]_1\lambda^2 E_4E_6 \\ \lambda^2 E_1[r_1r_5]_1 & \lambda^2 E_2[r_1r_6]_1 & \lambda E_3[r_2]_1 & \lambda E_4[r_2]_1 & -3\lambda E_3E_5[r_2]_1 & -3\lambda E_4E_6[r_2]_1 \end{bmatrix},$$

where

$$\begin{cases} E_1 := e^{r_1\lambda}, \quad E_2 := e^{-r_1\lambda}, \quad E_3 := e^{r_2\lambda}, \quad E_4 := e^{-r_2\lambda}, \\ E_5 := e_1(1) = e^{\frac{2}{3}\beta r_2}, \quad E_6 := e_2(1) = e^{-\frac{2}{3}\beta r_2}. \end{cases} \tag{3.22}$$

Hence,

$$T^R\widehat{\Phi}(x,\lambda) =$$

$$\begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda[r_2]_1 & \lambda[r_2]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda[r_2]_1 & \lambda[r_2]_1 \\ 0 & 0 & \lambda^2 E_3[r_2r_3]_1 & \lambda^2 E_4[r_2r_4]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & [r_2^2]_1\lambda^2 E_3E_5 & -[r_2^2]_1\lambda^2 E_4E_6 \\ \lambda^2 E_1[r_1r_5]_1 & \lambda^2 E_2[r_1r_6]_1 & \lambda E_3[r_2]_1 & \lambda E_4[r_2]_1 & -3\lambda E_3E_5[r_2]_1 & -3\lambda E_4E_6[r_2]_1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \Delta(\lambda) = \det(T^R\widehat{\Phi}(x,\lambda)) &= \lambda^9 r_1^2 r_2^5 \times \det\begin{bmatrix} [1]_1 & [1]_1 \\ [r_5]_1E_1 & [r_6]_1E_2 \end{bmatrix} \\ &\times \det\begin{bmatrix} [1]_1 & [1]_1 \\ [r_3]_1E_3 & [r_4]_1E_4 \end{bmatrix} \times \det\begin{bmatrix} [1]_1 & [1]_1 \\ -E_3E_5[1]_1 & E_4E_6[1]_1 \end{bmatrix} \\ &= r_1^2 r_2^5 \lambda^9 \Delta_1(\lambda)\Delta_2(\lambda)\Delta_3(\lambda), \end{aligned}$$

where

$$\begin{cases} \Delta_1(\lambda) := r_6E_2 - r_5E_1 + \mathcal{O}(\lambda^{-1}), \\ \Delta_2(\lambda) := r_4E_4 - r_3E_3 + \mathcal{O}(\lambda^{-1}), \\ \Delta_2(\lambda) := E_4E_6 + E_3E_5 + \mathcal{O}(\lambda^{-1}) \end{cases} \tag{3.23}$$

with r_i $(i = 3,4,5,6)$ being given in (3.21) and E_i $(i = 1,2,\dots,6)$ in (3.22), respectively. With all these preparations, we come to the proof of the asymptotic behavior of the eigenvalues.

THEOREM 3.3. *Let $r_1 \neq r_2$ and let $\Delta(\lambda)$ be the characteristic determinant of system* (2.26). *Then the following asymptotic expansion for $\Delta(\lambda)$ holds:*

$$\Delta(\lambda) = r_1^2 r_2^5 \lambda^9 \Delta_1(\lambda)\Delta_2(\lambda)\Delta_3(\lambda) \tag{3.24}$$

with $\Delta_i(\lambda)$ being given in (3.23). *If $k_i \neq r_i$ $(i = 1, 2)$, then there are three branches of asymptotic eigenvalues given by (as $|n| \to \infty$ and $n \in \mathbb{Z}$)*

$$\begin{cases} \lambda_{jn} = \mu_j + r_j^{-1} n\pi i + \mathcal{O}(n^{-1}) & \text{for } j = 1, 2, \\ \lambda_{3n} = \mu_3 + r_2^{-1}(n + \frac{1}{2})\pi i + \mathcal{O}(n^{-1}), \end{cases} \tag{3.25}$$

where

$$\mu_j := \begin{cases} \frac{1}{2r_j} \ln \frac{k_j - r_j}{k_j + r_j}, & k_j > r_j \\ \frac{1}{2r_j} \left(\ln \frac{r_j - k_j}{k_j + r_j} + \pi i \right), & k_j < r_j \end{cases} \qquad \text{for } j = 1, 2 \tag{3.26}$$

and

$$\mu_3 := -\frac{2}{3}\beta. \tag{3.27}$$

Moreover, we have, as $|n| \to \infty$,

$$\mathrm{Re}\lambda_{jn} \to \frac{1}{2r_j} \ln \left| \frac{k_j - r_j}{k_j + r_j} \right| < 0 \ \text{ for } j = 1, 2 \quad \text{and} \quad \mathrm{Re}\lambda_{3n} \to \mu_3 < 0. \tag{3.28}$$

Furthermore, if k_1 and k_2 satisfy the conditions

$$k_1 \neq \begin{cases} \dfrac{\alpha_1 + 1}{1 - \alpha_1} r_1 & \text{for } k_1 > r_1, \\ \dfrac{1 - \alpha_1}{\alpha_1 + 1} r_1 & \text{for } k_1 < r_1, \end{cases} \qquad \alpha_1 := \left| \frac{k_2 - r_2}{k_2 + r_2} \right|^{r_1/r_2}, \quad 0 < \alpha_1 < 1, \tag{3.29}$$

and

$$k_1 \neq \begin{cases} \dfrac{\alpha_2 + 1}{1 - \alpha_2} r_1 & \text{for } k_1 > r_1, \\ \dfrac{1 - \alpha_2}{\alpha_2 + 1} r_1 & \text{for } k_1 < r_1, \end{cases} \qquad \alpha_2 := e^{-\frac{4}{3}\beta r_1}, \quad 0 < \alpha_2 < 1, \tag{3.30}$$

then the zeros of $\Delta(\lambda)$ are simple when their moduli are sufficiently large.

