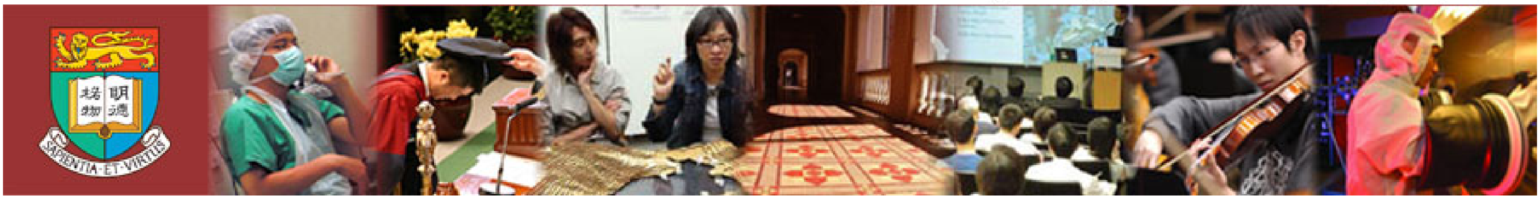

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Interactions of breathers and solitons in the extended Korteweg-de Vries equation

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Abstract

The extended Korteweg de Vries model governs the evolution of weakly dispersive waves under the combined influence of quadratic and cubic nonlinearities, and is relevant to finite-amplitude wave motions in the atmosphere and the ocean. Analytic expressions for a multi-soliton are obtained by the Hirota bilinear method, and are shown to agree with those for isolated solitary waves or breathers obtained earlier in the literature. In particular, the interaction of a breather and a soliton can now be studied. Both the soliton and the breather retain their identities after interaction except for some phase shifts. Detailed examination of the interaction process shows that the profile of the breather will depend critically on the polarity of the colliding soliton.

1. Introduction

The Korteweg - de Vries (KdV) equation, or one if its cousins, is now well established as a canonical model for the description of weakly nonlinear and weakly dispersive long waves in a variety of physical systems [1-10]. In particular, the extended Korteweg - de Vries (eKdV) equation,

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u_{xxx} = 0\,, \qquad (1)$$

has recently become a popular model for the description of internal solitary waves in shallow seas (see, for instance, the review article by Grimshaw [7] and the article by Grimshaw et al [10]). The competition among dispersion, quadratic and cubic nonlinearities constitutes the main interest here. The asymptotic derivation of (1) from the fully nonlinear equations of fluid dynamics has been described in various works [1-4]. Hence the focus of the present work is on the dynamics of the fundamental localized travelling waves. Like the KdV equation, the eKdV equation is integrable with a supporting Lax pair and an inverse scattering transform. Depending on the sign of the cubic nonlinearity (or $\beta\delta$ of Eq.(1)), the eKdV equation can support a single family of solitons ($\beta\delta < 0$), or two families of solitons of opposite polarities, and a family of breathers ($\beta\delta > 0$). Our interest here is in this latter case. While the soliton solutions are well-known [1], an explicit expression for the breather was obtained from the inverse scattering transform by Pelinovsky and Grimshaw [5], and later by Slyunyaev [9] from the coalescence of two solitons with complex parameters; the identification of this breather as a form of wave packet is discussed in Grimshaw et al [8]. Some aspects of the initial-value problem were considered for the special case when Eq. (1) reduces to the modified KdV (mKdV) equation (i.e. $\alpha = 0$ in Eq. (1)) by Clarke et al [6].

Our concern here is with the case $\beta\delta > 0$, which is considerably richer in its dynamics than the case $\beta\delta < 0$. In this latter case there is only a single family of solitons, all of the same polarity. But in the former case there are two families of solitons, with opposite polarities, and a family of breathers. One family of solitons has the polarity of $\alpha\delta$, and ranges from small-amplitude solitons of the familiar "sech^2" profile of the KdV equation to large amplitude solitons with the "sech" profile of the mKdV equation. The other family of solitons has polarity opposite to $\alpha\delta$, but ranges from large amplitude solitons with the "sech" profile of the mKdV equation, to an algebraic soliton with a certain minimum amplitude and mass (in absolute value). The breather solutions have a mass with the same polarity as this family of solitons, and the mass ranges from that of the afore-mentioned algebraic soliton to zero. Slyunyaev [9] obtained the two-soliton solution for this case when $\beta\delta > 0$, using the Darboux transformation, and showed that while the interaction of two solitons of the same polarity was qualitatively similar to the scenario for the KdV equation, the interaction of two solitons of opposite polarities produced some essentially different features. In this paper our main aim is to obtain explicit expressions describing the interaction of solitons and breathers. At the same time we also obtain the expressions for a three-soliton interaction. Unlike the previously mentioned works, we will use the Hirota bilinear transform [11-13], a method proven over several decades to be effective and powerful, to obtain our results.

