The HKU Scholars Hub   The University of Hong Kong   香港大學學術庫

Title	An inequality between the diameter and the inverse dual degree of a tree
Author(s)	Siu, MK; Zhang, ZF; Zhou, SM
Citation	Discrete Mathematics, 2002, v. 259, p. 351-358
Issued Date	2002
URL	http://hdl.handle.net/10722/48608
Rights	Creative Commons: Attribution 3.0 Hong Kong License

An inequality between the diameter and the inverse dual degree of a tree

Man-Keung Siu
Department of Mathematics, The University of Hong Kong
Hong Kong

Zhongfu Zhang
Department of Mathematics, Lanzhou Railway Institute
Lanzhou 730070, People's Republic of China

Sanming Zhou
Department of Mathematics, University of Western Australia
Nedlands, WA 6907, Australia

Abstract

Let $R(T), D(T)$ be respectively the radius and diameter of a nontrivial tree T and $I(T) = \sum_{u \in V(T)} 1/\overline{d}(u)$ be the inverse dual degree, where $\overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u)$ for each $u \in V(T)$. In this note we prove that

$$I(T) \geq \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even,} \end{cases}$$

with equality if and only if T is a path of at least 4 vertices. This inequality strengthens a conjecture of Graffiti.

Let $G = (V(G), E(G))$ be a simple, connected graph. The *distance* $d(u,v)$ between two vertices u, v of G is the minimal length of a path from u to v in G. The *diameter* $D(G)$ of G is the largest distance between any two vertices of G. The *radius* $R(G)$ of G is $\min_{u \in V(G)} \max_{v \in V(G)} d(u,v)$. If $S \subseteq V(G)$ and $u \in V(G) \setminus S$, then we denote $d(u, S) = \min_{v \in S} d(u,v)$. The *neighbours* of $u \in V(G)$ are vertices adjacent to u in G and the *neighbourhood* $N(u)$ of u in G is the set of

neighbours of u. Since G is simple, the degree of u is $d(u) = |N(u)|$. The *dual degree* of u and the *inverse dual degree* of G are respectively $\bar{d}(u) = (\sum_{v \in N(u)} d(v))/d(u)$ and $I(G) = \sum_{u \in V(G)} 1/\bar{d}(u)$ [2]. When ambiguity arises we use $d_G(u), \bar{d}_G(u)$, etc., to emphasize that the underlying graph is G.

The main purpose of this note is to prove an inequality between $D(T)$ and $I(T)$. As a consequence we get an inequality involving $R(T)$ and $I(T)$ which strengthens the following conjecture (see [1, 3, 4, 5] for results relating to Graffiti conjectures).

Graffiti Conjecture 577   For any (nontrivial) tree T, $I(T) \geq R(T)$.
(There are examples of G which are not trees such that $I(G) < R(G)$.)

In the following we suppose T is a (nontrivial) tree and $P = v_0 v_1 \ldots v_D$ is a path of maximal length in T, where $D = D(T)$. Then $d(v_0) = d(v_D) = 1$. Let

$$a \;=\; a(P) = |\{v \in V(T) \setminus V(P) : d(v) \geq 2\}|,$$

$$b \;=\; b(P) = \begin{cases} |\{i : d(v_i) \geq 3, 2 \leq i \leq D-2\}|, & \text{if } D \geq 4, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{and} \qquad c \;=\; c(P) = \begin{cases} |\{i : d(v_i) \geq 3, i = 1, D-1\}|, & \text{if } D \geq 2, \\ 0, & \text{if } D = 1. \end{cases}$$

Theorem $I(T) \geq D(T)/2 + a/3 + +b/10 + c/12 + 5/6.$

A *caterpillar* is a tree with the property that the removal of all degree-one vertices yields a path, called the spine. Note that if T is a caterpillar, then $v_1 \ldots v_{D-1}$ is the spine. To prove the theorem we need the following lemmas.

Lemma 1   *Suppose T is not a caterpillar (so in particular $D(T) \geq 4$) and u is a vertex not in P such that $d(u) \geq 2$ and $d(u, V(P))$ is as large as possible. Let T' be the subtree obtained from T by deleting all degree-one vertices adjacent to u. Then $D(T) = D(T')$ and $I(T) \geq I(T')+1/3.$*

Proof   We first note that all but one neighbours of u have degree one, for otherwise there would be a neighbour w of u not in P with $d(w) \geq 2$ and $d(w, V(P)) > d(u, V(P))$, violating the choice of u. Suppose $N(u) = \{u_1, \ldots, u_m, v\}$ where $d(u_i) = 1$, for $i \in \{1, \ldots, m\}$ and $d(v) = r$. Then