Proof. By $\Delta(\lambda) = 0$ and (3.24), it follows that

$$\Delta_1(\lambda)\Delta_2(\lambda)\Delta_3(\lambda) = 0 \tag{3.31}$$

and

$$\Delta_i(\lambda) = 0 \quad \text{for } i = 1, 2, 3.$$

Let $\Delta_1(\lambda) = 0$. Then we obtain

$$r_6 E_2 - r_5 E_1 + \mathcal{O}(\lambda^{-1}) = 0, \tag{3.32}$$

which is equivalent to (from (3.21) and (3.23))

$$(k_1 - r_1)e^{-r_1\lambda} - (k_1 + r_1)e^{r_1\lambda} + \mathcal{O}(\lambda^{-1}) = 0. \tag{3.33}$$

Since the solutions of the equation

$$(k_1 - r_1)e^{-r_1\lambda} - (k_1 + r_1)e^{r_1\lambda} = 0$$

are given by

$$\tilde{\lambda}_{1n} = \mu_1 + r_1^{-1}n\pi i, \quad n \in \mathbb{Z},$$

it follows from Rouché's theorem that the solutions to (3.33) are in the form

$$\lambda_{1n} = \tilde{\lambda}_{1n} + \mathcal{O}(n^{-1}) = \mu_1 + r_1^{-1}n\pi i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \tag{3.34}$$

Similarly, let $\Delta_2(\lambda) = 0$. Then the equation

$$(k_2 - r_2)e^{-r_2\lambda} - (k_2 + r_2)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1}) = 0 \tag{3.35}$$

has the solutions

$$\lambda_{2n} = \mu_2 + r_2^{-1}n\pi i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \tag{3.36}$$

Also, let $\Delta_3(\lambda) = 0$. The equation

$$e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1}) = 0 \tag{3.37}$$

has the solutions

$$\lambda_{3n} = \mu_3 + r_2^{-1}\Big(n + \frac{1}{2}\Big)\pi i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \tag{3.38}$$

Finally, by a direct computation, it follows from (3.29) and (3.30) that k_1 and k_2 satisfy the following conditions:

$$\frac{1}{r_1}\ln\left|\frac{k_1 - r_1}{k_1 + r_1}\right| \neq \frac{1}{r_2}\ln\left|\frac{k_2 - r_2}{k_2 + r_2}\right|, \quad \frac{1}{2r_1}\ln\left|\frac{k_1 - r_1}{k_1 + r_1}\right| \neq -\frac{2}{3}\beta.$$

Thus $\mu_1 \neq \mu_2$ and $\mu_1 \neq \mu_3$. The last assertion is then concluded. The proof is complete. ∎

THEOREM 3.4. *Suppose* $r_1 \neq r_2$ *and* $k_i \neq r_i$ *(*$i = 1, 2$*). Let* $\mathcal{A}$ *be defined by* (2.4) *and* (2.5)*. Then all eigenvalues of* $\mathcal{A}$ *have the asymptotic expressions given by* (3.25)*. Moreover, if* k_1 *and* k_2 *satisfy conditions* (3.29) *and* (3.30)*, then all eigenvalues of the system with sufficiently large moduli are simple.*

4. Asymptotic behavior of eigenfunctions.

In this section, we shall consider the asymptotic behavior for eigenfunctions of $\mathcal{A}$. It will be used in the proof of the Riesz basis in the last section.

THEOREM 4.1. *Suppose* $r_1 \neq r_2$ *and* $k_i \neq r_i$ $(i = 1, 2)$*. Let* $\sigma(\mathcal{A}) := \{\lambda_{1n}, \lambda_{2n}, \lambda_{3n}, n \in \mathbb{Z}\}$ *be the eigenvalues of* $\mathcal{A}$ *with* λ_{jn} *(*$j = 1, 2, 3$*) being given in* (3.25)*. Then the corresponding eigenfunctions*

$$\Big\{[w_{jn}, \lambda_{jn}w_{jn}, \xi_{jn}, \lambda_{jn}\xi_{jn}, s_{jn}, \lambda_{jn}s_{jn}]^\top, \quad j = 1, 2, 3,\ n \in \mathbb{Z}\Big\}$$

have the following asymptotic expressions for $|n| \to \infty$, $n \in \mathbb{Z}$:

(4.1)

$$\begin{cases} w'_{1n}(x) = \frac{1}{2}(e^{-r_1\lambda_{1n}x} + e^{r_1\lambda_{1n}x}) + \mathcal{O}(n^{-1}), \quad w'_{jn}(x) = \mathcal{O}(n^{-1}) \ \textit{for} \ j = 2,3, \\ \lambda_{1n}w_{1n}(x) = \frac{1}{2}r_1^{-1}(e^{r_1\lambda_{1n}x} - e^{-r_1\lambda_{1n}x}) + \mathcal{O}(n^{-1}), \ \lambda_{jn}w_{jn}(x) = \mathcal{O}(n^{-1}) \ \textit{for} \ j = 2,3, \\ \xi'_{2n}(x) = \frac{1}{2}(e^{-r_2\lambda_{2n}x} + e^{r_2\lambda_{2n}x}) + \mathcal{O}(n^{-1}), \quad \xi'_{jn}(x) = \mathcal{O}(n^{-1}) \ \textit{for} \ j = 1,3, \\ \lambda_{2n}\xi_{2n}(x) = \frac{1}{2}r_2^{-1}(e^{r_2\lambda_{2n}x} - e^{-r_2\lambda_{2n}x}) + \mathcal{O}(n^{-1}), \quad \lambda_{jn}\xi_{jn}(x) = \mathcal{O}(n^{-1}) \ \textit{for} \ j = 1,3, \\ s'_{3n}(x) = \frac{1}{2}(e_2(x)e^{-r_2\lambda_{3n}x} + e_1(x)e^{r_2\lambda_{3n}x}) + \mathcal{O}(n^{-1}), \ s'_{jn}(x) = \mathcal{O}(n^{-1}) \ \textit{for} \ j = 1,2, \\ \lambda_{3n}s_{3n}(x) = \frac{1}{2}r_2^{-1}(e_1(x)e^{r_2\lambda_{3n}x} - e_2(x)e^{-r_2\lambda_{3n}x}) + \mathcal{O}(n^{-1}), \\ \lambda_{jn}s_{jn}(x) = \mathcal{O}(n^{-1}) \quad \textit{for} \ j = 1,2, \end{cases}$$

where r_1, r_2 are given in (2.22) *and $e_1(x), e_2(x)$ are given in* (3.14)*, respectively. Moreover, $\{[w_{jn}, \lambda_{jn}w_{jn}, \ \xi_{jn}, \lambda_{jn}\xi_{jn}, s_{jn}, \lambda_{jn}s_{jn}]^\top (j = 1,2,3, \ n \in \mathbb{Z})\}$ are approximately normalized in $\mathcal{H}$ in the sense that there exist positive constants c_1 and c_2 independent of n such that $(j = 1,2,3)$*

$$c_1 \le \|w'_{jn}\|_{L^2}, \|\lambda_{jn}w_{jn}\|_{L^2}, \|\xi'_{jn}\|_{L^2}, \|\lambda_{jn}\xi_{jn}\|_{L^2}, \|s'_{jn}\|_{L^2}, \|\lambda_{jn}s_{jn}\|_{L^2} \le c_2 \tag{4.2}$$

for all integers n.

Proof. Note that the jth component of $\Phi(x) = [w_1(x), w_2(x), \xi_1(x), \xi_2(x), s_1(x), s_2(x)]^\top$ in (2.25) with respect to the eigenvalue λ can be obtained by taking the determinant of the matrices which are replaced one of the rows of $T^R\widehat{\Phi}$ in (3.20) by $e_j^\top(\widehat{\Phi}(x,\lambda))$ so that their determinants are not zero, where e_j is the jth column of the identity matrix. Indeed, we have from (3.18) that $\widehat{\Phi}(x,\lambda) = P(\lambda)\widehat{\Psi}(x,\lambda)$ and hence

$$\widehat{\Phi}(x,\lambda) = \begin{bmatrix} \widehat{\Phi}_{11}(x,\lambda) & O_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & \widehat{\Phi}_{22}(x,\lambda) & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} & \widehat{\Phi}_{33}(x,\lambda) \end{bmatrix}, \tag{4.3}$$

where

(4.4)

$$\widehat{\Phi}_{ii}(x,\lambda) := \begin{bmatrix} r_i\lambda e^{r_i\lambda x}[1 + \mathcal{O}(\lambda^{-1})] & r_i\lambda e^{-r_i\lambda x}[1 + \mathcal{O}(\lambda^{-1})] \\ r_i^2\lambda^2 e^{r_i\lambda x}[1 + \mathcal{O}(\lambda^{-1})] & -r_i^2\lambda^2 e^{-r_i\lambda x}[1 + \mathcal{O}(\lambda^{-1})] \end{bmatrix} \quad \text{for} \ \ i = 1,2$$

and

$$\widehat{\Phi}_{33}(x,\lambda) := \begin{bmatrix} r_2\lambda e^{r_2\lambda x}e_1(x)[1 + \mathcal{O}(\lambda^{-1})] & r_2\lambda e^{-r_2\lambda x}e_2(x)[1 + \mathcal{O}(\lambda^{-1})] \\ r_2^2\lambda^2 e^{r_2\lambda x}e_1(x)[1 + \mathcal{O}(\lambda^{-1})] & -r_2^2\lambda^2 e^{-r_2\lambda x}e_2(x)[1 + \mathcal{O}(\lambda^{-1})] \end{bmatrix} \tag{4.5}$$

with $e_i(x)$ $(i = 1,2)$ being given in (3.14).