The structure of this paper is as follows. The soliton and breather will first be calculated from the bilinear method and shown to agree with those obtained earlier in the literature. The power of the present approach is that a higher order soliton can be generated systematically, and hence the interaction of a soliton and a breather can be obtained from the expression for the 3-soliton solution.

2. Analysis

First we rescale Eq. (1) in the case $\beta\delta > 0$ to obtain,

$$u_t + \alpha u u_x + 6\delta u^2 u_x + \delta u_{xxx} = 0 . \qquad (2)$$

Here, without loss of generality, we may suppose that $\delta > 0$. To use the Hirota method we first obtain the bilinear forms,

$$(D_t + \delta D_x^3) G \cdot F = 0 \,, \tag{3}$$

$$6\delta D_x^2 G \cdot F + \alpha i D_x G \cdot F = 0 \,. \tag{4}$$

These are related to the eKdV equation (2) through the mapping

$$u = \left(\frac{1}{i} \log \frac{G}{F} \right)_x \,, \tag{5}$$

where, as usual, D is the Hirota operator defined by

$$D_x^m \, D_t^n \, g \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n g\,(x,t)\; f\,(x',t') \Big|_{x=x',t=t'}$$

A more familiar and convenient form is given in terms of the real-valued functions f and g,

$G = f + ig, \quad F = f - ig,$

$$\begin{aligned} &(D_t + \delta\, D_x^3)\; g \cdot f \;= 0, \\ &3\delta\, D_x^2 (\, f \cdot f + g \cdot g) - \alpha\, D_x g \cdot f \;= 0, \end{aligned} \tag{6}$$

$$u = \left[2 \tan^{-1} \left(\frac{g}{f} \right) \right]_x . \tag{7}$$

Multi-soliton, breather and other related solutions can now be obtained in explicit forms. They will be given below in order of increasing complexity.

(A) *Basic 1-soliton*:

The 1-soliton is

$$\begin{aligned} &u = \left(2 \tan^{-1} \left[\frac{1 + a_1 \exp(p_1 x - \Omega_1 t)}{1 + \exp(p_1 x - \Omega_1 t)} \right] \right)_x , \\ &\Omega_1 = \delta\, p_1^3, \qquad a_1 = \frac{\alpha + 6\delta p_1}{\alpha - 6\delta p_1}. \end{aligned} \tag{8}$$

This can be rewritten in the more familiar forms. For $\alpha > 6\delta p_1$, we have a solitary wave of elevation:

$$u = \frac{6\delta p_1^2}{\alpha + \sqrt{\alpha^2 + 36\delta^2 p_1^2}\cosh[p_1(x - \delta p_1^2 t - x_0)]},$$
$$\exp(px_0) = \frac{\alpha - 6\delta p_1}{\sqrt{\alpha^2 + 36\delta^2 p_1^2}}, \tag{9}$$

and hence the amplitude is

$$\frac{6\delta p_1^2}{\alpha + \sqrt{\alpha^2 + 36\delta^2 p_1^2}}.$$

For $\alpha < 6\delta p_1$, we have a solitary wave of depression

$$u = -\frac{6\delta p_1^2}{\sqrt{\alpha^2 + 36\delta^2 p_1^2}\cosh[p_1(x - \delta p_1^2 - x_0)] - \alpha}, \tag{10}$$

where x_0 is again given by Eq. (9). This difference in representation arises purely from the use of the hyperbolic cosine. No such difference will result if the exponential function form (Eq. (8)) is used.

(B) *2-soliton*:

The 2-soliton solution is

$$u = \left(2\tan^{-1}\left(\frac{g}{f}\right)\right)_x,$$

$$g = 1 + a_1\exp(\phi) + a_2\exp(\psi) + a_{12}\exp(\phi + \psi),$$
$$f = 1 + \exp(\phi) + \exp(\psi) + f_{12}\exp(\phi + \psi),$$

$$\phi = p_1 x - \delta p_1^3 t, \qquad \psi = p_2 x - \delta p_2^3 t, \qquad a_n = \frac{\alpha + 6\delta p_n}{\alpha - 6\delta p_n}, \qquad n = 1, 2, \tag{11}$$

$$f_{12} = \frac{(p_1 - p_2)^2[\alpha^2 - 6\alpha\delta(p_1 + p_2) - 36\delta^2 p_1 p_2]}{(p_1 + p_2)^2(\alpha - 6\delta p_1)(\alpha - 6\delta p_2)},$$

$$a_{12} = \frac{(p_1 - p_2)^2[\alpha^2 + 6\alpha\delta(p_1 + p_2) - 36\delta^2 p_1 p_2]}{(p_1 + p_2)^2(\alpha - 6\delta p_1)(\alpha - 6\delta p_2)}.$$

These expressions can be shown to agree with those obtained by Slyunyaev [9]. The phase shift after the interaction can be obtained by considering one of the phases, say ϕ, as fixed and then allowing the other phase (ψ) to go to plus or minus infinity.