$D(T) = D(T')$. Let $\sigma = \sum_{x \in N(v) \setminus \{u\}} d(x)$. Since $\sigma + 1 \geq r$, we have

$$
\begin{aligned}
&I(T) - I(T') \\
&= \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \left(\frac{1}{d_T(u)} - \frac{1}{d_{T'}(u)}\right) + \left(\frac{1}{d_T(v)} - \frac{1}{d_{T'}(v)}\right) \\
&= \frac{m}{m+1} + \left(\frac{m+1}{m+r} - \frac{1}{r}\right) + \left(\frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}\right) \\
&= 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1} + \left(\frac{m}{m+r} + \frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}\right) \\
&\geq 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1}.
\end{aligned}
$$

Note that $1/(m+x) - 1/x$ is an increasing function of x and $r \geq 2, m \geq 1$. We have from the inequality above that

$$
\begin{aligned}
&I(T) - I(T') \\
&\geq 1 + \frac{1}{m+2} - \frac{1}{2} - \frac{1}{m+1} \\
&= \frac{1}{2} - \frac{1}{(m+1)(m+2)} \\
&\geq \frac{1}{3}.
\end{aligned}
$$

<div align="right">Q.E.D.</div>

Lemma 2 *Suppose T is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) \geq 3$ (respectively $d(v_{D-1}) \geq 3$) and let T' be the subtree obtained from T by deleting all degree-one vertices adjacent to v_1 (respectively v_{D-1}) excepting v_0 (respectively v_D). Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/12$.*

Proof Suppose $d(v_1) = m + 2 \geq 3, d(v_2) = r, d(v_3) = s$. Then $r, s \geq 2$ and $D(T) = D(T')$. We have

$$
\begin{aligned}
&I(T) - I(T') \\
&= \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=0}^{2} \left(\frac{1}{d_T(v_i)} - \frac{1}{d_{T'}(v_i)}\right) \\
&= \frac{m}{m+2} + \left(\frac{1}{m+2} - \frac{1}{2}\right) + \left(\frac{m+2}{m+r+1} - \frac{2}{r+1}\right) + \left(\frac{r}{m+r+s} - \frac{r}{r+s}\right) \\
&= m\left[\left(\frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)}\right) + r\left(\frac{1}{(r+1)(m+r+1)} - \frac{1}{(r+s)(m+r+s)}\right)\right] \\
&\geq m\left(\frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)}\right) \\
&\geq m\left(\frac{1}{2(m+2)} - \frac{1}{3(m+3)}\right) \\
&= \frac{m(m+5)}{6(m+2)(m+3)} \\
&\geq \frac{1}{12}.
\end{aligned}
$$

<div align="right">Q.E.D.</div>

Lemma 3 *Suppose T is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) = d(v_{D-1}) = 2$ and let T' be the subtree obtained by deleting all degree-one neighbours of v_α, where v_α is the vertex nearest to one terminal vertex of P such that $d(v_i) \geq 3$. Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/10$.*

<div align="center">3</div>

Proof Without loss of generality we may assume $\alpha \leq \lfloor \frac{D}{2} \rfloor$. Let $u_1, \ldots, u_m$ be all the degree-one neighbours of v_α. We have $d(v_{\alpha-1}) = 2$. Let $d(v_{\alpha-2}) = r$ ($r = 1$ if $\alpha = 2$ and $r = 2$ otherwise), $d(v_{\alpha+1}) = s, d(v_{\alpha+2}) = t$. Clearly we have $D(T) = D(T')$. If $D \geq 5$, then $t \geq 2$, hence we have

$$
\begin{aligned}
&I(T) - I(T') \\
&= \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=\alpha-1}^{\alpha+1} \left(\frac{1}{d_T(v_i)} - \frac{1}{d_{T'}(v_i)} \right) \\
&= \frac{m}{m+2} + \left(\frac{2}{m+r+2} - \frac{2}{r+2} \right) + \left(\frac{m+2}{m+s+2} - \frac{2}{s+2} \right) \\
&\quad + \left(\frac{s}{m+s+t} - \frac{s}{s+t} \right) \\
&= \frac{m}{m+2} + \left(\frac{2}{m+3} - \frac{2}{3} \right) + \left(\frac{m+2}{m+s+2} - \frac{2}{s+2} \right) \\
&\quad + \left(\frac{s}{m+s+2} - \frac{s}{s+2} \right) \\
&= \frac{m(m+5)}{3(m+2)(m+3)} \\
&\geq \frac{1}{6} \geq \frac{1}{10}.
\end{aligned}
$$

If $D = 4$, then a straightforward calculation shows that

$$
\begin{aligned}
&I(T) - I(T') \\
&= \frac{1}{6} + \frac{4}{m+3} - \frac{2}{m+2} - \frac{2}{m+4} \\
&= \frac{1}{6} - \frac{4}{(m+2)(m+3)(m+4)} \\
&\geq \frac{1}{10}.
\end{aligned}
$$

Q.E.D.