Thus, the first component of $\Phi(x)$ is given by

$$\begin{aligned} w_1(x,\lambda) &= r_1^{-2} r_2^{-5} \lambda^{-8} \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda e^{r_1\lambda x}[r_1]_1 & \lambda e^{-r_1\lambda x}[r_1]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda^2 E_3[r_2 r_3]_1 & \lambda^2 E_4[r_2 r_4]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ -\lambda^2 E_3 E_5[r_2^2]_1 & \lambda^2 E_4 E_6[r_2^2]_1 \end{bmatrix} \\ &= \left(e^{-r_1\lambda x} - e^{r_1\lambda x} + \mathcal{O}(\lambda^{-1})\right)\left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right). \end{aligned}$$

By (3.23), (3.25), and (3.31), we conclude that

$$w_1(x,\lambda) = \begin{cases} r_7(\lambda)\left(e^{-r_1\lambda x} - e^{r_1\lambda x} + \mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda = \lambda_{1n}, \\ \mathcal{O}(\lambda^{-1}) & \text{if } \lambda = \lambda_{2n} \text{ or } \lambda_{3n}, \end{cases} \tag{4.6}$$

where $r_7(\lambda)$ is bounded in λ and has the form

$$r_7(\lambda) := \left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda}\right)\left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda}\right). \tag{4.7}$$

Similarly, we have

$$\begin{aligned} w_2(x,\lambda) &= r_1^{-2} r_2^{-5} \lambda^{-8} \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda^2 e^{r_1\lambda x}[r_1^2]_1 & -\lambda^2 e^{-r_1\lambda x}[r_1^2]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda^2 E_3[r_2 r_3]_1 & \lambda^2 E_4[r_2 r_4]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ -\lambda^2 E_3 E_5[r_2^2]_1 & \lambda^2 E_4 E_6[r_2^2]_1 \end{bmatrix} \\ &= -r_1\lambda\left(e^{-r_1\lambda x} + e^{r_1\lambda x} + \mathcal{O}(\lambda^{-1})\right)\left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \end{aligned}$$

and

$$w_2(x,\lambda) = \begin{cases} -\lambda r_1 r_7(\lambda)\left(e^{-r_1\lambda x} + e^{r_1\lambda x} + \mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda = \lambda_{1n}, \\ r_1\lambda[\mathcal{O}(\lambda^{-1})] & \text{if } \lambda = \lambda_{2n} \text{ or } \lambda_{3n}. \end{cases} \tag{4.8}$$

Also, along the same line,

$$\begin{aligned} w_3(x,\lambda) &= r_1^{-2} r_2^{-5} \lambda^{-8} \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda^2 E_1[r_1 r_5]_1 & \lambda^2 E_2[r_1 r_6]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda e^{r_2\lambda x}[r_2]_1 & \lambda e^{-r_2\lambda x}[r_2]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ -\lambda^2 E_3 E_5[r_2^2]_1 & \lambda^2 E_4 E_6[r_2^2]_1 \end{bmatrix} \\ &= \left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda} + \mathcal{O}(\lambda^{-1})\right)\left(e^{-r_2\lambda x} - e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \end{aligned}$$

and from (3.23), (3.25), and (3.31), we obtain that

$$w_3(x,\lambda) = \begin{cases} r_8(\lambda)\left(e^{-r_2\lambda x} - e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda = \lambda_{2n}, \\ \mathcal{O}(\lambda^{-1}) & \text{if } \lambda = \lambda_{1n} \text{ or } \lambda_{3n} \end{cases} \tag{4.9}$$

with

$$r_8(\lambda) := \left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda}\right)\left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda}\right). \tag{4.10}$$

Furthermore,

$$\begin{aligned} w_4(x,\lambda) &= r_1^{-2} r_2^{-5}\lambda^{-8} \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda^2 E_1[r_1 r_5]_1 & \lambda^2 E_2[r_1 r_6]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda^2 e^{r_2\lambda x}[r_2^2]_1 & -\lambda^2 e^{-r_2\lambda x}[r_2^2]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ -\lambda^2 E_3 E_5 [r_2^2]_1 & \lambda^2 E_4 E_6 [r_2^2]_1 \end{bmatrix} \\ &= -r_2\lambda\left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda} + \mathcal{O}(\lambda^{-1})\right)\left(e^{-r_2\lambda x} + e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(1)e^{-r_2\lambda} + e_1(1)e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \end{aligned}$$

and

$$w_4(x,\lambda) = \begin{cases} -\lambda r_2 r_8(\lambda)\left(e^{-r_2\lambda x} + e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda = \lambda_{2n}, \\ r_2\lambda[\mathcal{O}(\lambda^{-1})] & \text{if } \lambda = \lambda_{1n} \text{ or } \lambda_{3n}. \end{cases} \tag{4.11}$$

Also, the fifth component of $\Phi(x)$ can be given by

$$\begin{aligned} w_5(x,\lambda) &= -r_1^{-2} r_2^{-4}\lambda^{-8} \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda^2 E_1[r_1 r_5]_1 & \lambda^2 E_2[r_1 r_6]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda^2 E_3[r_2 r_3]_1 & \lambda^2 E_4[r_2 r_4]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda e^{r_2\lambda x} e_1(x)[r_2]_1 & \lambda e^{-r_2\lambda x} e_2(x)[r_2]_1 \end{bmatrix} \\ &= -\left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda} + \mathcal{O}(\lambda^{-1})\right)\left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(x)e^{-r_2\lambda x} - e_1(x)e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right), \end{aligned}$$

and we conclude from (3.23), (3.25), and (3.31) that

$$w_5(x,\lambda) = \begin{cases} r_9(\lambda)\left(e_2(x)e^{-r_2\lambda x} - e_1(x)e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda = \lambda_{3n}, \\ \mathcal{O}(\lambda^{-1}) & \text{if } \lambda = \lambda_{1n} \text{ or } \lambda_{2n} \end{cases} \tag{4.12}$$

with

$$r_9(\lambda) := -\left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda} + \mathcal{O}(\lambda^{-1})\right)\left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right). \tag{4.13}$$