(C) *3-soliton*: The 3-soliton solution is

$$\begin{aligned} g &= 1 + \sum a_i \exp(\phi_i) + \sum_{i<j} a_{ij} \exp(\phi_i + \phi_j) + a_{123} \exp(\phi_1 + \phi_2 + \phi_3), \\ f &= 1 + \sum \exp(\phi_i) + \sum_{i<j} f_{ij} \exp(\phi_i + \phi_j) + f_{123} \exp(\phi_1 + \phi_2 + \phi_3). \end{aligned} \tag{12}$$

$$\phi_n = p_n x - \delta p_n^3 t, \qquad n = 1, 2, 3, \qquad a_n = \frac{\alpha + 6\delta p_n}{\alpha - 6\delta p_n},$$

$$f_{ij} = \frac{(p_i - p_j)^2[\alpha^2 - 6\alpha\delta(p_i + p_j) - 36\delta^2 p_i p_j]}{(p_i + p_j)^2(\alpha - 6\delta p_i)(\alpha - 6\delta p_j)},$$

$$a_{ij} = \frac{(p_i - p_j)^2[\alpha^2 + 6\alpha\delta(p_i + p_j) - 36\delta^2 p_i p_j]}{(p_i + p_j)^2(\alpha - 6\delta p_i)(\alpha - 6\delta p_j)},$$

$$f_{123} = \frac{(p_1 - p_2)^2(p_2 - p_3)^2(p_3 - p_1)^2 R_f}{(p_1 + p_2)^2(p_2 + p_3)^2(p_3 + p_1)^2(\alpha - 6\delta p_1)(\alpha - 6\delta p_2)(\alpha - 6\delta p_3)},$$

$$\begin{aligned} R_f &= \alpha^3 - 6\alpha^2\delta(p_1 + p_2 + p_3) \\ &\quad - 36\alpha\delta^2(p_1 p_2 + p_2 p_3 + p_3 p_1) + 216\delta^3 p_1 p_2 p_3, \end{aligned}$$

$$a_{123} = \frac{(p_1 - p_2)^2(p_2 - p_3)^2(p_3 - p_1)^2 R_a}{(p_1 + p_2)^2(p_2 + p_3)^2(p_3 + p_1)^2(\alpha - 6\delta p_1)(\alpha - 6\delta p_2)(\alpha - 6\delta p_3)},$$

$$\begin{aligned} R_a &= \alpha^3 + 6\alpha^2\delta(p_1 + p_2 + p_3) \\ &\quad - 36\alpha\delta^2(p_1 p_2 + p_2 p_3 + p_3 p_1) - 216\delta^3 p_1 p_2 p_3. \end{aligned}$$

A remark on the interaction of solitons is now needed. In general, the interaction of two solitons in the classical Korteweg-de Vries equation is accomplished either by a merger of the two peaks, if the amplitudes of the two solitons differ sufficiently, or by an exchange of identities (but without actual merger), if the two are sufficiently similar to each other. For the present eKdV model, a similar scenario also holds. A major difference, and additional flexibility, is the possibility of solitons with opposite polarities. Earlier work has demonstrated that a depression soliton can suffer a downward impulse during interaction with an elevation soliton.

With an analytical expression for three solitons, we focus on situations where solitons with opposite polarities are present. Figure 1 illustrates the interactions of three solitons, (a relatively small) one of elevation and two of depression with similar amplitudes. The two depression solitons would normally interact by a simple exchange of identities without merger. However, as they travel past the elevation soliton, each one will receive a downward push in turn. This whole sequence of events will thus create a few cycles of changes for the relative amplitudes of the two depression solitons (Figure 1).

The other case, two solitons of elevation and one of depression, is less remarkable as the positive solitons are typically much smaller than the depression soliton.