Now let us prove the main theorem. If $T = P_n$, the path with n vertices, then

$$
I(P_n) - D(P_n)/2 = \begin{cases} 3/2, & n = 2 \\ 1, & n = 3 \\ 5/6, & n \geq 4. \end{cases}
$$

If $D(T) = 2$, then T is a star with $a = b = 0$, $c = 1$ and $I(T) - D(T)/2 = 1 \geq c/12 + 5/6$. If $D(T) = 3, T \neq P_4$, then $a = b = 0$ and T has exactly two vertices with degree ≥ 2. Suppose the degrees of them are $l+1, m+1$. Then $\max\{l, m\} \geq 2$ and

$$
\begin{aligned}
&I(T) - D(T)/2 \\
&= \frac{\ell+m+2}{l+m+1} + \frac{l}{l+1} + \frac{m}{m+1} - \frac{3}{2} \\
&= \frac{1}{l+m+1} - \frac{1}{l+1} - \frac{1}{m+1} + \frac{3}{2} \\
&\geq \frac{c}{12} + \frac{5}{6}.
\end{aligned}
$$

In the following we suppose T is not a path and $D = D(T) \geq 4$. If T is not a caterpillar, let u be the vertex not in P such that $d(u) \geq 2$ and $d(u, V(P))$ is as large as possible. Then all but one neighbours of u have degree one. Removing from T all the degree-one neighbours of

u we get a subtree T_1 with $D(T) = D(T_1), I(T) \geq I(T_1) + 1/3$, according to Lemma 1. If T_1 is not a caterpillar, then repeat this procedure until a caterpillar is obtained. It is clear that after a steps we get a sequence $T = T_0, T_1, \ldots, T_a$ such that each T_{i+1} is a subtree of T_i and $D(T_i) = D(T_{i+1})$ and $I(T_i) \geq I(T_{i+1}) + 1/3$. So we have $D(T) = D(T_a)$ and $I(T) \geq I(T_a) + a/3$.

If $d(v_1) \geq 3$ in T_a, then delete all the degree-one neighbours of v_1 except v_0. We get T_{a+1} with the same diameter as T such that $I(T_a) \geq I(T_{a+1}) + 1/12$, according to Lemma 2. If $d(v_{D-1}) \geq 3$, we do the same thing. In this way c subtrees are added to the sequence above and we get $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}$ with $D(T) = D(T_{a+c})$ and $I(T) = I(T_{a+c}) + a/3 + c/12$.

Now we have $d_{T_{a+c}}(v_1) = d_{T_{a+c}}(v_{D-1}) = 2$ and $d_{T_{a+c}}(v_i) = d(v_i)$, $i \notin \{1, D-1\}$. If T_{a+c} is not a path, then according to Lemma 3 we can delete all degree-one neighbours of some v_α and obtain a subtree T_{a+c+1} with $I(T_{a+c}) \geq I(T_{a+c+1}) + 1/10$. Repeat the procedure until we obtain a path P. When the process stops we get a sequence $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}, \ldots, T_{a+c+b} = P$ with $I(T) \geq I(T_{a+c}) + a/3 + c/12 \geq I(P) + a/3 + b/10 + c/12$. Since $I(P) = D(P)/2 + 5/6$, as we have just proved it for paths, and since $D(P) = D(T)$, we get $I(T) \geq D(T)/2 + a/3 + b/10 + c/12 + 5/6$. This completes the proof.

Note that $R(T) = \lceil D(T)/2 \rceil$ for any tree T and a, b, c are non-negative integers. Hence we have the following corollary.

Corollary For any (nontrivial) tree T

$$I(T) \geq \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even,} \end{cases}$$

with equality if and only if T is a path of at least four vertices.

This corollary tells us that $I(T) - R(T)$ is bounded below. We point out that it is unbounded above. In fact, for the full binary tree T of height $h \geq 3$ we have $I(T) - R(T) = 2^{h+2}/5 - h - 1/4$, which can be arbitrarily large as h tends to infinity.

References

[1] F.R.K. Chung, The average distance and the independence number, J. Graph Theory 12(1988)229-235.

[2] S. Fajtlowicz, Written on the wall, a list of conjectures of Graffiti, preprint, University of Houston, USA.

[3] S. Fajtlowicz, On conjectures of Graffiti, II, Congr. Numer. 60(1987)187-197.

[4] S. Fajtlowicz, On conjectures of Graffiti, Disc. Math. 72(1988)113-118.

[5] O. Favaron, M.Mahéo and J.-F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti–II), Disc. Math. 111(1993)197-220.