For the last component of $\Phi(x)$, one has

$$\begin{aligned} w_6(x,\lambda) &= r_1^{-2} r_2^{-4}\lambda^{-8} \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ -\lambda^2 e^{r_2\lambda x} e_1(x)[r_2^2]_1 & \lambda^2 e^{-r_2\lambda x} e_2(x)[r_2^2]_1 \end{bmatrix} \\ &\quad \times \det \begin{bmatrix} \lambda[r_2]_1 & \lambda[r_2]_1 \\ \lambda^2 E_3[r_2 r_3]_1 & \lambda^2 E_4[r_2 r_4]_1 \end{bmatrix} \times \det \begin{bmatrix} \lambda[r_1]_1 & \lambda[r_1]_1 \\ \lambda^2 E_1[r_1 r_5]_1 & \lambda^2 E_2[r_1 r_6]_1 \end{bmatrix} \\ &= r_2\lambda\left(r_6 e^{-r_1\lambda} - r_5 e^{r_1\lambda} + \mathcal{O}(\lambda^{-1})\right)\left(r_4 e^{-r_2\lambda} - r_3 e^{r_2\lambda} + \mathcal{O}(\lambda^{-1})\right) \\ &\quad \times \left(e_2(x)e^{-r_2\lambda x} + e_1(x)e^{r_2\lambda x} + \mathcal{O}(\lambda^{-1})\right) \end{aligned}$$

and

(4.14)
$$w_6(x,\lambda)=\begin{cases}-\lambda r_2 r_9(\lambda)\left(e_2(x)e^{-r_2\lambda x}+e_1(x)e^{r_2\lambda x}+\mathcal{O}(\lambda^{-1})\right) & \text{if } \lambda=\lambda_{3n},\\ r_2\lambda[\mathcal{O}(\lambda^{-1})] & \text{if } \lambda=\lambda_{1n} \text{ or } \lambda_{2n}.\end{cases}$$

On the basis of above computations, (4.1) can then be deduced from (4.6)–(4.14) by setting

$$w_n(x)=-\frac{w_1(x,\lambda)}{2r_1\lambda r_7(\lambda)},\quad \xi_n(x)=-\frac{w_3(x,\lambda)}{2r_2\lambda r_8(\lambda)},\quad s_n(x)=-\frac{w_5(x,\lambda)}{2r_2\lambda r_9(\lambda)} \tag{4.15}$$

in (4.6)–(4.14), respectively. Finally, it follows from (3.25) that

$$\begin{cases}\|e^{-r_j\lambda_{jn}x}\|_{L^2}=\dfrac{1-e^{-2r_j\mu_j}}{2r_j\mu_j}+\mathcal{O}(n^{-1}) \quad \text{for } j=1,2,\\ \|e^{r_j\lambda_{jn}x}\|_{L^2}=\dfrac{e^{2r_j\mu_j}-1}{2r_j\mu_j}+\mathcal{O}(n^{-1}) \quad \text{for } j=1,2,\\ \|e^{-r_2\lambda_{3n}x}\|_{L^2}=\dfrac{1-e^{-2r_2\mu_3}}{2r_2\mu_3}+\mathcal{O}(n^{-1}),\\ \|e^{r_2\lambda_{3n}x}\|_{L^2}=\dfrac{e^{2r_2\mu_3}-1}{2r_2\mu_3}+\mathcal{O}(n^{-1}),\end{cases} \tag{4.16}$$

where μ_j $(j=1,2,3)$ are given in (3.26) and (3.27). These together with (4.1) yield (4.2). The proof is complete. □

5. The Riesz basis property and exponential stability of the system. In the previous sections, we have obtained the asymptotic expressions of eigenpairs of $\mathcal{A}$ and concluded that there are three asymptotes for the spectrum $\sigma(\mathcal{A})$ with their asymptotic expressions in (3.25). In this section, we shall prove that there exists a sequence of generalized eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$. Furthermore, the exponential stability of the system can be determined by its spectrum distribution.

For these purposes, we introduce another equivalent inner product on $\mathcal{H}$. Let $Y_j:=[w_j,z_j,\xi_j,\varphi_j,s_j,h_j]^\top\in\mathcal{H}$ $(j=1,2)$ define a new inner product in $\mathcal{H}$ by