(D) *Breather*:

Similar to the technique used by Slyunaev [9], the breather is obtained from the 2-soliton solution (11) by choosing a pair of complex conjugate wave numbers ($p_1 = m + i\,n,\ p_2 = m - i\,n,\ m, n$ real)

$$u = \left[2\tan^{-1}\left(\frac{g}{f} \right) \right]_x , \qquad (13)$$

$$g = \exp(-\theta) - \frac{n^2}{m^2}\left[\frac{\alpha^2 + 12\,\alpha\delta m - 36\,\delta^2(m^2+n^2)}{(\alpha - 6\,\delta m)^2 + 36\,\delta^2 n^2} \right]\exp(\theta) + 2(\xi\cos\Theta - \eta\sin\Theta),$$

$$f = \exp(-\theta) - \frac{n^2}{m^2}\left[\frac{\alpha^2 - 12\,\alpha\delta m - 36\,\delta^2(m^2+n^2)}{(\alpha - 6\,\delta m)^2 + 36\,\delta^2 n^2} \right]\exp(\theta) + 2\cos\Theta,$$

$$\theta = m[x - \delta(m^2 - 3n^2)t]\ , \qquad \Theta = n[x - \delta(3m^2 - n^2)t]\ ,$$

$$\xi = \frac{\alpha^2 - 36\delta^2(m^2 + n^2)}{(\alpha - 6\delta m)^2 + 36\delta^2 n^2}, \qquad \eta = \frac{12\alpha\delta n}{(\alpha - 6\delta m)^2 + 36\delta^2 n^2}.$$

(E) *Soliton-breather interaction*:

Similarly, the interaction of a breather and a soliton can be studied by choosing the three wave numbers of the 3-soliton expression Eq. (12) as $p_1 = m + \mathrm{i}\, n$, $p_2 = m - \mathrm{i}\, n$, $p_3 = p$. Due to the complexity of the expressions involved, the details of the interaction can best be studied by a computer algebra program. However, one can first show that the soliton retains its identity after interaction, except for a phase shift. For $\phi_1, \phi_2 \to -\infty$, u is given by

$$u = \left(2 \tan^{-1} \left[\frac{1 + a_3 \exp(p_3 x - \delta p_3^3 t)}{1 + \exp(p_3 x - \delta p_3^3 t)} \right] \right)_x . \tag{14}$$

but for $\phi_1, \phi_2 \to +\infty$, u is given by

$$u = \left(2 \tan^{-1} \left[\frac{a_{12} + a_{123} \exp(p_3 x - \delta p_3^3 t)}{f_{12} + f_{123} \exp(p_3 x - \delta p_3^3 t)} \right] \right)_x . \tag{15}$$

The maximum amplitude in the second expression (Eq. (15)) is

$$\frac{(a_{123} f_{12} - a_{12} f_{123}) p_3}{a_{12} a_{123} + f_{12} f_{123} + (\sqrt{a_{12}^2 + f_{12}^2} \sqrt{a_{123}^2 + f_{123}^2})},$$

which gives, after some algebra,

$$\frac{6\delta p_3^2}{\alpha + \sqrt{\alpha^2 + 36\delta^2 p_3^2}},$$

for the case of an elevation soliton. This is the same as that obtained by Eq. (9). Thus the amplitude is preserved and the soliton is changed only by a phase shift. Similar reasoning will apply to the case of a depression soliton. Likewise the breather also remains unchanged except for a phase shift.

For a more precise description of the stages of interaction between a soliton and a breather, we shall further subdivide into two cases, solitons of depression and those of elevation. For the first case (Figure 2), we choose, as a typical example, a breather travelling to the left colliding with a depression soliton going to the right. The breather may be regarded instantaneously as two ‘small hills’ on both sides of a deep valley, with polarities reversed half a cycle later. In the initial stage of the collision, the soliton will collide with the ‘small hill’ on the left, while the central valley remains almost frozen. After crossing the small hill on the left, the soliton then interacts with this central valley by exchanging identities, i.e., without the two valleys actually merging. Finally the central valley detaches, becomes the depression soliton, and propagates steadily to the right, while the entire remaining structures exhibit well defined oscillatory features and travel to the left as a breather. The breather and soliton retain their identities after all these intermediate stages, save for certain specified phase shifts.

For the second case, an elevation soliton moving to the right collides with a breather travelling to the left (Figure 3). We again regard the breather as two small hills on the adjacent sides of a deep central valley, with polarities reversed half a cycle later. The interaction starts here with the small hill on the left running into the elevation soliton. Without actual merger, the two entities exchange identities while the central valley again remains almost frozen. The elevation soliton seems to avoid any major contact or interference with the central structure, as this structure just maintains its original oscillations from valley to peak. Instead, the small hill on the right enlarges, detaches, moves to the right, and becomes an elevation soliton. In accordance with the analytical theory, the breather and soliton retain their identities after all these actions, except for some phase shifts.