(5.1)
$$[Y_1,\ Y_2]_H:=\langle w_1',w_2'\rangle_{L^2}+\langle z_1,z_2\rangle_{L^2}+\langle \xi_1',\xi_2'\rangle_{L^2}+\langle\varphi_1,\varphi_2\rangle_{L^2}+\langle s_1',s_2'\rangle_{L^2}+\langle h_1,h_2\rangle_{L^2},$$

and write its induced norm of (5.1) by $\|\cdot\|_H$. One can easily check that $\mathcal{H}$ is a Hilbert space under this new inner product. From now on, we shall consider our problem in $\mathcal{H}$ associated with this new inner product of (5.1). For convenience, define another Hilbert space

$$\mathcal{L}:=\left(L^2(0,1)\right)^6 \tag{5.2}$$

with an inner product (for any $X_j:=[w_j,z_j,\xi_j,\varphi_j,s_j,h_j]^\top\in\mathcal{L},\ j=1,2$)

$$\langle X_1,\ X_2\rangle_{\mathcal{L}}:=\langle w_1,w_2\rangle_{L^2}+\langle z_1,z_2\rangle_{L^2}+\langle\xi_1,\xi_2\rangle_{L^2}+\langle\varphi_1,\varphi_2\rangle_{L^2}+\langle s_1,s_2\rangle_{L^2}+\langle h_1,h_2\rangle_{L^2}$$

and define the subspaces of $\mathcal{H}$ and $\mathcal{L}$, respectively, by

$$\begin{cases} \mathcal{H}_1 := \{Y \in \mathcal{H} \mid Y = [w, z, 0, 0, 0, 0]^\top\}, \\ \mathcal{H}_2 := \{Y \in \mathcal{H} \mid Y = [0, 0, \xi, \varphi, 0, 0]^\top\}, \\ \mathcal{H}_3 := \{Y \in \mathcal{H} \mid Y = [0, 0, 0, 0, s, h]^\top\}, \end{cases} \qquad \begin{cases} \mathcal{L}_1 := \{X \in \mathcal{L} \mid X = [w, z, 0, 0, 0, 0]^\top\}, \\ \mathcal{L}_2 := \{X \in \mathcal{L} \mid X = [0, 0, \xi, \varphi, 0, 0]^\top\}, \\ \mathcal{L}_3 := \{X \in \mathcal{L} \mid X = [0, 0, 0, 0, s, h]^\top\}. \end{cases} \tag{5.3}$$

Obviously, we have

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \quad \text{and} \quad \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3, \tag{5.4}$$

where the sign $\oplus$ denotes the direct sum in the sense of orthogonality with respect to the inner products $[\cdot, \cdot]_H$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ in $\mathcal{H}$ and $\mathcal{L}$, respectively.

Before continuing, let us recall some forms of notation. For a closed operator $\mathbf{A}$ in a Hilbert space $\mathbf{H}$, a nonzero element $\phi \in \mathbf{H}$ is called a generalized eigenvector of $\mathbf{A}$, corresponding to an eigenvalue λ of $\mathbf{A}$, if there is an integer $\nu \geq 1$ such that $(\lambda I - \mathbf{A})^\nu \phi = 0$. If $\nu = 1$, then ϕ is an eigenvector. A sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathbf{H}$ is called a Riesz basis for $\mathbf{H}$ if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $\mathbf{H}$ and a linear bounded invertible operator T such that

$$T\phi_n = e_n, \quad n = 1, 2, \dots .$$

Let $\{\lambda_n\}_{n=1}^\infty = \sigma(\mathbf{A})$ be the spectrum of $\mathbf{A}$. Suppose each λ_n has finite algebraic multiplicity m_n, and let $\{\psi_{n_i}\}_1^{m_n}$ be the set of generalized eigenvectors of $\mathbf{A}$ corresponding to λ_n. If $\{\psi_{n_i} \mid 1 \leq i \leq m_n,\ n = 1, 2, \dots\}$ form a Riesz basis for $\mathbf{H}$, then the C_0-semigroup generated by $\mathbf{A}$ can be represented as

$$e^{\mathbf{A}t}x = \sum_{n=1}^\infty e^{\lambda_n t} \sum_{j=1}^{m_n} a_{nj} f_{nj}(t) \psi_{nj} \quad \forall\, x = \sum_{n=1}^\infty \sum_{j=1}^{m_n} a_{nj} \psi_{nj} \in \mathbf{H}, \tag{5.5}$$

where $f_{nj}(t)$ are the polynomials of t with order not greater than m_n. In particular, if m_n has a uniform upper bound and $\{\psi_{n_i}\}_1^{m_n}$ is the eigenvector (not generalized eigenvector) set of $\mathcal{A}$ with respect to λ_n for all sufficiently large n, then the spectrum determined growth condition holds, i.e., $\omega(\mathbf{A}) = s(\mathbf{A})$, where $\omega(\mathbf{A})$ is the growth bound of $e^{\mathbf{A}t}$, and $s(\mathbf{A})$ is the spectral bound of $\mathbf{A}$ (see [2]).

To establish the Riesz basis property for the root space of the operator $\mathcal{A}$, we recall a result of Bari's theorem in [11].

THEOREM 5.1. *Let $\mathbf{H}$ be a separable Hilbert space and let $\{e_n;\ n \in \mathbb{Z}\}$ be an orthonormal basis for $\mathbf{H}$. If $\{f_n;\ n \in \mathbb{Z}\}$ is an ω-independent sequence that is quadratically close to $\{e_n;\ n \in \mathbb{Z}\}$, then $\{f_n;\ n \in \mathbb{Z}\}$ is a Riesz basis for $\mathbf{H}$.*

LEMMA 5.2. *Let $\{\phi_n(x);\ n \in \mathbb{N}\}$ and $\{1, \psi_n(x);\ n \in \mathbb{N}\}$ be two subsets in $L^2(0,1)$ defined by, respectively,*

$$\phi_n(x) := \sin n\pi x \quad \text{and} \quad \psi_n(x) := \cos n\pi x \quad \forall\, x \in (0,1),\ n \in \mathbb{N}.$$

Then $\{\phi_n(x);\ n \in \mathbb{N}\}$ and $\{1, \psi_n(x);\ n \in \mathbb{N}\}$ are two orthogonal bases in $L^2(0,1)$. Moreover, for any scalars $\alpha, \beta \neq 0 \in \mathbb{C}$ the vector family $\{\Psi_n := [\cosh(\alpha + in\pi)x, \beta \sinh(\alpha+in\pi)x]^\top, n \in \mathbb{Z}\}$ forms a Riesz basis on the Hilbert space $L^2(0,1) \times L^2(0,1)$.

Proof. The first assertion is a direct result in [11] and it is easily verified that

$$\left\{ \begin{bmatrix} 0 \\ \sin n\pi x \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \cos n\pi x \\ 0 \end{bmatrix} \right\}_{n \in \mathbb{N}}$$

constitutes a Riesz basis on $L^2(0,1)\times L^2(0,1)$. So the sequence

$$\left\{\begin{bmatrix}\cos n\pi x\\ \sin n\pi x\end{bmatrix},\quad \begin{bmatrix}1\\0\end{bmatrix},\quad \begin{bmatrix}\cos n\pi x\\ -\sin n\pi\end{bmatrix}\right\}_{n\in\mathbb{N}}$$

also forms a Riesz basis on $L^2(0,1)\times L^2(0,1)$. Let T be an invertible matrix function in $(0,1)$ given by

$$T := \begin{bmatrix}\cosh\alpha x & i\sinh\alpha x\\ \beta\sinh\alpha x & i\beta\cosh\alpha x\end{bmatrix}\quad \text{with } \ |T| = i\beta \ \text{ for each } x\in(0,1).$$

Then we obtain, for $n\in\mathbb{N}$,

$$\begin{bmatrix}\cosh(\alpha+in\pi)x\\ \beta\sinh(\alpha+in\pi)x\end{bmatrix} = T\begin{bmatrix}\cos n\pi x\\ \sin n\pi x\end{bmatrix},\quad \begin{bmatrix}\cosh(\alpha-in\pi)x\\ \beta\sinh(\alpha-in\pi)x\end{bmatrix} = T\begin{bmatrix}\cos n\pi x\\ -\sin n\pi x\end{bmatrix},$$

and

$$\begin{bmatrix}\cosh\alpha x\\ \beta\sinh\alpha x\end{bmatrix} = T\begin{bmatrix}1\\0\end{bmatrix}.$$

Thus, the sequence $\{\Psi_n := [\cosh(\alpha+in\pi)x,\ \beta\sinh(\alpha+in\pi)x]^\top, n\in\mathbb{Z}\}$ also forms a Riesz basis on the Hilbert space $L^2(0,1)\times L^2(0,1)$. The second assertion is concluded. ◻

THEOREM 5.3. *Suppose $r_1\neq r_2$ and $k_i\neq r_i$ $(i=1,2)$. Let $\mathcal{A}$ be defined by* (2.4) *and* (2.5)*, and let*

$$\Psi_{jn} := [w'_{jn},\ \lambda_{jn}w_{jn},\ \xi'_{jn},\ \lambda_{jn}\xi_{jn},\ s'_{jn},\ \lambda_{jn}s_{jn}]^\top,\quad (j=1,2,3,\ n\in\mathbb{Z}), \tag{5.6}$$

where the entries are given as (4.1) *corresponding to the eigenvalues λ_{jn}. Then $\{\Psi_{1n},\Psi_{2n},\Psi_{3n};\ n\in\mathbb{Z}\}$ forms a Riesz basis in Hilbert space $\mathcal{L}$ provided that $\{\Psi_{jn},\ j=1,2,3,\ n\in\mathbb{Z}\}$ is ω-linearly independent in $\mathcal{L}$.*

Proof. Let three vector families be given by

$$\Phi_{1n} := \left[\cosh(r_1\mu_1+in\pi)x,\ r_1^{-1}\sinh(r_1\mu_1+in\pi)x,0,0,0,0\right]^\top,$$

$$\Phi_{2n} := \left[0,0,\cosh(r_2\mu_2+in\pi)x,\ r_2^{-1}\sinh(r_2\mu_2+in\pi)x,0,0\right]^\top,$$

$$\Phi_{3n} := \left[0,0,0,0,\cosh i\Big(n+\frac{1}{2}\Big)\pi x,\ r_3^{-1}\sinh i\Big(n+\frac{1}{2}\Big)\pi x\right]^\top.$$

Then one concludes from Lemma 5.2 that the families $\{\Phi_{jn},\ n\in\mathbb{Z}\}$ $(j=1,2,3)$ are the Riesz bases for $\mathcal{L}_j$, respectively. Also, by using the asymptotic expressions of both eigenvalues (3.25) and their eigenfunctions (4.1), it follows that

$$\begin{cases}\|w'_{1n}-\cosh(r_1\mu_1+in\pi)x\|_{L^2} = \mathcal{O}(n^{-1}),\\ \|r_1\lambda_{1n}w_{1n}-\sinh(r_1\mu_1+in\pi)x\|_{L^2} = \mathcal{O}(n^{-1}),\\ \|\xi'_{2n}-\cosh(r_2\mu_2+in\pi)x\|_{L^2} = \mathcal{O}(n^{-1}),\\ \|r_2\lambda_{2n}\xi_{1n}-\sinh(r_2\mu_2+in\pi)x\|_{L^2} = \mathcal{O}(n^{-1}),\\ \|s'_{3n}-\cosh i(n+\frac{1}{2})\pi x\|_{L^2} = \mathcal{O}(n^{-1}),\\ \|r_3\lambda_{3n}s_{3n}-\sinh i(n+\frac{1}{2})\pi x\|_{L^2} = \mathcal{O}(n^{-1}).\end{cases} \tag{5.7}$$