(F) *A double pole solution*:

Finally, a double pole solution for the eKdV Eq. (2) is calculated as a special 2-soliton solution. Conceptually, in the inverse scattering transform, the double pole solution will arise when two simple poles in the reflection coefficients coalesce to form a double pole. In the Hirota bilinear mechanism, such solutions can arise from a 'coalescence of wave numbers'. Similar solutions for the modified KdV, the nonlinear Schrodinger and the sine Gordon models can be found in [14-17]. By choosing nearly identical wave numbers and special phase factors, this double pole solution is calculated as

$$u = \left[2\tan^{-1}\left(\frac{g}{f} \right) \right]_x \, ,$$

$$g = 1 - \left[\frac{\alpha^2 + 12\alpha\delta m - 36\delta^2 m^2}{m^2(\alpha - 6\delta m)^2} \right] \exp\left[2m(x - \delta m^2 t) \right] + \frac{2\left[(\alpha^2 - 36\delta^2 m^2)(x - 3\delta m^2 t) + 12\alpha\delta \right] \exp\left[m(x - \delta m^2 t) \right]}{(\alpha - 6\delta m)^2} \, ,$$

$$f = 1 - \left[\frac{\alpha^2 - 12\alpha\delta m - 36\delta^2 m^2}{m^2(\alpha - 6\delta m)^2} \right] \exp\left[2m(x - \delta m^2 t) \right] + 2(x - 3\delta m^2 t)\exp\left[m(x - \delta m^2 t) \right] . \qquad (16)$$

A pair of elevation and depression waves travels to the right as an approximate 'bound state' (Figure 4). The elevation wave lags the depression wave in the left far field, but the role is reversed in the right far field. At time *t* about zero, the profile is very similar to a breather. Analytically this double pole solution consists of mixed algebraic – exponential expressions. Physically it can be regarded as a breather of nearly zero frequency.

The major difference between a double pole solution and a 2-soliton is that the peaks in the former are separated like the logarithm of time *t*, and hence the separation distance is effectively constant. For a two-soliton pattern, the peaks will diverge like the difference in velocities multiplied by time.

3. Conclusions

The interaction between a soliton and a breather for the case of the extended Korteweg – de Vries equation with positive cubic nonlinearity has been described. Our analytical descriptions are obtained using the Hirota bilinear method. Not surprisingly, both the soliton and the breather retain their identities except for a phase shift. The breather can be regarded as two small hills of elevation on the adjacent sides of a deep depression (a 'central valley'), with polarities reversed half a cycle later. The collisions of a breather with a soliton of elevation, or with a soliton of depression, are studied. During the collision phase, the 'central valley' may maintain its oscillations, or rendered 'frozen', depending on the polarity of the colliding soliton and the physical parameters of the breather, e.g., its frequency. Such motions are highly time dependent, and will have important implications for the physical processes which can be modeled with this eKdV equation. As an example, the dynamics of the currents and density perturbations in an evolving internal oceanic tide can be modeled in this way, and one of the implications of this present study is the following assertion. It is highly likely that at least some of the temporal and spatial variability that has been observed in oceanic internal soliton fields may be due to the presence of breathers and their interactions with solitons.

The next phase of this project is to conduct a similar investigation for the case of negative cubic nonlinearity. In this regime, plateau-type solitons and bore-like structures arise. The dynamics of such entities will also be of great interest.

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Figures Captions

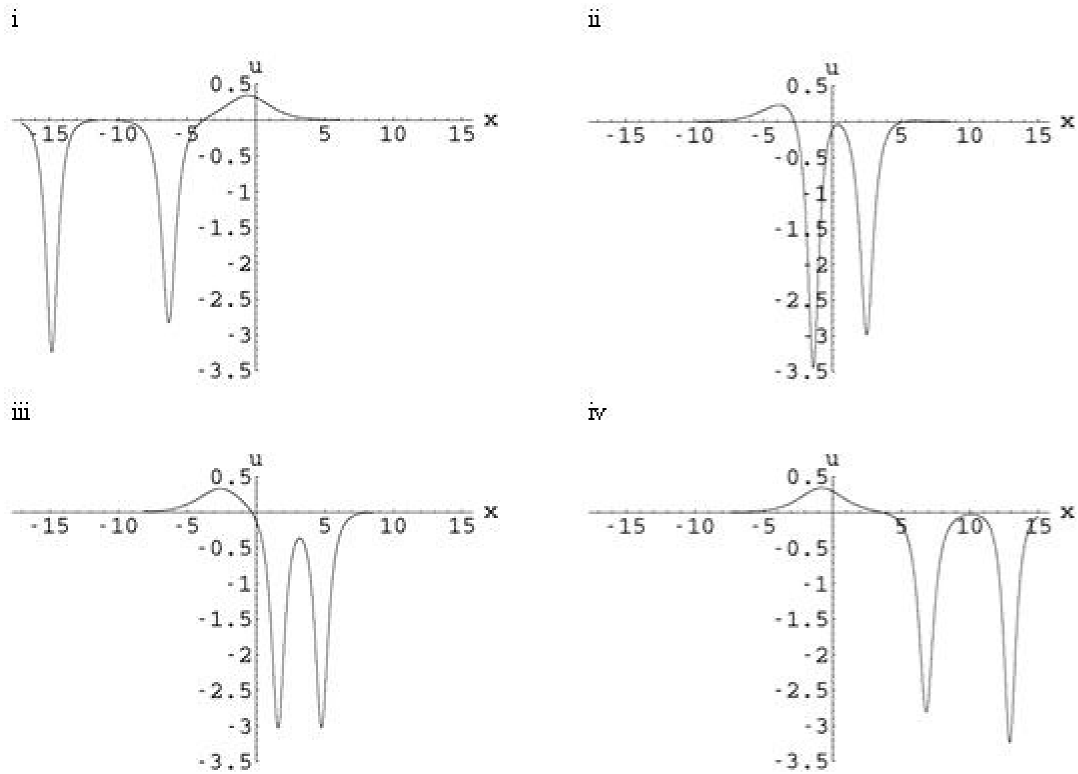
(1) Figure 1: Planar view of the interaction among 3 solitons with $\alpha = 6$, $\delta = 1$, $p_1 = 2$, $p_2 = 1.5$ and $p_3 = 0.9$, at (i) $t = -3.6$, (ii) $t = -0.35$, (iii) $t = 0.45$, and (iv) $t = 2.6$.

(2a) Figure 2a: General profile for the interaction of a breather and a depression soliton with $\alpha = 6$, $\delta = 1$, $m = 0.7$, $n = 0.7$, and $p = 0.9$.

(2b) Figure 2b: Interaction between a breather and a depression soliton at (i) $t = -2$, (ii) $t = -1$, (iii) $t = 0$, (iv) $t = 1$, (v) $t = 2$, and (vi) $t = 3$.

(3) Figure 3: Interaction between a breather and an elevation soliton, $\alpha = 6$, $\delta = 1$, $m = 0.5$, $n = 0.5$, and $p = 0.9$ at (i) $t = -2$, (ii) $t = -1$, (iii) $t = 0$, (iv) $t = 1$, (v) $t = 2$, and (vi) $t = 3$.

(4) Figure 4: A double-pole solution or a breather of nearly zero frequency of Eq. (16) with $\alpha = 6$, $\delta = 1$, $m = 0.7$ at (i) $t = -50$, (ii) $t = 0$, and (iii) $t = 50$.



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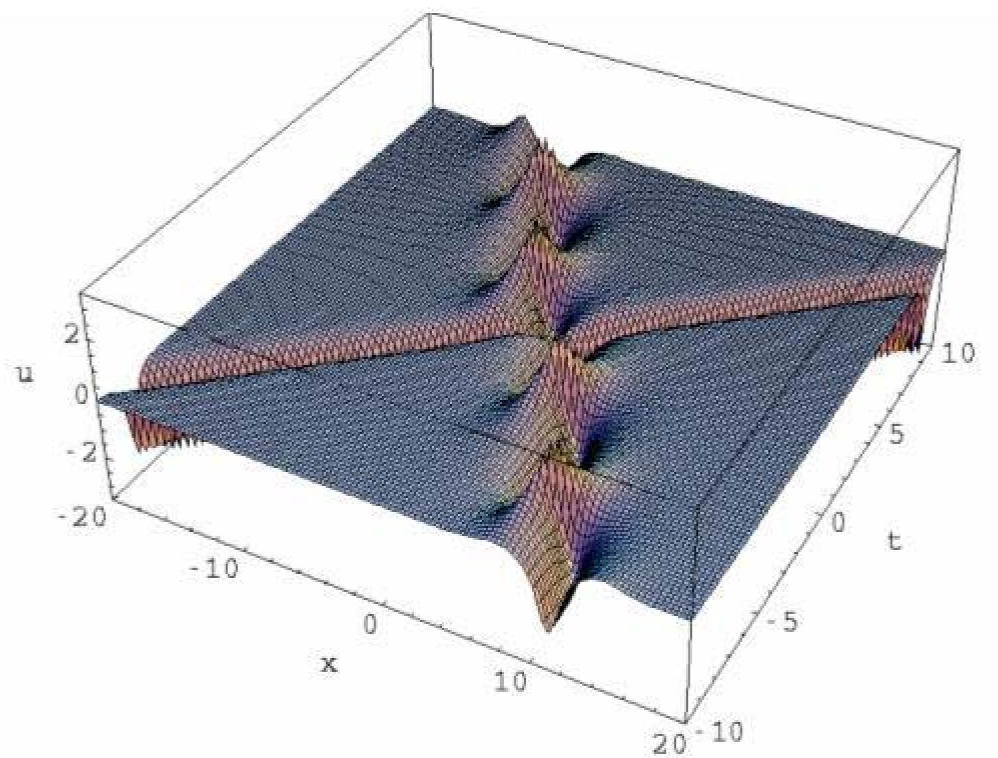
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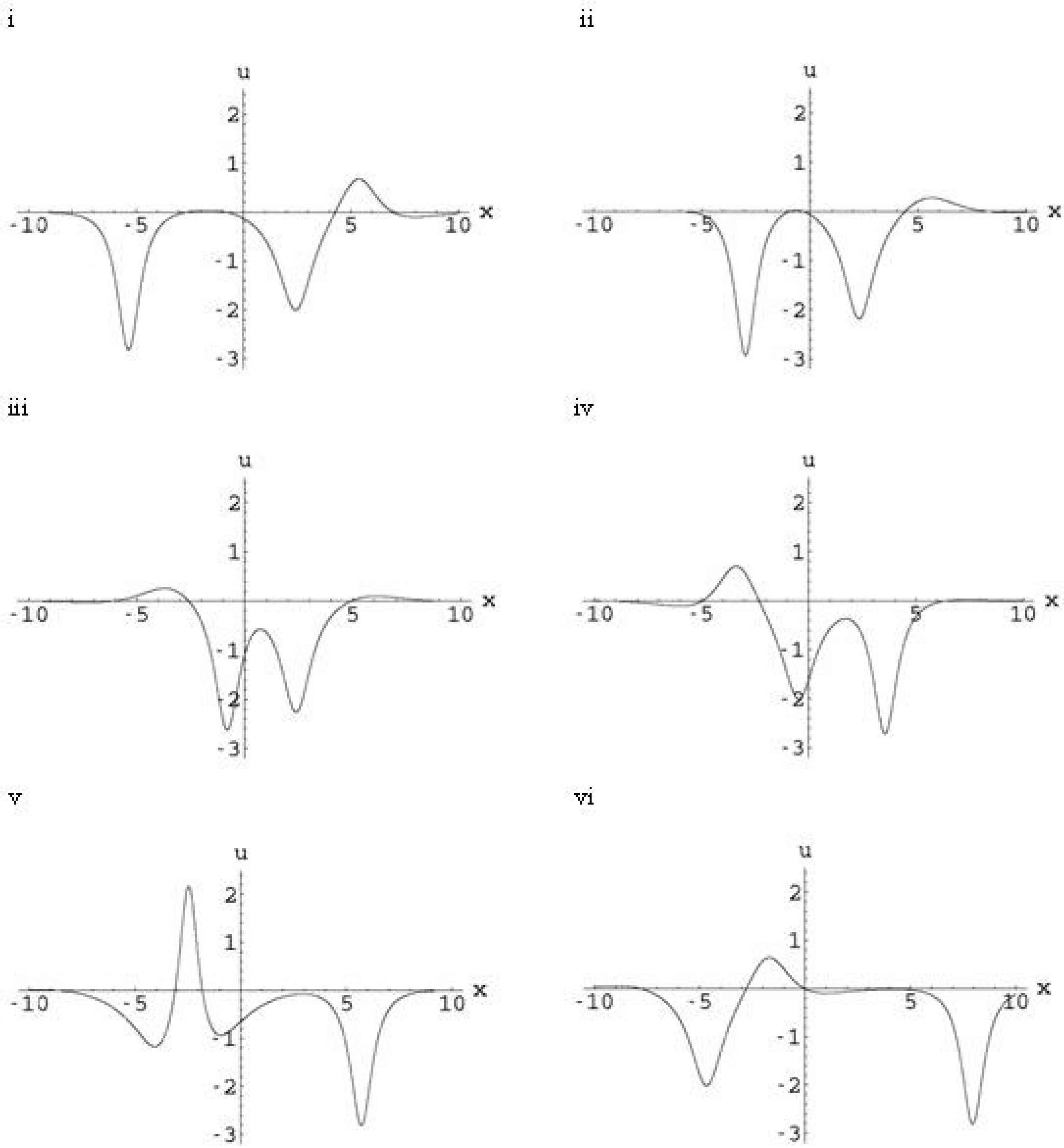
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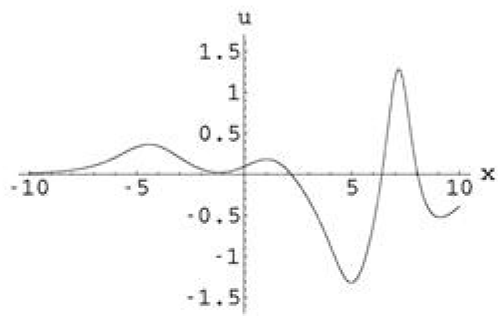