Hence, we obtain

$$\begin{aligned} \|\Psi_{1n} - \Phi_{1n}\|_{\mathcal{L}_1} &= \mathcal{O}(n^{-1}), \\ \|\Psi_{2n} - \Phi_{2n}\|_{\mathcal{L}_2} &= \mathcal{O}(n^{-1}), \\ \|\Psi_{3n} - \Phi_{3n}\|_{\mathcal{L}_3} &= \mathcal{O}(n^{-1}). \end{aligned} \tag{5.8}$$

Therefore, by Theorem 5.1 (Bari's theorem), $\{\Psi_{1n}, \Psi_{2n}, \Psi_{3n}; n \in \mathbb{Z}\}$ forms a Riesz basis for $\mathcal{L}$ provided that $\{\Psi_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ is ω-linearly independent in $\mathcal{L}$. ∎

As a consequence of this theorem, we obtain the main results of the paper.

THEOREM 5.4. *Suppose $r_1 \neq r_2$ and $k_i \neq r_i$ $(i = 1, 2)$. Let $\mathcal{A}$ be defined by* (2.4) *and* (2.5)*. Then there exists a sequence of generalized eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$.*

Proof. In Theorem 4.1, we have obtained the asymptotic expressions of the eigenfunctions of $\mathcal{A}$ corresponding to the eigenvalues with large moduli. Without loss of generality, we may assume that

$$Y_{jn} := [w_{jn}, \lambda_{jn} w_{jn}, \xi_{jn}, \lambda_{jn}\xi_{jn}, s_{jn}, \lambda_{jn} s_{jn}]^\top \quad \text{for} \ \ j = 1, 2, 3 \ \ \text{and} \ \ n \in \mathbb{Z}$$

is an eigenfunction corresponding to the eigenvalue λ_{jn} in which the entries have the asymptotic expansions given in (4.1). Then $\{Y_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ is ω-linearly independent in $\mathcal{H}$. The proof will be completed via an isomorphic mapping between two Hilbert spaces $\mathcal{H}$ and $\mathcal{L}$ that maps Y_{jn} to Ψ_{jn}, where $\{\Psi_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ is a sequence given by (5.6).

To do this, for any $F := [f_1, f_2, g_1, g_2, u_1, u_2]^\top \in \mathcal{H}$, we define a linear bounded operator $\mathcal{T} : \mathcal{H} \to \mathcal{L}$ by

$$\mathcal{T}F := [f_1', f_2, g_1', g_2, u_1', u_2]^\top := \widehat{F}.$$

Since $[F,\ Y_{jn}]_H = \langle \widehat{F},\ \Psi_{jn}\rangle_{\mathcal{L}}$, it is easy to prove that $\mathcal{T}$ is isomorphic and satisfies

$$\|\mathcal{T}F\|_{\mathcal{L}} = \|\widehat{F}\|_{\mathcal{L}} = \|F\|_H. \tag{5.9}$$

In particular, for $j = 1, 2, 3$ and $n \in \mathbb{Z}$,

$$\mathcal{T}Y_{jn} = \Psi_{jn} \quad \text{and} \quad \|Y_{jn}\|_H = \|\Psi_{jn}\|_{\mathcal{L}}.$$

Moreover, $\{Y_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ is ω-linearly independent in $\mathcal{H}$, so is $\{\Psi_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ in $\mathcal{L}$. Therefore, by Theorem 5.3, $\{\Psi_{jn};\ j = 1, 2, 3, n \in \mathbb{Z}\}$ forms a Riesz basis in $\mathcal{L}$. Hence, $\{Y_{jn}; j = 1, 2, 3,\ n \in \mathbb{Z}\}$ forms a Riesz basis for $\mathcal{H}$. The proof is complete. ∎

THEOREM 5.5. *Suppose $r_1 \neq r_2$ and $k_i \neq r_i$ $(i = 1, 2)$. Let $\mathcal{A}$ be defined by* (2.4) *and* (2.5)*, and let $T(t)$ be a C_0-semigroup generated by $\mathcal{A}$ in $\mathcal{H}$. Then $T(t)$ is exponentially stable, and in fact it is a C_0-group in $\mathcal{H}$.*

Proof. As a direct consequence of Theorems 3.4 and 5.4, the spectrum determined growth condition $\omega(\mathbf{A}) = s(\mathbf{A})$ for $T(t)$ holds. Furthermore, Corollary 2.2 implies that there is no eigenvalue on the imaginary axis. This, together with (3.25) and the spectrum determined growth condition, shows that $T(t)$ is an exponentially stable semigroup on $\mathcal{H}$. Moreover, $T(t)$ is also a C_0-group in $\mathcal{H}$. This is because of the fact that the spectrum of $\mathcal{A}$ distributes in a vertical strip due to Theorems 3.3 and 3.4. ∎

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