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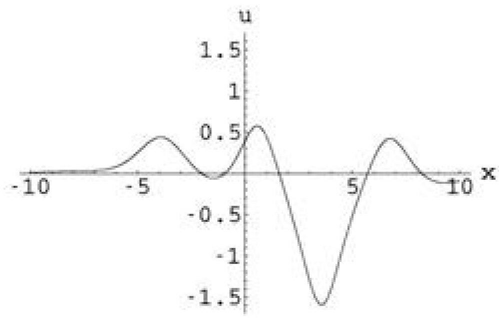

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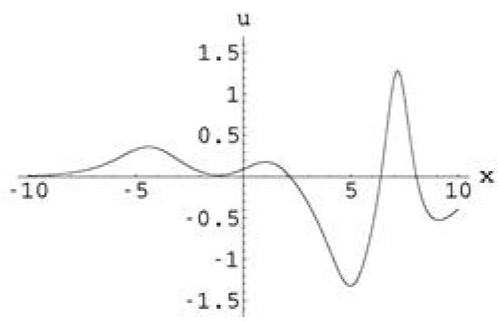

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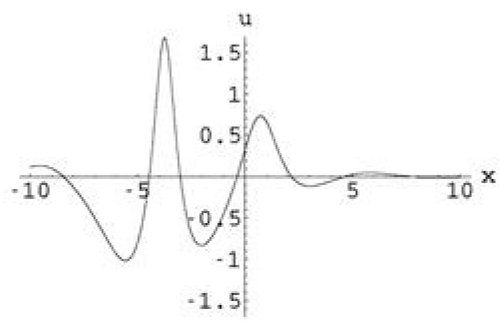

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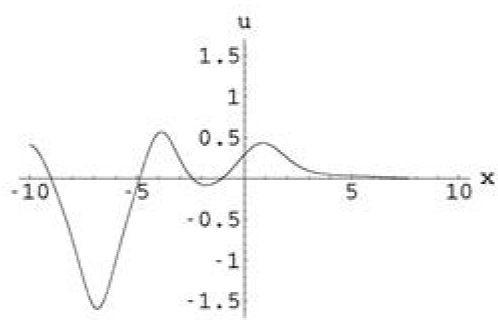

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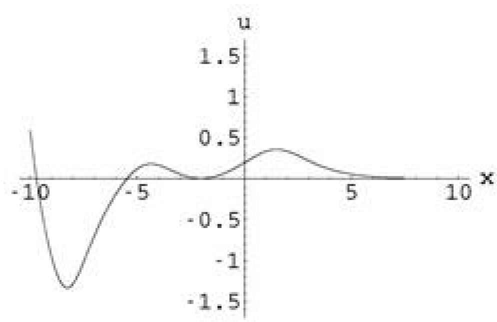

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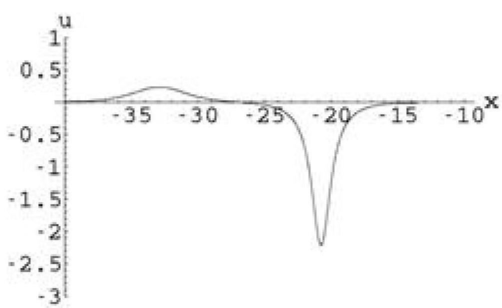
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x
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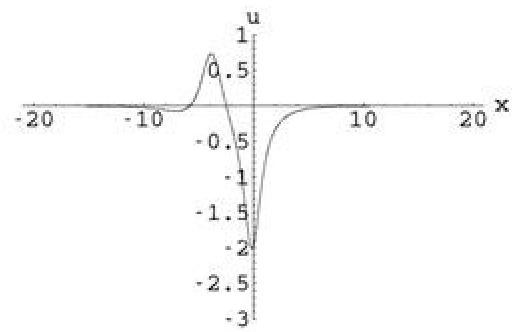
-1
-1.5
-2
-2.5
-3

ii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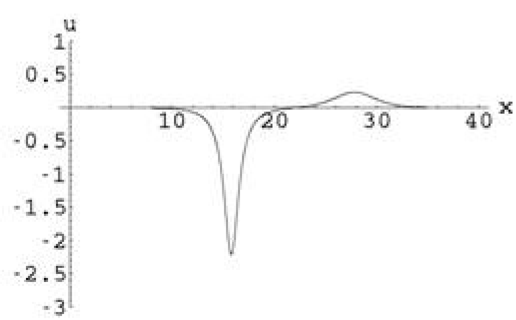
iii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